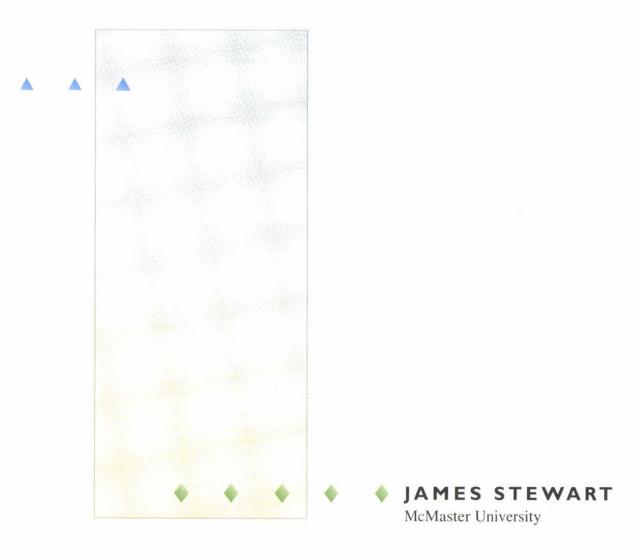
# CALCULUS

**Concepts AND Contexts** 

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For Geoff, Brad, Anna, and Jon. For Debbie, Lorraine, Alan, and Matt. For absent friends.

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# Preface

Mathematics departments are engaged in a debate over calculus reform. Our debate, of course, is part of an ongoing one that started 300 years ago when the first calculus textbook was published. But the current debate has sometimes turned acrimonious. Tales abound of departments badly split, with instructors favoring reform and those favoring a traditional approach not speaking to one another.

What is arousing such passions? What do people *mean* when they say they are in favor of calculus reform? As a result of talking with many instructors and reading hundreds of survey responses, I have learned that different people mean different things. They have passionately held opinions; there is common ground on some issues, but instructors are diametrically opposed on other issues. Let's look at some of the suggested key components of calculus reform.

Several survey respondents think that *technology* is the most important issue. Certainly, those of us who have watched our students use graphing calculators or computers know how enlivening this can be. We have seen from the looks on their faces how these devices engage our students' attention and enable them to become active learners. But these machines have been used by many schools with traditional curricula. For example, several traditional calculus texts (including my own *Calculus, Third Edition*) make extensive use of technology. Furthermore, I know of some very innovative reform calculus courses that use virtually no technology. So, while technology can be a critical component for implementing the goals of reform, I don't believe that technology itself characterizes reform.

Many people cite the *Rule of Three* as a key principle: "Topics should be presented geometrically, numerically, and algebraically." The implication is that, in the past, the algebraic point of view has been predominant and the graphical and numerical aspects have been given short shrift. More recently, the Rule of Three has been expanded to become the *Rule of Four* by emphasizing the verbal, or descriptive, point of view as well. But again, I think that my traditional book *Calculus, Third Edition* incorporates visualization and the Rule of Three. So I believe that the Rule of Three (or Four), important guiding principle though it is, still does not capture the most critical aspect of reform by itself.

Some respondents think that the enhanced attention to applications is a key feature and that instructors now have more freedom to choose applications for which they themselves have enthusiasm. While this aspect is certainly true, it is just as important in a traditional course.

So what *do* I think is the essence of calculus reform? In a word: *concepts*. We sometimes forget that the impetus for the current reform movement came from the Tulane Conference in January, 1986. I believe that the primary goal of reform should be what that conference formulated as their first recommendation:

Focus on conceptual understanding

What technology, the Rule of Four, and other aspects of reform have done is to enable instructors to use new tools and approaches to conceptual reasoning and skills. Visualization, numerical and graphical experimentation, and other approaches have changed how we teach conceptual reasoning in fundamental ways.

I think that nearly everybody—from the radical reformer to the staunch traditionalist—supports the central goal of focusing on conceptual understanding. So why are there so many heated discussions in mathematics departments? I believe that the explanation lies in what is involved in implementing this goal. If we are serious about emphasizing conceptual understanding, then we have to expect faculty and students to give clear explanations of what symbols mean and why things work the way they do. That is simply not going to happen unless we take the time to work patiently with students. We need to slow down, provide multiple approaches, and not rush through the material when a new concept is introduced. It follows that some less conceptual traditional calculus topics will not be covered in many courses. And that is where the controversy arises.

Most of the existing reform projects have greatly reduced the coverage of techniques of integration and I agree that this is appropriate. (This book has no full chapter on methods of integration, but substitution and parts are covered in Chapter 5 and partial fractions in Appendix F.) I have also streamlined the coverage of many other topics in order to free up time to achieve conceptual understanding. But I have not gone as far as some other reform texts in deleting traditional topics. In particular, I have decided to retain related rates problems, l'Hospital's Rule, and series of constants. My premise in writing this book has been that it is possible to achieve conceptual understanding and still retain the best traditions of traditional calculus. I hope that this book will support a wider range of approaches to teaching calculus and improving students' conceptual understanding in diverse college and university settings.

Features · · · ·	
Conceptual Exercis	The most important way to foster conceptual understanding is through the prob- lems that we assign. To that end I have devised various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of
Pages 109, 128, 140, 380, 577	the section. (See, for instance, the first couple of exercises in Sections 2.2, 2.4, 2.5, 5.3, and 8.2.) Similarly, review sections begin with a Concept Check and a True-False Quiz. Other exercises test conceptual understanding through graphs (see
Pages 156, 170	Exercises 1–3 in Section 2.7 and Exercises 29–36 in Section 2.8). Another type of exercise uses verbal description to test conceptual understanding (see Exercise 8
Pages 129, 171, 180, 437	in Section 2.4; Exercise 46 in Section 2.8; Exercises 5, 9, and 10 in Section 2.10; and Exercise 53 in Section 5.9). I particularly value problems that combine and compare graphical, numerical, and algebraic approaches (see Exercise 30 in Sec-
Pages 141, 200, 544	tion 2.5, Exercise 39 in Section 3.1, and Exercise 2 in Section 7.6).
Real-World Da	ta My assistants and I spent a great deal of time looking in libraries, contacting com- panies and government agencies, and searching the Internet for interesting real- world data to introduce, motivate, and illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such nu-
Pages 12, 16	merical data or graphs. See, for instance, Figures 1, 11, and 12 in Section 1.1 (seis- mograms from the Northridge earthquake), Figure 5 in Section 5.3 (San Francisco
Pages 379, 360, 381	power consumption), Exercise 10 in Section 5.1 (velocity of the space shuttle <i>Endeavour</i> ), and Exercise 56 in Section 5.3 (Consumer Price Index).

Projects One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. *Applied Projects* involve applications that are designed to appeal to the imagination of students. The project after Section 7.4 asks whether a ball thrown upward takes longer to reach its maximum height or to fall back to its original height. (The answer might surprise you.) *Laboratory Projects* involve technology; the project following Section 3.5 shows how to use Bézier curves to design shapes that represent letters for a laser printer. *Writing Projects* ask students to compare present-day methods with those of the founders of calculus—Fermat's method for finding tangents, for instance. Suggested references are supplied. *Discovery Projects* anticipate results to be discussed later or cover optional topics (hyperbolic functions) or encourage discovery through pattern recognition (see the project following Section 5.7).

Rigor I include fewer proofs than in my more traditional books, but I think it is still worthwhile to expose students to the idea of proof and to make a clear distinction between a proof and a plausibility argument. The important thing, I think, is to show how to deduce something that seems less obvious from something that seems more obvious. A good example is the use of the Mean Value Theorem to prove the Evaluation Theorem (Part 2 of the Fundamental Theorem of Calculus). I have chosen, on the other hand, not to prove the convergence tests but rather to argue intuitively that they are true.

Technology The availability of technology makes it not less important but more important to clearly understand the concepts that underlie the images on the screen. But, when properly used, graphing calculators and computers are powerful tools for discovering and understanding those concepts. I assume that the student has access to either a graphing calculator or a computer algebra system. The icon indicates an example or exercise that definitely requires the use of such technology, but that is not to say that it can't be used on the other exercises as well. The symbol is reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or the TI-92) are required. But technology doesn't make pencil and paper obsolete. Hand calculation and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where the hand or the machine is appropriate.

Problem Solving Students usually have difficulties with problems for which there is no single well-defined procedure for obtaining the answer. I think nobody has improved very much on George Polya's four-stage problem-solving strategy and, accordingly, I have included a version of his problem-solving principles at the end of Chapter 1. They are applied, both explicitly and implicitly, throughout the book. After the other chapters I have placed sections called *Focus on Problem Solving*, which feature examples of how to tackle challenging calculus problems. In selecting the varied problems for these sections I kept in mind the following advice from David Hilbert: "A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts." When I put these challenging problems on assignments and tests I grade them in a different way. Here I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant.

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Page 237

Page 157

Page 414

Pages 186, 263, 344, 442, 499, 556, 640

Content .

The book begins with A Preview of Calculus, which gives an overview of the subject and includes a list of questions to motivate the study of calculus.

Chapter | From the beginning, multiple representations of functions are stressed; verbal, nu-Functions and Models merical, visual, and algebraic. The standard functions, including exponential and logarithmic functions, are reviewed here from these four points of view. Parametric curves are introduced in the first chapter, partly so that curves can be drawn easily, with technology, whenever needed throughout the text. This early placement also enables inverse functions to be graphed in Section 1.6, tangents to parametric curves to be treated in Section 3.5, and graphing such curves to be covered in Section 4.4. All students should read the general discussion of modeling at the beginning of Section 1.7 as a background to the models that pervade the book. The remainder of the section (on curve fitting) is optional, but some instructors may wish to exploit the ability of the newest calculators to model data. A small number of later exercises make use of this material (see, for example, Exercises 59 and 60 in Section 3.5 and Exercise 54 in Section 4.2).

Chapter 2 The material on limits is motivated by a prior discussion of the tangent and veloc-Limits and Derivatives ity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. (The precise  $\varepsilon$ - $\delta$  definition of a limit is provided in Appendix D for those who wish to cover it.) It is important not to rush through Sections 2.7-2.10, which deal with derivatives (especially with functions defined graphically and numerically) before the differentiation rules are covered in Chapter 3. Here the examples and exercises explore the meanings of derivatives in various contexts. Section 2.10 foreshadows, in an intuitive way and without differentiation formulas, the material on shapes of curves that is studied in greater depth in Chapter 4.

Chapter 3 All the basic functions are differentiated here. When derivatives are computed in Differentiation Rules applied situations, students are asked to explain their meanings. Optional topics (hyperbolic functions, an early introduction to Taylor polynomials) are explored in Discovery and Laboratory Projects.

Chapter 4 The basic facts concerning extreme values and shapes of curves are derived using Applications of Differentiation the Mean Value Theorem as the starting point. Graphing with technology emphasizes the interaction between calculus and calculators and the analysis of families of curves. Some substantial optimization problems are provided, including an explanation of why you need to raise your head 42° to see the top of a rainbow.

> Chapter 5 The area problem and the distance problem serve to motivate the definite integral. Integrals I have decided to make the definition of an integral easier to understand by using subintervals of equal width. Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables. There is no separate chapter on techniques of integration, but substitution and parts are covered here and partial fractions are treated in Appendix F. The use of computer algebra systems is discussed in Section 5.7.

Chapter 6 General methods, not formulas, are emphasized. The goal is for students to be able Applications of Integration to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral. There are more applications here than can realistically be covered in a given course. Instructors should select applications suitable for their students and for which they themselves have enthusiasm.

Pages 67, 233, 300

Page 75

Pages 236, 281

Pages 151-181

Page 175

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Pages 410-412

Chapter 7 Modeling is the theme that unifies this introductory treatment of differential equa-Differential Equations Direction fields and Euler's method are studied before separable equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. Predator-prey models are used to illustrate systems of differential equations.

Chapter 8 Tests for the convergence of series are considered briefly, with intuitive rather than Sequences and Series formal justifications. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics. Error estimates include those from graphing devices.



# Ancillaries

*Calculus: Concepts and Contexts, Single Variable* is supported by a complete set of ancillaries developed under my direction. Each piece has been designed to enhance student understanding and to facilitate creative instruction.

A DATA DATA DATA DATA DATA DATA DATA

The following resources are available, free of charge, to adopters of the text.

Instructor's Guide by Harvey B. Keynes, James Stewart, Douglas Shaw, and Robert Hesse Offering suggestions on how to implement ideas about reform into your calculus course, this Guide serves as a practical roadmap to topics and projects in the text. Each section of the main text is discussed from several viewpoints and contains suggested time to allot, points to stress, text discussion topics, core materials for lecture, workshop/discussion suggestions, group work exercises in a form suitable for handout, and suggested homework problems.

## Complete Solutions Manual by Jeffery A. Cole Provides detailed solutions to all exercises in the text.

Transparencies by James Stewart Thirty full-color transparencies featuring 80 of the more complex diagrams from the text for use in the classroom.

Test Items by William Tomhave and Xueqi Zeng Organized according to the main text, this complete set of Test Items contains both multiple-choice and open-ended questions, offering a range of model problems, including short-answer questions that focus narrowly on one basic concept; items that integrate two or more concepts and require more detailed analysis and written response; and application problems, including situations that use real data generated in laboratory settings.

Electronic Test Items by William Tomhave, Xueqi Zeng, and Charles Heuer This computerized version of the printed Test Items allows instructors to insert their own questions and customize ones that are provided. Some test items will be algorithmically generated. A complete range of student ancillaries is also available:

Study Guide by Robert Burton and Dennis Garity

Offering additional explanations and worked-out examples, and formatted to provide guided practice, each section in this Study Guide corresponds to a section in the text. Every section contains a short list of key concepts; a short list of skills to master; a brief introduction to the ideas of the section; an elaboration of the concepts and skills, including extra worked-out examples; and links in the margin to earlier and later material in the text and Study Guide.

### Student Solutions Manual by Jeffery A. Cole

Contains detailed solutions to all odd-numbered exercises in the text.

Lab Manuals Each of these comprehensive lab manuals will help students learn to effectively use the technology tools available to them. Each lab contains clearly explained exercises and a variety of labs and projects to accompany the text.

# CalcLabs with Maple®

by Al Boggess, David Barrow, Maury Rahe, Jeff Morgan, Samia Massoud, Philip Yasskin, Michael Stecher, Art Belmonte, and Kirby Smith

#### CalcLabs with Mathematica<sup>®</sup>

by David Barrow, Art Belmonte, Nancy Blachman, Al Boggess, Samia Massoud, Jeff Morgan, Maury Rahe, Kirby Smith, Michael Stecher, Colin Williams, and Philip Yasskin

### CalcLabs with Derive®

by David Barrow, Art Belmonte, Al Boggess, Samia Massoud, Jeff Morgan, Maury Rahe, Kirby Smith, Michael Stecher, and Philip Yasskin

CalcLabs with the TI-82/83 by Jeff Morgan and Selwyn Hollis

CalcLabs with the TI-85/86 by David Rollins

CalcLabs with the TI-92 by Selwyn Hollis

A Companion to Calculus by Dennis Ebersole, Doris Schattschneider, Alicia Sevilla, and Kay Somers

Written to improve algebra and problem-solving skills of students taking a calculus course, every chapter in this companion is keyed to a calculus topic, providing conceptual background and specific algebra techniques needed to understand and solve calculus problems related to that topic. It is designed for calculus courses that integrate the review of precalculus concepts (web site http://www.hvcc.edu/faculty/ amm/fipse/fipse.htm) or for individual use.

### Scientific Notebook™ by TCI Software Research

Featuring a built-in version of the Maple<sup>®</sup> computer algebra system, *Scientific Notebook*<sup>TM</sup> provides students with the computational power necessary to solve the most complex homework problems. It is easy to link to resources within a document, in other documents, or in documents on the World Wide Web. This combination gives students a unique tool for exploring, explaining, and understanding key mathematical and scientific concepts.

System requirements: *Scientific Notebook* runs on any Windows<sup>®</sup> 95 or Windows NT<sup>®</sup> 4.0 system. It requires 10 MB of hard disk space, a CD-ROM drive, and an Internet connection to access the *Scientific Notebook* Resource Center.

Doing Calculus with Scientific Notebook™

by Darel W. Hardy and Carol L. Walker

Contains activities that will help you develop a clearer understanding of calculus.

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JAMES STEWART

# To the Student

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix I. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from mine, don't immediately assume you're wrong. There may be an algebraic or trigonometric identity that connects the answers. For example, if the answer given in the back of the book is  $\sqrt{2} - 1$  and you obtain  $1/(1 + \sqrt{2})$ , then you're right and rationalizing the denominator will show that the answers are equivalent.

The icon  $\frac{16}{100}$  indicates an example or exercise that definitely requires the use of either a graphing calculator or a computer with graphing software. (Section 1.3 discusses the use of these graphing devices and some of the pitfalls that you may encounter.) But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol  $\frac{16}{100}$  is reserved for problems in which the full resources of a computer algebra system (like Derive, Maple, Mathematica, or TI-92) are required. You will also encounter the symbol  $\frac{100}{100}$  which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. I hope you will discover that it is not only useful but also intrinsically beautiful.



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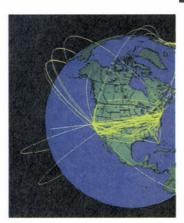
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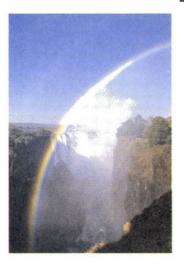
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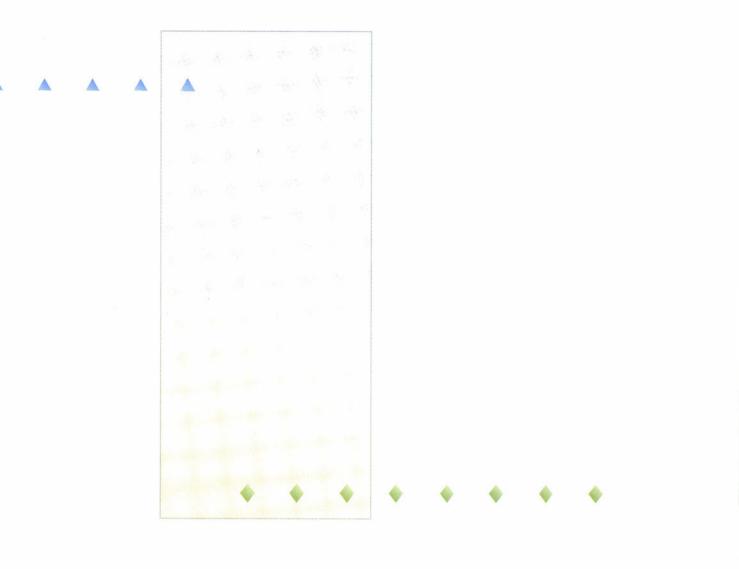
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# CALCULUS

# **Concepts AND Contexts**

# SINGLE • VARIABLE • •



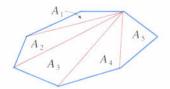
• • • By the time you finish this course, you will be able to use the ideas of calculus to decide where to sit in a movie theater, explain the shapes of cans, position a shortstop, and explain the formation and location of rainbows. See the list of questions on page 9.

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study. Here we give a glimpse of some the main ideas of calculus by showing how limits arise when we attempt to solve a variety of problems.

# A Preview of Calculus

# The Area Problem

A.



 $A = A_1 + A_2 + A_3 + A_4 + A_5$ 

FIGURE I

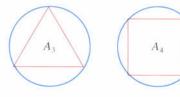
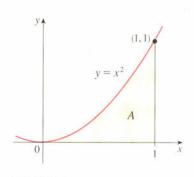
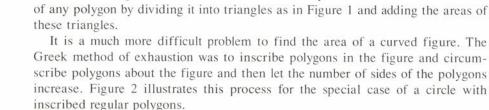


FIGURE 2







 $A_6$ 

Let  $A_n$  be the area of the inscribed polygon with n sides. As n increases, it appears that  $A_n$  becomes closer and closer to the area of the circle. We say that the area of the circle is the *limit* of the areas of the inscribed polygons, and we write

 $A_7$ 

 $A_{12}$ 

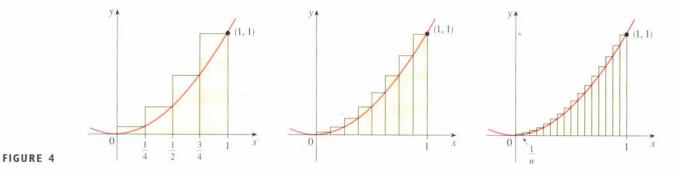
...

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A

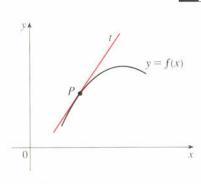
$$A = \lim_{n \to \infty} A_n$$

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century B.C.) used exhaustion to prove the familiar formula for the area of a circle:  $A = \pi r^2$ .

We will use a similar idea in Chapter 5 to find areas of regions of the type shown in Figure 3. We will approximate the desired area A by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate A as the limit of these sums of areas of rectangles.



The area problem is the central problem in the branch of calculus called *integral calculus*. The techniques that we will develop in Chapter 5 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.



#### **FIGURE 5** The tangent line at *P*

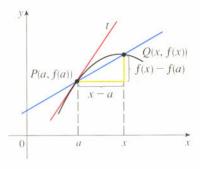


FIGURE 6

The secant line PQ

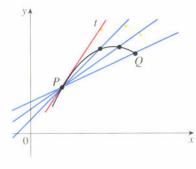


FIGURE 7 Secant lines approaching the tangent line

#### The Tangent Problem

2

Consider the problem of trying to find the equation of the tangent line *t* to a curve with equation y = f(x) at a given point *P*. (We will give a precise definition of a tangent line in Chapter 2. For now you can think of it as a line that touches the curve at *P* as in Figure 5.) Since we know that the point *P* lies on the tangent line, we can find the equation of *t* if we know its slope *m*. The problem is that we need two points to compute the slope and we know only one point, *P*, on *t*. To get around the problem we first find an approximation to *m* by taking a nearby point *Q* on the curve and computing the slope  $m_{PQ}$  of the secant line *PQ*. From Figure 6 we see that

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Now imagine that Q moves along the curve toward P as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope  $m_{PQ}$  of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$n = \lim_{Q \to P} m_{PQ}$$

1

and we say that *m* is the limit of  $m_{PQ}$  as *Q* approaches *P* along the curve. Since *x* approaches *a* as *Q* approaches *P*, we could also use Equation 1 to write

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Specific examples of this procedure will be given in Chapter 2.

The tangent problem has given rise to the branch of calculus called *differential calculus*, which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601–1665) and were developed by the English mathematicians John Wallis (1616–1703), Isaac Barrow (1630–1677), and Isaac Newton (1642–1727) and the German mathematician Gottfried Leibniz (1646–1716).

The two branches of calculus and their chief problems, the area problem and the tangent problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent problem and the area problem are inverse problems in a sense that will be described in Chapter 5.

## Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information indicate to us? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let us analyze the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at 1-second intervals as in the following chart:

t = Time elapsed (s)	0	1	2	3	4	5
d = Distance (ft)	0	2	10	25	43	78

As a first step toward finding the velocity after 2 seconds have elapsed, let us find the average velocity during the time interval  $2 \le t \le 4$ :

average velocity = 
$$\frac{\text{distance traveled}}{\text{time elapsed}}$$
  
=  $\frac{43 - 10}{4 - 2}$   
= 16.5 ft/s

Similarly, the average velocity in the time interval  $2 \le t \le 3$  is

average velocity = 
$$\frac{25 - 10}{3 - 2}$$
 = 15 ft/s

We have the feeling that the velocity at the instant t = 2 cannot be much different from the average velocity during a short time interval starting at t = 2. So let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following chart:

1	2.0	2.1	2.2	2.3	2,4	2.5
d	10.00	11.02	12.16	13.45	14.96	16.80

Then we can compute, for instance, the average velocity over the time interval [2, 2.5]:

average velocity = 
$$\frac{16.80 - 10.00}{2.5 - 2} = 13.6$$
 ft/s

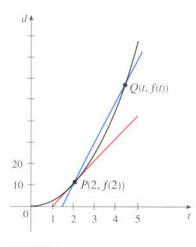
The results of such calculations are shown in the following chart:

Time interval	[2,3]	[2, 2.5]	[2,2,4]	[2,2,3]	[2,2.2]	[2,2,1]
Average velocity (ft/s)	15.0	13.6	12.4	11.5	10.8	10.2

The average velocities over successively smaller intervals appear to be getting closer to a number near 10, and so we expect that the velocity at exactly t = 2 is about 10 ft/s. In Chapter 2 we will define the instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

In Figure 8 we show a graphical representation of the motion of the car by plotting the distance traveled as a function of time. If we write d = f(t), then f(t) is the number of feet traveled after t seconds. The average velocity in the time interval [2, t] is

average velocity = 
$$\frac{\text{distance traveled}}{\text{time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$



which is the same as the slope of the secant line PQ in Figure 8. The velocity v when t = 2 is the limiting value of this average velocity as t approaches 2; that is,

$$v = \lim_{t \to 2} \frac{f(t) - f(2)}{t - 2}$$

and we recognize from Equation 2 that this is the same as the slope of the tangent line to the curve at P.

Thus, when we solve the tangent problem in differential calculus, we are also solving problems concerning velocities. The same techniques also enable us to solve problems involving rates of change in all of the natural and social sciences.

#### The Limit of a Sequence

In the fifth century B.C. the Greek philosopher Zeno of Elea posed four problems, now known as Zeno's paradoxes, that were intended to challenge some of the ideas concerning space and time that were held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: Suppose that Achilles starts at position  $a_1$  and the tortoise starts at position  $t_1$  (see Figure 9). When Achilles reaches the point  $a_2 = t_1$ , the tortoise is farther ahead at position  $t_2$ . When Achilles reaches  $a_3 = t_2$ , the tortoise is at  $t_3$ . This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.



FIGURE 9

One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles  $(a_1, a_2, a_3, \ldots)$  or the successive positions of the tortoise  $(t_1, t_2, t_3, \ldots)$  form what is known as a sequence.

In general, a sequence  $\{a_n\}$  is a set of numbers written in a definite order. For instance, the sequence

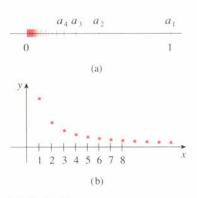
$$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$$

can be described by giving the following formula for the *n*th term:

$$a_n = \frac{1}{n}$$

We can visualize this sequence by plotting its terms on a number line as in Figure 10(a) or by drawing its graph as in Figure 10(b). Observe from either picture that the terms of the sequence  $a_n = 1/n$  are becoming closer and closer to 0 as n increases. In fact we can find terms as small as we please by making n large enough. We say that the limit of the sequence is 0, and we indicate this by writing

$$\lim_{n \to \infty} \frac{1}{n} = 0$$





In general, the notation

$$\lim_{n\to\infty}a_n=L$$

is used if the terms  $a_n$  approach the number L as n becomes large. This means that the numbers  $a_n$  can be made as close as we like to the number L by taking n sufficiently large.

The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

> $a_1 = 3.1$   $a_2 = 3.14$   $a_3 = 3.141$   $a_4 = 3.1415$   $a_5 = 3.14159$   $a_6 = 3.141592$   $a_7 = 3.1415926$   $\vdots$ lim  $a_n = \pi$

then

The terms in this sequence are rational approximations to  $\pi$ .

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences  $\{a_n\}$  and  $\{t_n\}$ , where  $a_n < t_n$  for all *n*. It can be shown that both sequences have the same limit:

$$\lim_{n\to\infty}a_n=p=\lim_{n\to\infty}t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

#### The Sum of a Series

Another of Zeno's paradoxes, as passed on to us by Aristotle, is the following: "A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended." (See Figure 11.)



FIGURE II

#### 8 A PREVIEW OF CALCULUS

Of course, we know that the man can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

**3** 
$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

Zeno was arguing that it does not make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the symbol  $0.\overline{3} = 0.3333...$  means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \cdots$$

and so, in some sense, it must be true that

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \dots = \frac{1}{3}$$

More generally, if  $d_n$  denotes the *n*th digit in the decimal representation of a number, then

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \ldots + \frac{d_n}{10^n} + \ldots$$

Therefore, some infinite sums, or infinite series as they are called, have a meaning. But we must define carefully what the sum of an infinite series is.

Returning to the series in Equation 3, we denote by  $s_n$  the sum of the first *n* terms of the series. Thus

$$s_{1} = \frac{1}{2} = 0.5$$

$$s_{2} = \frac{1}{2} + \frac{1}{4} = 0.75$$

$$s_{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$$

$$s_{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375$$

$$s_{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.96875$$

$$s_{6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.984375$$

$$s_{7} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 0.9921875$$

$$\vdots$$

$$s_{10} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} \approx 0.99902344$$

$$\vdots$$

$$s_{16} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{16}} \approx 0.99998474$$

Observe that as we add more and more terms, the partial sums become closer and closer to 1. In fact, it can be shown that by taking n large enough (that is, by adding

sufficiently many terms of the series), we can make the partial sum  $s_n$  as close as we please to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1$ 

In other words, the reason the sum of the series is 1 is that

$$\lim_{n\to\infty}s_n=1$$

In Chapter 8 we will discuss these ideas further. We will then use Newton's idea of combining infinite series with differential and integral calculus.

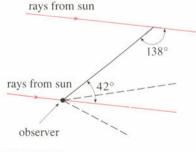
#### Summary

We have seen that the concept of a limit arises in trying to find the area of a region, the slope of a tangent to a curve, the velocity of a car, or the sum of an infinite series. In each case the common theme is the calculation of a quantity as the limit of other, easily calculated quantities. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits.

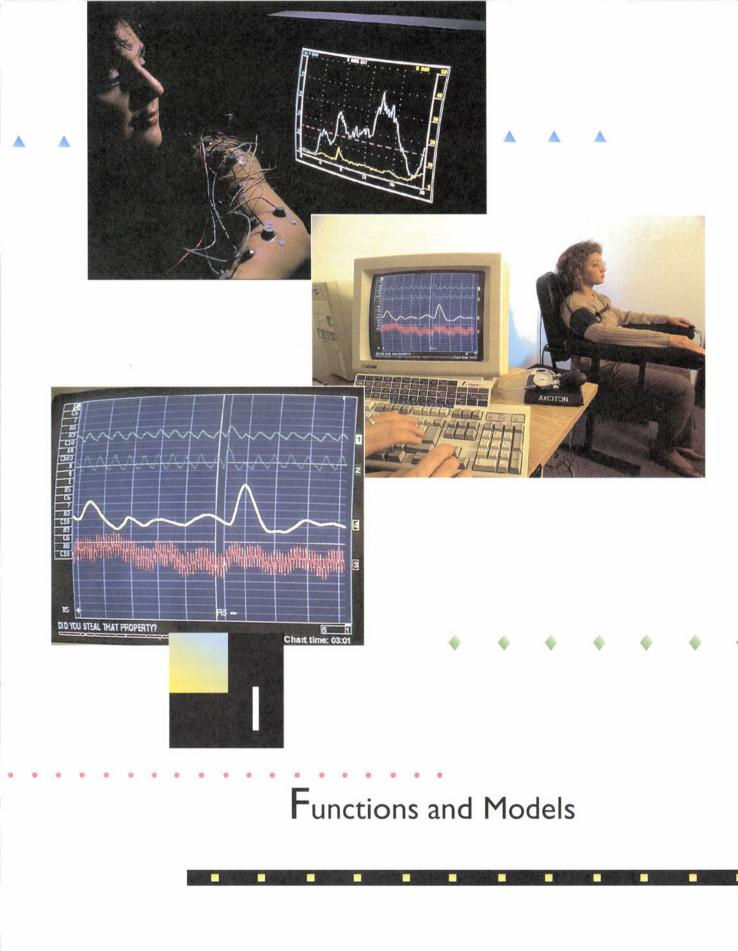
Sir Isaac Newton invented his version of calculus in order to explain the motion of the planets around the sun. Today calculus is used in calculating the orbits of satellites and spacecraft, in predicting population sizes, in estimating how fast coffee prices rise, in forecasting weather, in measuring the cardiac output of the heart, in calculating life insurance premiums, and in a great variety of other areas. We will explore some of these uses of calculus in this book.

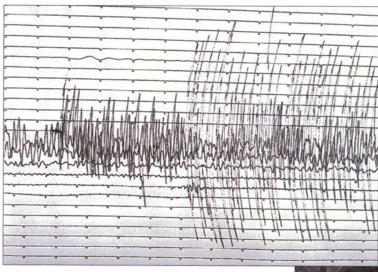
In order to convey a sense of the power of the subject, we end this preview with a list of some of the questions that you will be able to answer using calculus:

- How can we explain the fact, illustrated in Figure 12, that the angle of elevation from an observer up to the highest point in a rainbow is 42°? (See page 282.)
- **2.** How can we explain the shapes of cans on supermarket shelves? (See page 321.)
- 3. Where is the best place to sit in a movie theater? (See page 472.)
- 4. How far away from an airport should a pilot start descent? (See page 238).
- **5.** How can we fit curves together to design shapes to represent letters on a laser printer? (See page 237.)
- **6.** Where should an infielder position himself to catch a baseball thrown by an outfielder and relay it to home plate? (See page 537.)
- **7.** Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height? (See page 526.)









• • • The most natural and convenient representation of many functions is graphical. Shown here are graphs recorded by instruments—an electrocardiograph for heartbeats, a polygraph for lie-detection, and a seismograph for earthquake activity (in this case, the Loma Prieta earthquake that destroyed the Bay Bridge from San Francisco to Oakland in 1989).



The fundamental objects that we deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of realworld phenomena. We also discuss the use of graphing calculators and graphing software for computers and see that parametric equations provide the best method for graphing certain types of curves.



# Four Ways to Represent a Function

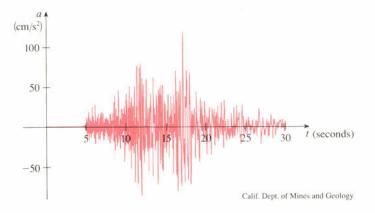
Functions arise whenever one quantity depends on another. Consider the following four situations.

- **A.** The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation  $A = \pi r^2$ . With each positive number r there is associated one value of A, and we say that A is a *function* of r.
- **B.** The human population of the world P depends on the time t. The table gives estimates of the world population P(t) at time t, for certain years. For instance,

$$P(1950) \approx 2,520,000,000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

- **C.** The cost *C* of mailing a first-class letter depends on the weight *w* of the letter. Although there is no simple formula that connects *w* and *C*, the post office has a rule for determining *C* when *w* is known.
- **D.** The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.





Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

A function f is a rule that assigns to each element x in a set A exactly one element, called f(x), in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The number f(x) is the **value** of f at x and is read "f of x." The **range** of f is the set of all possible values of f(x)

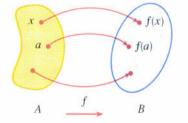
Population (millions)
1650
1750
1860
2070
2300
2520
3020
3700
4450
5300
5770

as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

It is helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output f(x) according to the rule of the function. Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the  $\sqrt{x}$  key on your calculator is such a function. First you input x into the display. Then you press the key labeled  $\sqrt{x}$ . If x < 0, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If  $x \ge 0$ , then an approximation to  $\sqrt{x}$  will appear in the display. Thus, the  $\sqrt{x}$  key on your calculator is not quite the same as the exact mathematical function f defined by  $f(x) = \sqrt{x}$ .

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of A to an element of B. The arrow indicates that f(x) is associated with x, f(a) is associated with a, and so on.



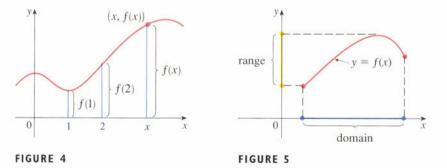
**FIGURE 3** Arrow diagram for *f* 

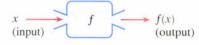
The most common method for visualizing a function is its graph. If f is a function with domain A, then its graph is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that y = f(x) and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is y = f(x), we can read the value of f(x) from the graph as being the height of the graph above the point x (see Figure 4). The graph of f also allows us to picture the domain and range of f on the x-axis and y-axis as in Figure 5.

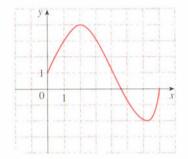






**EXAMPLE I** The graph of a function f is shown in Figure 6.

- (a) Find the values of f(1) and f(5).
- (b) What are the domain and range of f?



#### FIGURE 6

#### SOLUTION

(a) We see from Figure 6 that the point (1, 3) lies on the graph of f, so the value of f at 1 is f(1) = 3. (In other words, the point on the graph that lies above x = 1 is three units above the x-axis.)

When x = 5, the graph lies about 0.7 unit below the x-axis, so we estimate that  $f(5) \approx -0.7$ .

(b) We see that f(x) is defined when  $0 \le x \le 7$ , so the domain of f is the closed interval [0,7]. Notice that f takes on all values from -2 to 4, so the range of f is

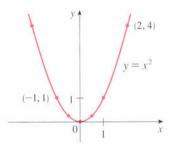
$$\{y \mid -2 \le y \le 4\} = [-2, 4]$$

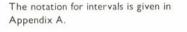
**EXAMPLE 2** Sketch the graph and find the domain and range of each function. (a) f(x) = 2x - 1 (b)  $g(x) = x^2$ 

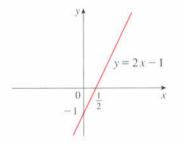
#### SOLUTION

(a) The equation of the graph is y = 2x - 1, and we recognize this as being the equation of a line with slope 2 and y-intercept -1. (Recall the slope-intercept form of the equation of a line: y = mx + b. See Appendix B.) This enables us to sketch the graph of f in Figure 7. The expression 2x - 1 is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by  $\mathbb{R}$ . The graph shows that the range is also  $\mathbb{R}$ .

(b) Since  $g(2) = 2^2 = 4$  and  $g(-1) = (-1)^2 = 1$ , we could plot the points (2, 4) and (-1, 1), together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is  $y = x^2$ , which represents a parabola (see Appendix B). The domain of g is  $\mathbb{R}$ . The range of g consists of all values of g(x), that is, all numbers of the form  $x^2$ . But  $x^2 \ge 0$  for all numbers x and any positive number y is a square. So the range of g is  $\{y \mid y \ge 0\} = [0, \infty)$ . This can also be seen from Figure 8.









#### Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it is often useful to go from one representation to another to gain additional insight into the function. (In Example 1, for instance, we started with algebraic formulas and then obtained the graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

- A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula A(r) = πr<sup>2</sup>, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is {r | r > 0} = (0,∞), and the range is also (0,∞).
- **B.** We are given a description of the function in words: P(t) is the human population of the world at time *t*. The table of values of world population on page 12 provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population P(t) at any time *t*. But it is possible to find an expression for a function that *approximates* P(t). In fact, using methods explained in Section 1.7, we obtain the approximation

$$P(t) \approx f(t) = (0.008306312) \cdot (1.013716)^{t}$$

and Figure 10 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

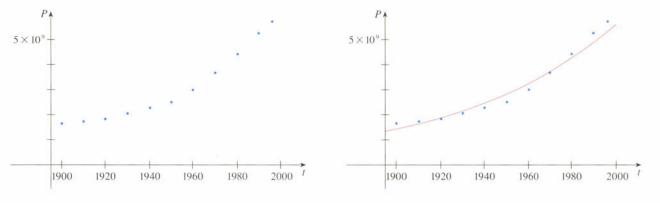


FIGURE 9

FIGURE 10

A function defined by a table of values is called a tabular function.

w (ounces)	C(w) (dollars)
$0 < w \leq 1$	0.32
$1 \leq w \leq 2$	0.55
$2 \leq w \leq 3$	0.78
$3 \le w \le 4$	1.01
$4 \leq w \leq 5$	1.24
1	3
	4

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that is still possible to perform the operations of calculus on such a function.

- **C.** Again the function is described in words: C(w) is the cost of mailing a first-class letter with weight w. The rule that the U.S. Postal Service used in 1996 is as follows: The cost is 32 cents for up to one ounce, plus 23 cents for each successive ounce up to 11 ounces. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).
- **D.** The graph shown in Figure 1 is the most natural representation of the vertical acceleration function a(t). It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie detection.) Figures 11 and 12 show the graphs of the north-south and east-west accelerations for the Northridge earthquake; when used in conjunction with Figure 1, they provide a great deal of information about the earthquake.

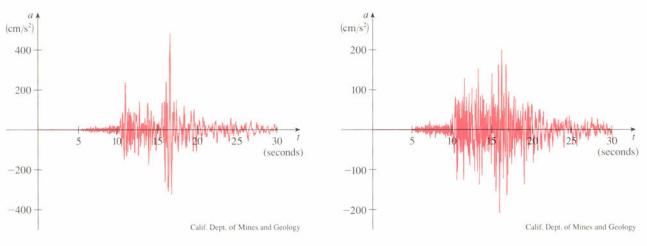


FIGURE II North-south acceleration for the Northridge earthquake

FIGURE 12 East-west acceleration for the Northridge earthquake

In the next example we sketch the graph of a function that is defined verbally.

**EXAMPLE 3** When you turn on a hot water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

**SOLUTION** The initial temperature of the running water is close to room temperature because of the water that has been in the pipes. When the water from the hot water tank starts coming out, T increases quickly. In the next phase, T is constant at the temperature of the water in the tank. When the tank is drained,



FIGURE 13

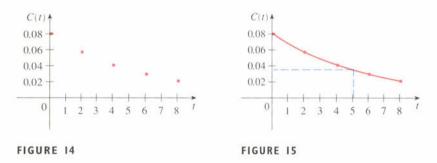
1	C(t)
0	0.0800
2	0.0570
4	0.0408
6	0.0295
8	0.0210

T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 13.

A more accurate graph of the function in Example 3 could be obtained by using a thermometer to measure the temperature of the water at 10-second intervals. In general, scientists collect experimental data and use them to sketch the graphs of functions, as the next example illustrates.

**EXAMPLE 4** The data shown in the margin come from an experiment on the lactonization of hydroxyvaleric acid at 25 °C. They give the concentration C(t) of this acid (in moles per liter) after t minutes. Use these data to draw an approximation to the graph of the concentration function. Then use this graph to estimate the concentration after 5 minutes.

**SOLUTION** We plot the five points corresponding to the data from the table in Figure 14. The curve-fitting methods of Section 1.7 could be used to choose a model and graph it. But the data points in Figure 14 look quite well behaved, so we simply draw a smooth curve through them by hand as in Figure 15.



Then we use the graph to estimate that the concentration after 5 min is

$$C(5) \approx 0.035 \text{ mole/liter}$$

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in calculus problems that ask for the maximum or minimum values of quantities.

**EXAMPLE 5** • A rectangular storage container with an open top has a volume of  $10 \text{ m}^3$ . The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

**SOLUTION** We draw a diagram as in Figure 16 and introduce notation by letting w and 2w be the width and length of the base, respectively, and h be the height.

The area of the base is  $(2w)w = 2w^2$ , so the cost, in dollars, of the material for the base is  $10(2w^2)$ . Two of the sides have area *wh* and the other two have area 2wh, so the cost of the material for the sides is 6[2(wh) + 2(2wh)]. The total cost is therefore

$$C = 10(2w^{2}) + 6[2(wh) + 2(2wh)] = 20w^{2} + 36wh$$

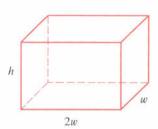


FIGURE 16

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To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m<sup>3</sup>. Thus

w(2w)h = 10

which gives

 $h = \frac{10}{2w^2} = \frac{5}{w^2}$ 

 $C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$ 

 $C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$ 

In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving as discussed on page 87, particularly Step 1: Understanding the Problem.

expresses 
$$C$$
 as a function of  $w$ .

Therefore, the equation

**EXAMPLE 6** Find the domain of each function.

Substituting this into the expression for C, we have

(a) 
$$f(x) = \sqrt{x+2}$$
 (b)  $g(x) = \frac{1}{x^2 - x}$ 

### SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of *f* consists of all values of *x* such that  $x + 2 \ge 0$ . This is equivalent to  $x \ge -2$ , so the domain is the interval  $[-2, \infty)$ .

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that g(x) is not defined when x = 0 or x = 1. Thus, the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty,0) \cup (0,1) \cup (1,\infty)$$

The graph of a function is a curve in the *xy*-plane. But the question arises: Which curves in the *xy*-plane are graphs of functions? This is answered by the following test.

**The Vertical Line Test** A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 17. If each vertical line x = a intersects a curve only once, at (a, b), then exactly one functional value is defined by f(a) = b. But if a line x = a intersects the curve twice, at (a, b) and (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

SECTION I.I FOUR WAYS TO REPRESENT A FUNCTION 19

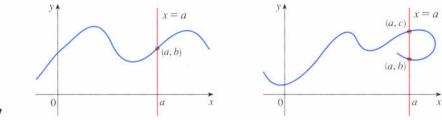
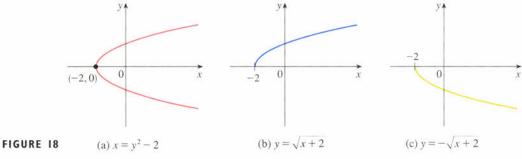


FIGURE 17

For example, the parabola  $x = y^2 - 2$  shown in Figure 18(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of two functions of x. Notice that  $x = y^2 - 2$  implies  $y^2 = x + 2$ , so  $y = \pm \sqrt{x + 2}$ . So the upper and lower halves of the parabola are the graphs of the functions  $f(x) = \sqrt{x+2}$ [from Example 6(a)] and  $q(x) = -\sqrt{x+2}$  [see Figures 18(b) and (c)]. We observe that if we reverse the roles of x and y, then the equation  $x = h(y) = y^2 - 2$  does define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h.



### **Piecewise Defined Functions**

The functions in the following four examples are defined by different formulas in different parts of their domains.

**EXAMPLE 7** • A function f is defined by

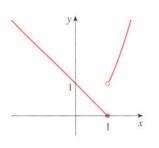
$$f(x) = \begin{cases} 1 - x & \text{if } x \le 1\\ x^2 & \text{if } x > 1 \end{cases}$$

Evaluate f(0), f(1), and f(2) and sketch the graph.

**SOLUTION** Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that  $x \leq 1$ , then the value of f(x) is 1 - x. On the other hand, if x > 1, then the value of f(x) is  $x^2$ .

> Since  $0 \le 1$ , we have f(0) = 1 - 0 = 1. Since  $1 \le 1$ , we have f(1) = 1 - 1 = 0. Since 2 > 1, we have  $f(2) = 2^2 = 4$ .

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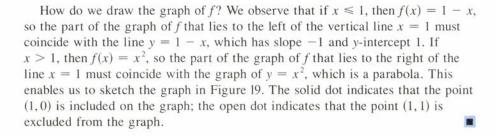


FIGURE 19

For a more extensive review of absolute values, see Appendix A.

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by |a|, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

 $|a| \ge 0$  for every number a

For example,

$$|3| = 3$$
  $|-3| = 3$   $|0| = 0$   
 $|\sqrt{2} - 1| = \sqrt{2} - 1$   $|3 - \pi| = \pi - 3$ 

In general, we have

$$|a| = a$$
 if  $a \ge 0$   
 $|a| = -a$  if  $a < 0$ 

(Remember that if a is negative, then -a is positive.)

**EXAMPLE 8** Sketch the graph of the absolute value function f(x) = |x|. SOLUTION From the preceding discussion we know that

$$x = \begin{cases} x & \text{if } x \ge 0\\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line y = x to the right of the y-axis and coincides with the line y = -x to the left of the y-axis (see Figure 20).

**EXAMPLE 9** Find a formula for the function f graphed in Figure 21.

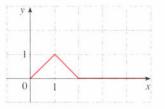
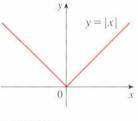


FIGURE 21

**SOLUTION** The line through (0,0) and (1,1) has slope m = 1 and y-intercept b = 0, so its equation is y = x. Thus, for the part of the graph of f that joins (0,0) to (1,1), we have

$$f(x) = x$$
 if  $0 \le x \le 1$ 





The line through (1, 1) and (2, 0) has slope m = -1, so its point-slope form is

Point-slope form of the equation of a line:

 $y - y_1 = m(x - x_1)$ See Appendix B.

So we have

$$y - 0 = (-1)(x - 2)$$
 or  $y = 2 - x$   
 $f(x) = 2 - x$  if  $1 < x \le 2$ 

We also see that the graph of f coincides with the x-axis for x > 2. Putting this information together, we have the following three-piece formula for *f*:

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1\\ 2 - x & \text{if } 1 < x \le 2\\ 0 & \text{if } x > 2 \end{cases}$$

**EXAMPLE 10** In Example C at the beginning of this section we considered the  $\cot C(w)$  of mailing a first-class letter with weight w. In effect, this is a piecewise defined function because, from the table of values, we have

$$C(w) = \begin{cases} 0.32 & \text{if } 0 < w \le 1\\ 0.55 & \text{if } 1 < w \le 2\\ 0.78 & \text{if } 2 < w \le 3\\ 1.01 & \text{if } 3 < w \le 4 \end{cases}$$

The graph is shown in Figure 22. You can see why functions similar to this one are called step functions-they jump from one value to the next. Such functions will be studied in Chapter 2.

# **Symmetry**

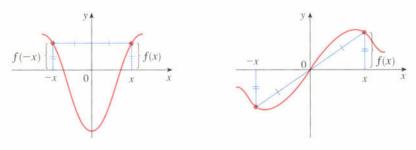
If a function f satisfies f(-x) = f(x) for every number x in its domain, then f is called an **even function**. For instance, the function  $f(x) = x^2$  is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 23). This means that if we have plotted the graph of f for  $x \ge 0$ , we obtain the entire graph simply by reflecting about the y-axis. If f satisfies f(-x) = -f(x) for every number x in its domain, then f is called an odd **function**. For example, the function  $f(x) = x^3$  is odd because

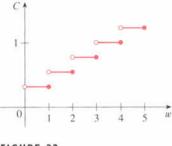
$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 24). If we













already have the graph of *f* for  $x \ge 0$ , we can obtain the entire graph by rotating through 180° about the origin.

**EXAMPLE II** Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) 
$$f(x) = x^5 + x$$
 (b)  $g(x) = 1 - x^4$  (c)  $h(x) = 2x - x^2$   
SOLUTION  
(a)  $f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$   
 $= -x^5 - x = -(x^5 + x)$   
 $= -f(x)$ 

Therefore, f is an odd function.

(b) 
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)  $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$ 

Since  $h(-x) \neq h(x)$  and  $h(-x) \neq -h(x)$ , we conclude that h is neither even nor odd.

The graph of the functions in Example 11 are shown in Figure 25. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

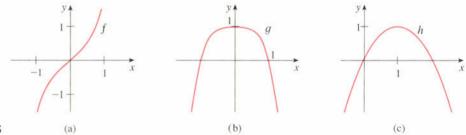
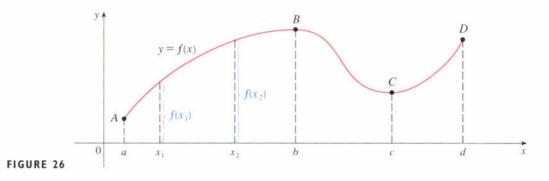
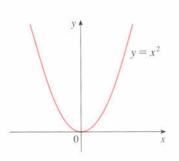


FIGURE 25

#### Increasing and Decreasing Functions

The graph shown in Figure 26 rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval [a, b], decreasing on [b, c], and increasing again on [c, d]. Notice that if  $x_1$  and  $x_2$  are any two numbers between a and b with  $x_1 < x_2$ , then  $f(x_1) < f(x_2)$ . We use this as the defining property of an increasing function.







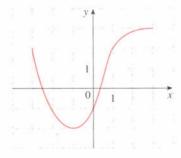


I. The graph of a function *f* is given.

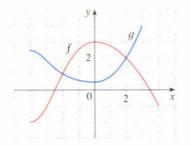
(a) State the value of f(-1).

Exercises

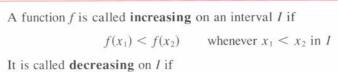
- (b) Estimate the value of f(2).
- (c) For what values of x is f(x) = 2?
- (d) Estimate the values of x such that f(x) = 0.
- (e) State the domain and range of f.
- (f) On what interval is f increasing?



- **2.** The graphs of f and g are given.
  - (a) State the values of f(-4) and g(3).
  - (b) For what values of x is f(x) = g(x)?
  - (c) Estimate the solution of the equation f(x) = -1.
  - (d) On what interval is f decreasing?
  - (e) State the domain and range of f.
  - (f) State the domain and range of g.



**3.** Figures 1, 11, and 12 were recorded by an instrument operated by the California Department of Mines and



 $f(x_1) > f(x_2)$  whenever  $x_1 < x_2$  in I

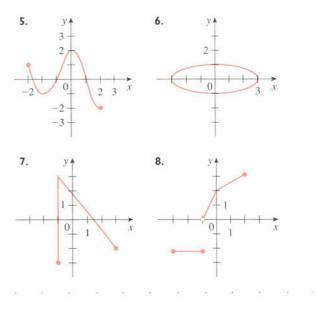
In the definition of an increasing function it is important to realize that the inequality  $f(x_1) < f(x_2)$  must be satisfied for *every* pair of numbers  $x_1$  and  $x_2$  in I with  $x_1 < x_2$ .

You can see from Figure 27 that the function  $f(x) = x^2$  is decreasing on the interval  $(-\infty, 0]$  and increasing on the interval  $[0, \infty)$ .

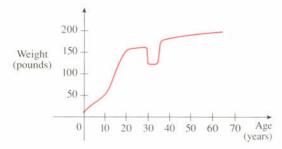
Geology at the University Hospital of the University of Southern California in Los Angeles. Use them to estimate the ranges of the vertical, north-south, and east-west ground acceleration functions at USC during the Northridge earthquake.

4. In this section we discussed examples of ordinary, everyday functions: population is a function of time, postage cost is a function of weight, water temperature is a function of time. Given three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

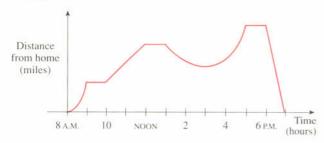
**5–8** Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.



**9.** The graph shown gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?



10. The graph shown gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



- 11. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.
- **12.** Sketch a rough graph of the number of hours of daylight as a function of the time of year.
- **13.** Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.
- 14. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.
- **15.** A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.
- **16.** An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If *t* represents the time in minutes since the plane has left the terminal building, let x(t) be the horizontal distance traveled and y(t) be the altitude of the plane.
  - (a) Sketch a possible graph of x(t).

- (b) Sketch a possible graph of y(t).
- (c) Sketch a possible graph of the ground speed.
- (d) Sketch a possible graph of the vertical velocity.
- 17. Temperature readings T (in °F) were recorded every two hours from midnight to noon in Atlanta, Georgia, on March 18, 1996. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12
Т	58	57	53	50	51	57	61

- (a) Use the readings to sketch a rough graph of *T* as a function of *t*.
- (b) Use the graph to estimate the temperature at 11 A.M.
- The population P (in thousands) of San Jose, California, from 1984 to 1994 is shown in the table. (Midyear estimates are given.)

ī	1984	1986	1988	1990	1992	1994
P	695	716	733	782	800	817

- (a) Draw a graph of *P* as a function of time.
- (b) Use the graph to estimate the population in 1991.
- **19.** If  $f(x) = 2x^2 + 3x 4$ , find  $f(0), f(2), f(\sqrt{2}), f(1 + \sqrt{2}), f(-x), f(x + 1), 2f(x), and f(2x).$
- **20.** A spherical balloon with radius *r* inches has volume  $V(r) = \frac{4}{3}\pi r^3$ . Find a function that represents the amount of air required to inflate the balloon from a radius of *r* inches to a radius of r + 1 inches.

**21-22** Find f(2 + h), f(x + h), and  $\frac{f(x + h) - f(x)}{h}$ , where  $h \neq 0$ .

**21.**  $f(x) = x - x^2$  **22.**  $f(x) = \frac{x}{x+1}$ 

**23–25** ■ Find the domain of the function.

**23.** 
$$f(x) = \frac{x^4}{x^2 + x - 6}$$
 **24.**  $h(x) = \sqrt[4]{7 - 3x}$   
**25.**  $f(t) = \sqrt[3]{t - 1}$ 

26. Find the domain and range and sketch the graph of the function  $h(x) = \sqrt{4 - x^2}$ .

**27–36** Find the domain and sketch the graph of the function.

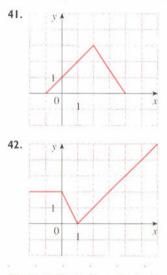
**27.** f(x) = 3 - 2x**28.**  $f(x) = x^2 + 2x - 1$ **29.** G(x) = |x| + x**30.** H(x) = |2x|**31.** f(x) = x/|x|**32.**  $f(x) = \frac{x^2 + 5x + 6}{x + 2}$ 

**33.** 
$$f(x) = \begin{cases} x & \text{if } x \le 0\\ x+1 & \text{if } x > 0 \end{cases}$$
  
**34.** 
$$f(x) = \begin{cases} 2x+3 & \text{if } x < -1\\ 3-x & \text{if } x \ge -1 \end{cases}$$
  
**35.** 
$$f(x) = \begin{cases} x+2 & \text{if } x \le -1\\ x^2 & \text{if } x > -1 \end{cases}$$
  
**36.** 
$$f(x) = \begin{cases} -1 & \text{if } x \le -1\\ 3x+2 & \text{if } |x| < 1\\ 7-2x & \text{if } x \ge 1 \end{cases}$$

6

37-42 = Find an expression for the function whose graph is the given curve.

- **37.** The line segment joining the points (-2, 1) and (4, -6)
- **38.** The line segment joining the points (-3, -2) and (6, 3)
- **39.** The bottom half of the parabola  $x + (y 1)^2 = 0$
- **40.** The top half of the circle  $(x 1)^2 + y^2 = 1$



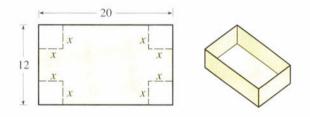
43-47 = Find a formula for the described function and state its domain.

- **43.** A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.
- **44.** A rectangle has area 16 m<sup>2</sup>. Express the perimeter of the rectangle as a function of the length of one of its sides.
- **45.** Express the area of an equilateral triangle as a function of the length of a side.
- **46.** Express the surface area of a cube as a function of its volume.
- **47.** An open rectangular box with volume 2 m<sup>3</sup> has a square base. Express the surface area of the box as a function of the length of a side of the base.

**48.** A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

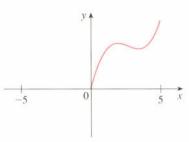


49. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



- **50.** A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost *C* (in dollars) of a ride as a function of the distance *x* traveled (in miles) for 0 < x < 2, and sketch the graph of this function.
- 51. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
  - (a) Sketch the graph of the tax rate *R* as a function of the income *I*.
  - (b) How much tax is assessed on an income of \$14,000? On \$26,000?
  - (c) Sketch the graph of the total assessed tax *T* as a function of the income *I*.
- **52.** The functions in Example 10 and Exercises 50 and 51(a) are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.
- **53.** (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?
  - (b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

**54.** A function f has domain [-5, 5] and a portion of its graph is shown.



(a) Complete the graph of f if it is known that f is even.



the rest of the second second

**55–60** Determine whether f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

**55.** 
$$f(x) = x^{-2}$$
  
**56.**  $f(x) = x^{-3}$   
**57.**  $f(x) = x^2 + x$   
**58.**  $f(x) = x^4 - 4x^2$   
**59.**  $f(x) = x^3 - x$   
**60.**  $f(x) = 3x^3 + 2x^2 + 1$ 



# New Functions from Old Functions

In solving calculus problems you will find that it is helpful to be familiar with the graphs of some commonly occurring functions. In this section we classify various types of functions and then we show how to transform them by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

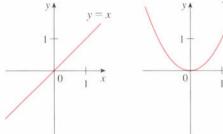
# Types of Functions

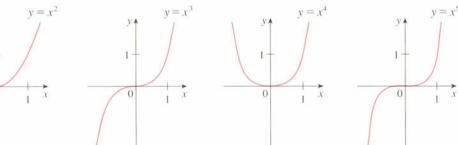
**Constant Functions** The constant function f(x) = c has domain  $\mathbb{R}$  and its range consists of the single number c. Its graph is a horizontal line and is illustrated in Figure 1 for c = 2.

**Power Functions** A function of the form  $f(x) = x^a$ , where *a* is a constant, is called a **power function**. We consider several cases.

#### (a) a = n, a positive integer

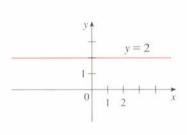
The graphs of  $f(x) = x^n$  for n = 1, 2, 3, 4, and 5 are shown in Figure 2. We already know the shape of the graphs of y = x (a line through the origin with slope 1) and  $y = x^2$  (a parabola, see Example 2 in Section 1.1).





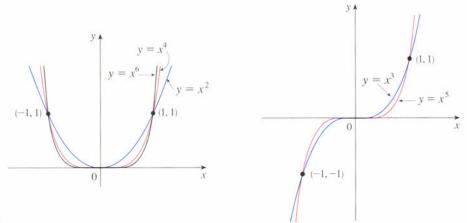
**FIGURE 2** Graphs of  $f(x) = x^n$  for n = 1, 2, 3, 4, 5

The general shape of the graph of  $f(x) = x^n$  depends on whether *n* is even or odd. If *n* is even, then  $f(x) = x^n$  is an even function and its graph is similar to the parabola  $y = x^2$ . If *n* is odd, then  $f(x) = x^n$  is an odd function and its graph is



#### FIGURE I

similar to that of  $y = x^3$ . Notice from Figure 3, however, that as *n* increases, the graph of  $y = x^n$  becomes flatter near 0 and steeper when  $|x| \ge 1$ . (If x is small, then  $x^2$  is smaller,  $x^3$  is even smaller,  $x^4$  is smaller still, and so on.)



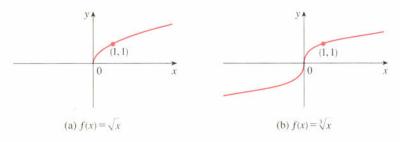
#### FIGURE 3

# (b) a = -1

The graph of the reciprocal function  $f(x) = x^{-1} = 1/x$  is shown in Figure 4. Its graph has the equation y = 1/x or xy = 1. This is an equilateral hyperbola with the coordinate axes as its asymptotes.

# (c) a = 1/n, n a positive integer

The function  $f(x) = x^{1/n} = \sqrt[n]{x}$  is a **root function**. For n = 2 it is the square root function  $f(x) = \sqrt{x}$  whose domain is  $[0, \infty)$  and whose graph is the upper half of the parabola  $x = y^2$  [see Figure 5(a)]. For other even values of *n*, the graph of  $y = \sqrt[n]{x}$  is similar to that of  $y = \sqrt{x}$ . For n = 3 we have the cube root function  $f(x) = \sqrt[3]{x}$  whose domain is  $\mathbb{R}$  (recall that every real number has a cube root) and whose graph is shown in Figure 5(b). The graph of  $y = \sqrt[n]{x}$  for *n* odd (*n* > 3) is similar to that of  $y = \sqrt[3]{x}$ .





**Polynomials** A function *P* is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where *n* is a nonnegative integer and the numbers  $a_0, a_1, a_2, \ldots, a_n$  are constants called the **coefficients** of the polynomial. The domain of any polynomial is  $\mathbb{R} = (-\infty, \infty)$ . If the leading coefficient  $a_n \neq 0$ , then the **degree** of the polynomial is *n*. For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

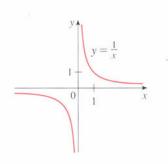


FIGURE 4

The coordinate geometry of lines is reviewed in Appendix B.

A polynomial of degree 1 is of the form P(x) = ax + b and is called a **linear** function because its graph is the line y = ax + b (slope a, y-intercept b). A characteristic feature of linear functions is that they grow at a constant rate. For instance, Figure 6 shows a graph of the linear function f(x) = 3x - 2 and a table of sample values. Notice that whenever x increases by 0.1, the value of f(x) increases by 0.3. So f(x) increases three times as fast as x. Thus, the slope of the graph y = 3x - 2, namely 3, can be interpreted as the rate of change of y with respect to x.

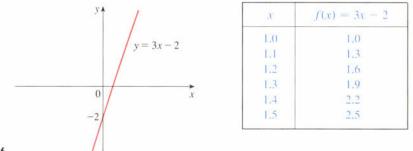


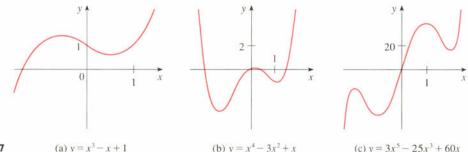
FIGURE 6

A polynomial of degree 2 is of the form  $P(x) = ax^2 + bx + c$  and is called a quadratic function. The graph of P is always a parabola obtained by shifting the parabola  $y = ax^2$ . (See Example 3.)

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d$$

and is called a **cubic function**. Figure 7 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.



**FIGURE 7** 

(b)  $y = x^4 - 3x^2 + x$ 

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 3.3 we will explain why economists often use a polynomial P(x) to represent the cost of producing x units of a commodity.

**Rational Functions** A **rational function** *f* is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that

#### SECTION 1.2 NEW FUNCTIONS FROM OLD FUNCTIONS 29

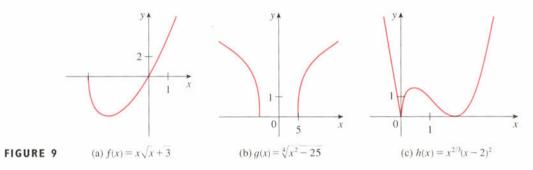
 $Q(x) \neq 0$ . For example, the function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

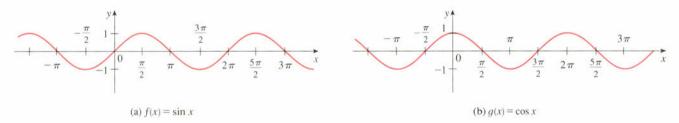
is a rational function with domain  $\{x \mid x \neq \pm 2\}$ . Its graph is shown in Figure 8.

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 4 we will see that their graphs can assume a variety of shapes. Figure 9 illustrates some of the possibilities.



**Trigonometric Functions** Trigonometry and the trigonometric functions are reviewed on the front endpapers and in Appendix C. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function  $f(x) = \sin x$ , it is understood that  $\sin x$  means the sine of the angle whose radian measure is x. Thus, the graphs of the sine and cosine functions are as shown in Figure 10.





Notice that for both the sine and cosine functions the domain is  $(-\infty, \infty)$  and the range is the closed interval [-1, 1]. Thus, for all values of x we have

$$-1 \le \sin x \le 1$$
  $-1 \le \cos x \le 1$ 

Also, the zeros of the sine function occur at the integer multiples of  $\pi$ ; that is,

 $\sin x = 0$  when  $x = n\pi$  *n* an integer

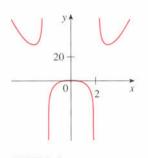


FIGURE 8  $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$  An important property of the sine and cosine functions is that they are periodic functions and have period  $2\pi$ . This means that, for all values of *x*,

 $\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$ 

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves.

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 11. It is undefined when  $\cos x = 0$ , that is, when  $x = \pm \pi/2, \pm 3\pi/2, \ldots$ . Its range is  $(-\infty, \infty)$ . Notice that the tangent function has period  $\pi$ :

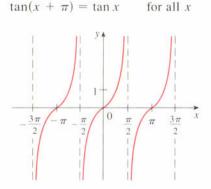


FIGURE II  $y = \tan x$ 

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix C.

**Exponential Functions** These are the functions of the form  $f(x) = a^x$ , where the base *a* is a positive constant. The graphs of  $y = 2^x$  and  $y = (0.5)^x$  are shown in Figure 12. In both cases the domain is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ .

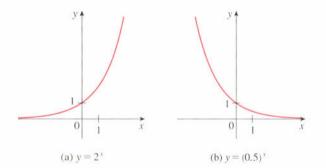
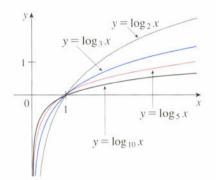


FIGURE 12

Exponential functions will be studied in detail in Section 1.5 and we will see in Section 1.7 and in later chapters that they are useful for modeling many natural phenomena, such as population growth (if a > 1) and radioactive decay (if a < 1).

**Logarithmic Functions** These are the functions  $f(x) = \log_a x$ , where the base *a* is a positive constant. They are the inverse functions of the exponential functions and will be studied in Section 1.6. Figure 13 shows the graphs of four logarithmic functions with various bases. In each case the domain is  $(0, \infty)$ , the range is  $(-\infty, \infty)$ , and the function increases slowly when x > 1.



### FIGURE 13

**Transcendental Functions** These are functions that are not algebraic. The set of transcendental functions includes the trigonometric, inverse trigonometric, exponential, and logarithmic functions, but it also includes a vast number of other functions that have never been named. In Chapter 8 we will study transcendental functions that are defined as sums of infinite series.

**EXAMPLE 1** Classify the following functions as one of the types of functions that we have discussed.

- (a)  $f(x) = 5^x$  (b)  $g(x) = x^5$
- (c)  $h(x) = \frac{1+x}{1-\sqrt{x}}$  (d)  $u(t) = 1 t + 5t^4$

#### SOLUTION

- (a)  $f(x) = 5^x$  is an exponential function. (The x is the exponent.)
- (b)  $g(x) = x^5$  is a power function. (The x is the base.)

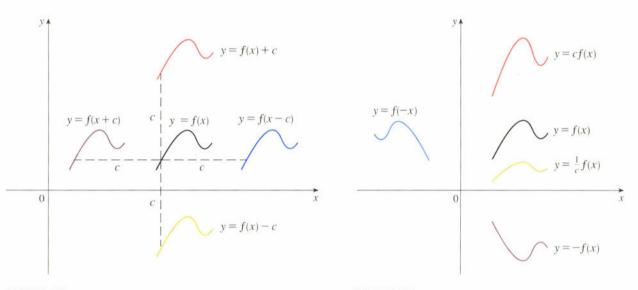
(c) 
$$h(x) = \frac{1+x}{1-\sqrt{x}}$$
 is an algebraic function.

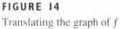
(d)  $u(t) = 1 - t + 5t^4$  is a polynomial of degree 4.

## Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions and thereby reduce the amount of work in graphing. Let us first consider **translations**. If c is a positive number, then the graph of y = f(x) + c is just the graph of y = f(x) shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if g(x) = f(x - c), where c > 0, then the value of g at x is the same as the value of f at x - c (c units to the left of x). Therefore, the graph of y = f(x - c) is just the graph of y = f(x) shifted c units to the right (see Figure 14).

Vertical and Horizontal Shifts Suppose c > 0. To obtain the graph of y = f(x) + c, shift the graph of y = f(x) a distance c units upward y = f(x) - c, shift the graph of y = f(x) a distance c units downward y = f(x - c), shift the graph of y = f(x) a distance c units to the right y = f(x + c), shift the graph of y = f(x) a distance c units to the left

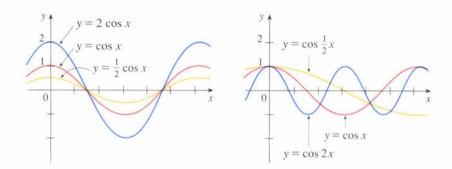




**FIGURE 15** Stretching and reflecting the graph of *f* 

Now let's consider the **stretching** and **reflecting** transformations. If c > 1, then the graph of y = cf(x) is the graph of y = f(x) stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c). The graph of y = -f(x) is the graph of y = f(x) reflected about the x-axis because the point (x, y) is replaced by the point (x, -y). (See Figure 15 and the following chart, where the results of other stretching, compressing, and reflecting transformations are also given.)

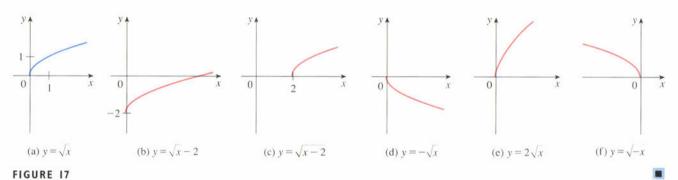
Vertical and Horizontal Stretching and Reflecting Suppose c > 1. To obtain the graph of y = cf(x), stretch the graph of y = f(x) vertically by a factor of c y = (1/c)f(x), compress the graph of y = f(x) vertically by a factor of c y = f(cx), compress the graph of y = f(x) horizontally by a factor of c y = f(x/c), stretch the graph of y = f(x) horizontally by a factor of c y = -f(x), reflect the graph of y = f(x) about the x-axis y = f(-x), reflect the graph of y = f(x) about the y-axis Figure 16 illustrates these stretching transformations when applied to the cosine function with c = 2.





**EXAMPLE 2** Given the graph of  $y = \sqrt{x}$ , use transformations to graph  $y = \sqrt{x} - 2$ ,  $y = \sqrt{x-2}$ ,  $y = -\sqrt{x}$ ,  $y = 2\sqrt{x}$ , and  $y = \sqrt{-x}$ .

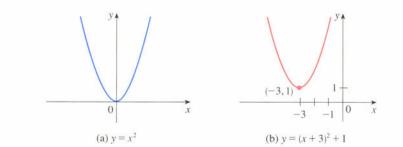
**SOLUTION** The graph of the square root function  $y = \sqrt{x}$ , obtained from Figure 5, is shown in Figure 17(a). In the other parts of the figure we sketch  $y = \sqrt{x} - 2$  by shifting 2 units downward,  $y = \sqrt{x} - 2$  by shifting 2 units to the right,  $y = -\sqrt{x}$  by reflecting about the x-axis,  $y = 2\sqrt{x}$  by stretching vertically by a factor of 2, and  $y = \sqrt{-x}$  by reflecting about the y-axis.



**EXAMPLE 3** Sketch the graph of the function  $f(x) = x^2 + 6x + 10$ . SOLUTION Completing the square, we write the equation of the graph as

$$y = x^{2} + 6x + 10 = (x + 3)^{2} + 1$$

This means we obtain the desired graph by starting with the parabola  $y = x^2$  and shifting 3 units to the left and then 1 unit upward (see Figure 18).





**EXAMPLE 4** Sketch the graphs of the following functions. (a)  $y = \sin 2x$ (b)  $y = 1 - \sin x$ 

#### SOLUTION

(a) We obtain the graph of  $y = \sin 2x$  from that of  $y = \sin x$  by compressing horizontally by a factor of 2 (see Figures 19 and 20).

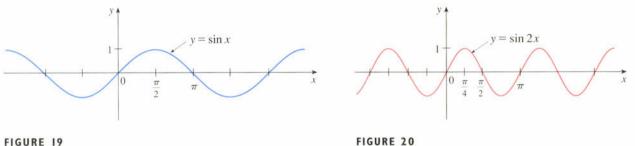
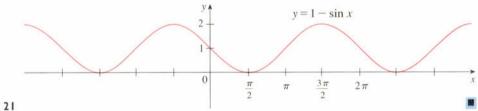


FIGURE 20

(b) To obtain the graph of  $y = 1 - \sin x$ , we again start with  $y = \sin x$ . We reflect about the x-axis to get the graph of  $y = -\sin x$  and then we shift 1 unit upward to get  $y = 1 - \sin x$  (see Figure 21).





Another transformation of some interest is taking the absolute value of a function. If y = |f(x)|, then according to the definition of absolute value, y = f(x)when  $f(x) \ge 0$  and y = -f(x) when f(x) < 0. This tells us how to get the graph of y = |f(x)| from the graph of y = f(x): The part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

**EXAMPLE 5** Sketch the graph of the function  $y = |x^2 - 1|$ .

**SOLUTION** We first graph the parabola  $y = x^2 - 1$  in Figure 22(a) by shifting the parabola  $y = x^2$  downward 1 unit. We see that the graph lies below the x-axis when -1 < x < 1, so we reflect that part of the graph about the x-axis to obtain the graph of  $y = |x^2 - 1|$  in Figure 22(b).

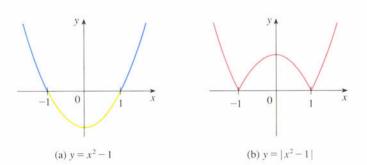


FIGURE 22

# Combinations of Functions

Two functions f and g can be combined to form new functions f + g, f - g, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

If we define the sum f + g by the equation

(f+g)(x) = f(x) + g(x)

then the right side of Equation 1 makes sense if both f(x) and g(x) are defined, that is, if x belongs to the domain of f and also to the domain of g. If the domain of f is A and the domain of g is B, then the domain of f + g is the intersection of these domains, that is,  $A \cap B$ .

Notice that the + sign on the left side of Equation 1 stands for the operation of addition of *functions*, but the + sign on the right side of the equation stands for addition of the *numbers* f(x) and g(x).

Similarly, we can define the difference f - g and the product fg, and their domains are also  $A \cap B$ . But in defining the quotient f/g we must remember not to divide by 0.

**Algebra of Functions** Let f and g be functions with domains A and B. Then the functions f + g, f - g, fg, and f/g are defined as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{domain} = A \cap B$$
  

$$(f-g)(x) = f(x) - g(x) \qquad \text{domain} = A \cap B$$
  

$$(fg)(x) = f(x)g(x) \qquad \text{domain} = A \cap B$$
  

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

**EXAMPLE 6** If  $f(x) = \sqrt{x}$  and  $g(x) = \sqrt{4 - x^2}$ , find the functions f + g, f - g, fg, and f/g.

**SOLUTION** The domain of  $f(x) = \sqrt{x}$  is  $[0, \infty)$ . The domain of  $g(x) = \sqrt{4 - x^2}$  consists of all numbers x such that  $4 - x^2 \ge 0$ , that is,  $x^2 \le 4$ . Taking square roots of both sides, we get  $|x| \le 2$ , or  $-2 \le x \le 2$ , so the domain of g is the interval [-2, 2]. The intersection of the domains of f and g is

$$[0,\infty) \cap [-2,2] = [0,2]$$

Thus, according to the definitions, we have

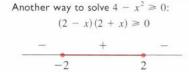
$$(f+g)(x) = \sqrt{x} + \sqrt{4-x^2}$$
  $0 \le x \le 2$ 

$$(f - g)(x) = \sqrt{x} - \sqrt{4 - x^2}$$
  $0 \le x \le 2$ 

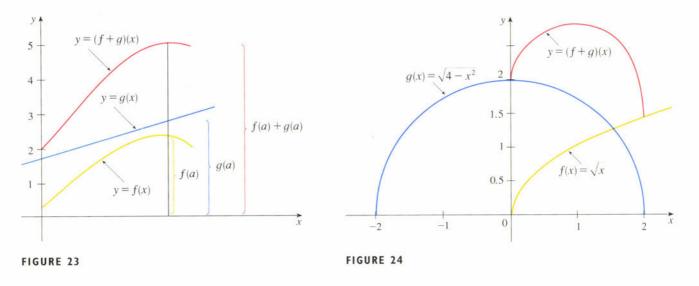
$$(fg)(x) = \sqrt{x}\sqrt{4 - x^2} = \sqrt{4x - x^3} \qquad 0 \le x \le 2$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{4-x^2}} = \sqrt{\frac{x}{4-x^2}} \qquad 0 \le x < 2$$

Notice that the domain of f/g is the interval [0, 2) because we must exclude the points where g(x) = 0, that is,  $x = \pm 2$ .



The graph of the function f + g is obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates as in Figure 23. Figure 24 shows the result of using this procedure to graph the function f + g from Example 6.



#### Composition of Functions

There is another way of combining two functions to get a new function. For example, suppose that  $y = f(u) = \sqrt{u}$  and  $u = g(x) = x^2 + 1$ . Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

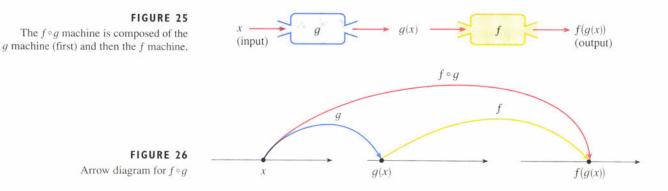
The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

In general, given any two functions f and g, we start with a number x in the domain of g and find its image g(x). If this number g(x) is in the domain of f, then we can calculate the value of f(g(x)). The result is a new function h(x) = f(g(x)) obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by  $f \circ g$  ("f circle g").

**Definition** Given two functions f and g, the **composite** function  $f \circ g$  (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of  $f \circ g$  is the set of all x in the domain of g such that g(x) is in the domain of f. In other words,  $(f \circ g)(x)$  is defined whenever both g(x) and f(g(x)) are defined. The best way to picture  $f \circ g$  is by a machine diagram (Figure 25) or an arrow diagram (Figure 26).



**EXAMPLE 7** If  $f(x) = x^2$  and g(x) = x - 3, find the composite functions  $f \circ g$  and  $g \circ f$ .

**SOLUTION** We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^{2}$$
$$(g \circ f)(x) = g(f(x)) = g(x^{2}) = x^{2} - 3$$

**NOTE** • You can see from Example 7 that, in general,  $f \circ g \neq g \circ f$ . Remember, the notation  $f \circ g$  means that the function g is applied first and then f is applied second. In Example 7,  $f \circ g$  is the function that *first* subtracts 3 and *then* squares;  $g \circ f$  is the function that *first* squares and *then* subtracts 3.

**EXAMPLE 8** If  $f(x) = \sqrt{2 - x}$  and  $g(x) = \sqrt{x}$ , find  $f \circ g$  and its domain.

SOLUTION

TION 
$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$$

For  $\sqrt{x}$  to be defined we must have  $x \ge 0$ . For  $\sqrt{2} - \sqrt{x}$  to be defined we must have  $2 - \sqrt{x} \ge 0$ , that is,  $\sqrt{x} \le 2$ , or  $x \le 4$ . Thus, we have  $0 \le x \le 4$ , so the domain of  $f \circ g$  is the closed interval [0, 4].

Suppose that we don't have explicit formulas for f and g but we do have tables of values or graphs for them. We can still graph the composite function  $f \circ g$ , as the following example shows.

**EXAMPLE 9** The graphs of f and g are as shown in Figure 27 and  $h = f \circ g$ . Estimate the value of h(0.5). Then sketch the graph of h.

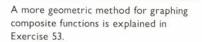
**SOLUTION** From the graph of g we estimate that  $g(0.5) \approx 0.8$ . Then from the graph of f we see that  $f(0.8) \approx -1.7$ . So

$$h(0.5) = f(g(0.5)) \approx f(0.8) \approx -1.7$$

In a similar way we estimate the values of *h* in the following table:

X	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2,0
g(x)	-1.5	-1.6	-1.3	-0.8	0.0	0.8	1.3	1.6	1.5
h(x) = f(g(x))	1.0	0.7	1.5	1.7	0.0	-1.7	-1.5	-0.7	-1.0

If  $0 \le a \le b$ , then  $a^2 \le b^2$ .



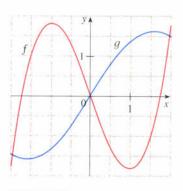
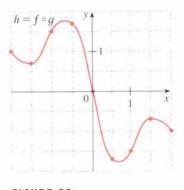


FIGURE 27



We use these values to graph the composite function h in Figure 28. If we want a more accurate graph, we could apply this procedure to more values of x.

It is possible to take the composition of three or more functions. For instance, the composite function  $f \circ g \circ h$  is found by first applying *h*, then *g*, and then *f* as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

**EXAMPLE 10** Find  $f \circ g \circ h$  if f(x) = x/(x + 1),  $g(x) = x^{10}$ , and h(x) = x + 3. SOLUTION

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3))$$
$$= f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to decompose a complicated function into simpler ones, as in the following example.

**EXAMPLE II** Given  $F(x) = \cos^2(x + 9)$ , find functions f, g, and h such that  $F = f \circ g \circ h$ .

**SOLUTION** Since  $F(x) = [\cos(x + 9)]^2$ , the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9$$
  $g(x) = \cos x$   $f(x) = x^2$ 

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9))$$
$$= [\cos(x + 9)]^2 = F(x)$$



Exercises

I-2 ■ Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a)  $f(x) = \sqrt[5]{x}$  (b)  $g(x) = \sqrt{1 - x^2}$ (c)  $h(x) = x^9 + x^4$  (d)  $r(x) = \frac{x^2 + 1}{x^3 + x}$ (e)  $s(x) = \tan 2x$  (f)  $t(x) = \log_{10} x$ 2. (a)  $y = \frac{x - 6}{x + 6}$  (b)  $y = x + \frac{x^2}{\sqrt{x - 1}}$ (c)  $y = 10^x$  (d)  $y = x^{10}$ (e)  $y = 2t^6 + t^4 - \pi$  (f)  $y = \cos \theta + \sin \theta$  **3–4** Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

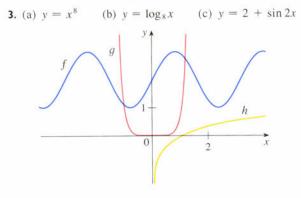
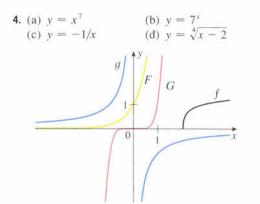


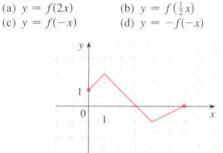
FIGURE 28



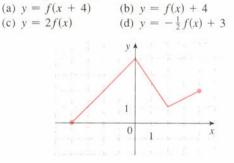
- 5. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.(a) Shift 3 units upward.
  - (a) Shift 5 units upward.
  - (b) Shift 3 units downward.
  - (c) Shift 3 units to the right.
  - (d) Shift 3 units to the left.
  - (e) Reflect about the *x*-axis.
  - (f) Reflect about the y-axis.
  - (g) Stretch vertically by a factor of 3.
  - (h) Shrink vertically by a factor of 3.
- **6.** Explain how the following graphs are obtained from the graph of y = f(x).

(a) $y = 5f(x)$	(b) $y = f(x - 5)$
(c) $y = -f(x)$	(d) $y = -5f(x)$
(e) $y = f(5x)$	(f) y = 5f(x) - 3

7. The graph of *f* is given. Use it to graph the following functions.



**8.** The graph of *f* is given. Draw the graphs of the following functions.

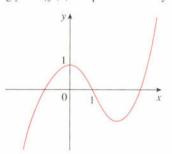


- 9. How is the graph of y = 2 sin x related to the graph of y = sin x? Use your answer and Figure 10(a) to sketch the graph of y = 2 sin x.
- 10. How is the graph of  $y = 1 + \sqrt{x}$  related to the graph of  $y = \sqrt{x}$ ? Use your answer and Figure 5(a) to sketch the graph of  $y = 1 + \sqrt{x}$ .

**11–26** Graph each function, not by plotting points, but by starting with the graph of one of the standard functions given in this section, and then applying the appropriate transformations.

<b>11.</b> $y = -1/x$	<b>12.</b> $y = 2 - \cos x$
<b>13.</b> $y = \tan 2x$	<b>14.</b> $y = \sqrt[3]{x+2}$
<b>15.</b> $y = \cos(x/2)$	<b>16.</b> $y = x^2 + 2x + 3$
<b>17.</b> $y = \frac{1}{x - 3}$	<b>18.</b> $y = -2 \sin \pi x$
<b>19.</b> $y = \frac{1}{3} \sin\left(x - \frac{\pi}{6}\right)$	<b>20.</b> $y = 2 + \frac{1}{x+1}$
<b>21.</b> $y = 1 + 2x - x^2$	<b>22.</b> $y = \frac{1}{2}\sqrt{x+4} - 3$
<b>23.</b> $y = 2 - \sqrt{x+1}$	<b>24.</b> $y = (x - 1)^3 + 2$
<b>25.</b> $y =  \cos x $	<b>26.</b> $y =   x  - 1 $

- **27.** (a) How is the graph of y = f(|x|) related to the graph of f?
  - (b) Sketch the graph of  $y = \sin |x|$ .
  - (c) Sketch the graph of  $y = \sqrt{|x|}$ .
- **28.** Use the given graph of f to sketch the graph of y = 1/f(x). Which features of f are the most important in sketching y = 1/f(x)? Explain how they are used.



**29–30** Find f + g, f - g, fg, and f/g and state their domains.

**29.**  $f(x) = x^3 + 2x^2$ ,  $g(x) = 3x^2 - 1$ **30.**  $f(x) = \sqrt{1+x}$ ,  $g(x) = \sqrt{1-x}$ 

**31–32** Use the graphs of f and g and the method of graphical addition to sketch the graph of f + g.

**31.** 
$$f(x) = x$$
,  $g(x) = 1/x$  **32.**  $f(x) = x^3$ ,  $g(x) = -x^2$ 

**33–36** Find the functions  $f \circ g$ ,  $g \circ f$ ,  $f \circ f$ , and  $g \circ g$  and their domains.

**33.** 
$$f(x) = 2x^2 - x$$
,  $g(x) = 3x + 2$   
**34.**  $f(x) = \sqrt{x - 1}$ ,  $g(x) = x^2$   
**35.**  $f(x) = \sqrt{x^2 - 1}$ ,  $g(x) = \sqrt{1 - x}$   
**36.**  $f(x) = \frac{1}{x - 1}$ ,  $g(x) = \frac{x - 1}{x + 1}$   
**37-38** Find  $f \circ g \circ h$ .

**37.** 
$$f(x) = \frac{1}{x}$$
,  $g(x) = x^3$ ,  $h(x) = x^2 + 2$ 

**38.** 
$$f(x) = \sqrt{x}$$
,  $g(x) = \frac{x}{x-1}$ ,  $h(x) = \sqrt[3]{x}$ 

**39–42** Express the function in the form  $f \circ g$ .

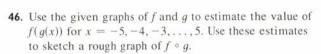
**39.** 
$$F(x) = (x - 9)^5$$
  
**40.**  $F(x) = \sin(\sqrt{x})$   
**41.**  $G(x) = \frac{x^2}{x^2 + 4}$   
**42.**  $G(x) = \frac{1}{x + 3}$ 

**43–44** Express the function in the form  $f \circ g \circ h$ .

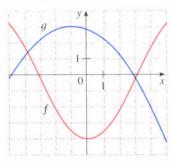
**43.** 
$$H(x) = 1 - 3^{x^2}$$
 **44.**  $H(x) = \sqrt[3]{\sqrt{x} - 1}$ 

- **45.** Use the given graphs of *f* and *g* to evaluate each expression, or explain why it is undefined.
  - (a) f(g(2)) (b) g(f(0)) (c)  $(f \circ g)(0)$ (d)  $(g \circ f)(6)$  (e)  $(g \circ g)(-2)$  (f)  $(f \circ f)(4)$

a



0



- **47.** A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
  - (a) Express the radius *r* of this circle as a function of the time *t* (in seconds).
  - (b) If A is the area of this circle as a function of the radius, find A  $\circ$  r and interpret it.
- **48.** An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time t = 0.
  - (a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.
  - (b) Express the distance *s* between the plane and the radar station as a function of *d*.
  - (c) Use composition to express s as a function of t.

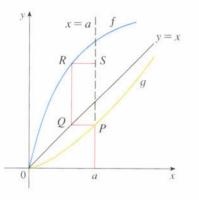
#### 49. The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0\\ 1 & \text{if } t \ge 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

- (a) Sketch the graph of the Heaviside function.
- (b) Sketch the graph of the voltage V(t) in a circuit if the switch is turned on at time t = 0 and 120 volts are applied instantaneously to the circuit. Write a formula for V(t) in terms of H(t).
- (c) Sketch the graph of the voltage V(t) in a circuit if the switch is turned on at time t = 5 seconds and 240 volts are applied instantaneously to the circuit. Write a formula for V(t) in terms of H(t). (Note that starting at t = 5 corresponds to a translation.)
- **50.** The Heaviside function defined in Exercise 49 can also be used to define the **ramp function** y = ctH(t), which represents a gradual increase in voltage or current in a circuit.
  - (a) Sketch the graph of the ramp function y = tH(t).
  - (b) Sketch the graph of the voltage V(t) in a circuit if the switch is turned on at time t = 0 and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for V(t)in terms of H(t) for  $t \le 60$ .
  - (c) Sketch the graph of the voltage V(t) in a circuit if the switch is turned on at time t = 7 seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for V(t) in terms of H(t) for  $t \le 32$ .
- **51.** Suppose g is an even function and let  $h = f \circ g$ . Is h always an even function?
- **52.** Suppose g is an odd function and let  $h = f \circ g$ . Is h always an odd function? What if f is odd? What if f is even?

**53.** Suppose we are given the graphs of f and g, as in the figure, and we want to find the point on the graph of  $h = f \circ g$  that corresponds to x = a. We start at the point (a, 0) and draw a vertical line that intersects the graph of g at the point P. Then we draw a horizontal line from P to the point Q on the line y = x.

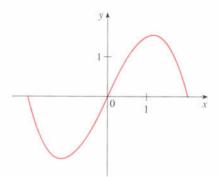


- (a) What are the coordinates of *P* and of *Q*?
- (b) If we now draw a vertical line from Q to the point R on the graph of f, what are the coordinates of R?



x = a

- (c) If we now draw a horizontal line from R to the point S on the line x = a, show that S lies on the graph of h.
- (d) By carrying out the construction of the path *PQRS* for several values of *a*, sketch the graph of *h*.
- 54. If f is the function whose graph is shown, use the method of Exercise 53 to sketch the graph of  $f \circ f$ . Start by using the construction for a = 0, 0.5, 1, 1.5, and 2. Sketch a rough graph for  $0 \le x \le 2$ . Then use the result of Exercise 52 to complete the graph.



# Graphing Calculators and Computers

In this section we assume that you have access to a graphing calculator or a computer with graphing software. We will see that the use of such a device enables us to graph more complicated functions and to solve more complex problems than would otherwise be possible. We also point out some of the pitfalls that can occur with these machines.

Graphing calculators and computers can give very accurate graphs of functions. But we will see in Chapter 4 that only through the use of calculus can we be sure that we have uncovered all the interesting aspects of a graph.

A graphing calculator or computer displays a rectangular portion of the graph of a function in a **display window** or **viewing screen**, which we refer to as a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the *x*-values to range from a minimum value of Xmin = a to a maximum value of Xmax = b and the *y*-values to range from a minimum of Ymin = c to a maximum of Ymax = d, then the portion of the graph lies in the rectangle

$$[a,b] \times [c,d] = \{(x,y) \mid a \le x \le b, c \le y \le d\}$$



y = d

x = b



shown in Figure 1. We refer to this rectangle as the [a,b] by [c,d] viewing rectangle.

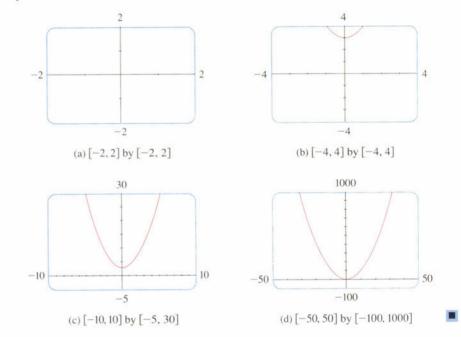
The machine draws the graph of a function f much as you would. It plots points of the form (x, f(x)) for a certain number of values of x between a and b. If an x-value is not in the domain of f, or if f(x) lies outside the viewing rectangle, the

machine moves on to the next x-value. It connects each point to the preceding plotted point to form a representation of the graph of f.

**EXAMPLE 1** Draw the graph of the function  $f(x) = x^2 + 3$  in each of the following viewing rectangles. (a) [-2, 2] by [-2, 2] (b) [-4, 4] by [-4, 4](c) [-10, 10] by [-5, 30] (d) [-50, 50] by [-100, 1000]

**SOLUTION** For part (a) we select the range by setting Xmin = -2, Xmax = 2, Ymin = -2, and Ymax = 2. The resulting graph is shown in Figure 2(a). The display window is blank! A moment's thought provides the explanation: Notice that  $x^2 \ge 0$  for all x, so  $x^2 + 3 \ge 3$  for all x. Thus, the range of the function  $f(x) = x^2 + 3$  is  $[3, \infty)$ . This means that the graph of f lies entirely outside the viewing rectangle [-2, 2] by [-2, 2].

The graphs for the viewing rectangles in parts (b), (c), and (d) are also shown in Figure 2. Observe that we get a more complete picture in parts (c) and (d), but in part (d) it is not clear that the y-intercept is 3.





We see from Example 1 that the choice of a viewing rectangle can make a big difference in the appearance of a graph. Sometimes it is necessary to change to a larger viewing rectangle to obtain a more complete picture, a more global view, of the graph. But too large a viewing rectangle can also be misleading. In the next example we see that knowledge of the domain and range of a function sometimes provides us with enough information to select a good viewing rectangle.

**EXAMPLE 2** Determine an appropriate viewing rectangle for the function  $f(x) = \sqrt{8 - 2x^2}$  and use it to graph *f*.

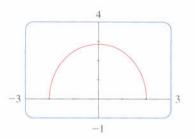
**SOLUTION** The expression for f(x) is defined when

Therefore, the domain of f is the interval [-2, 2]. Also,

$$0 \le \sqrt{8 - 2x^2} \le \sqrt{8} = 2\sqrt{2} \approx 2.83$$

so the range of f is the interval  $[0, 2\sqrt{2}]$ .

We choose the viewing rectangle so that the *x*-interval is somewhat larger than the domain and the *y*-interval is larger than the range. Taking the viewing rectangle to be [-3, 3] by [-1, 4], we get the graph shown in Figure 3.



5

-5

FIGURE 3

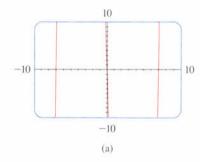
FIGURE 4

-5

**EXAMPLE 3** Graph the function  $y = x^3 - 49x$ .

**SOLUTION** Here the domain is  $\mathbb{R}$ , the set of all real numbers. That does not help us choose a viewing rectangle. Let's experiment. If we start with the viewing rectangle [-5, 5] by [-5, 5], we get the graph in Figure 4, which is nearly blank. The reason is that for all the *x*-values that the calculator chooses between -5 and 5, except 0, the values of f(x) are greater than 5 or less than -5, so the corresponding points on the graph lie outside the viewing rectangle.

If we use the zoom-out feature of a graphing calculator to change the viewing rectangle to [-10, 10] by [-10, 10], we get the picture shown in Figure 5(a). The graph appears to consist of vertical lines, but we know that can't be correct. If we look carefully while the graph is being drawn, we see that the graph leaves the screen and reappears during the graphing process. This indicates that we need to see more in the vertical direction, so we change the viewing rectangle to [-10, 10] by [-100, 100]. The resulting graph is shown in Figure 5(b). It still doesn't quite reveal all the main features of the function, so we try [-10, 10] by [-200, 200] in Figure 5(c). Now we are more confident that we have arrived at an appropriate viewing rectangle. In Chapter 4 we will be able to see that the graph shown in Figure 5(c) does indeed reveal all the main features of the function.



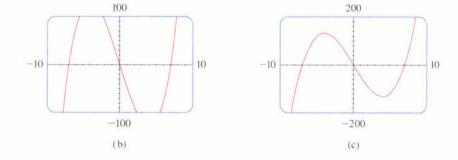
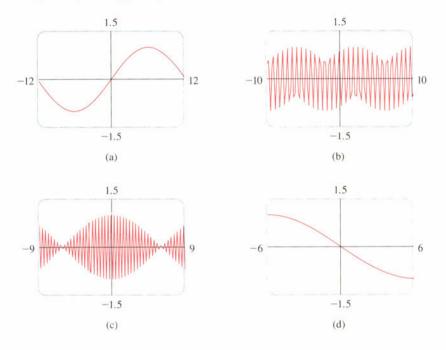


FIGURE 5  $f(x) = x^3 - 49x$ 

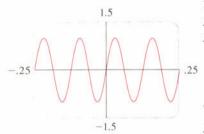
**EXAMPLE 4** Graph the function  $f(x) = \sin 50x$  in an appropriate viewing rectangle.

**SOLUTION** Figure 6(a) shows the graph of *f* produced by a graphing calculator using the viewing rectangle [-12, 12] by [-1.5, 1.5]. At first glance the graph appears to be reasonable. But if we change the viewing rectangle to the ones shown in the following parts of Figure 6, the graphs look very different. Something strange is happening.



The appearance of the graphs in Figure 6 depends on the machine used. The graphs you get with your own graphing device might not look like these figures, but they will also be quite inaccurate.

**FIGURE 6** Graphs of  $f(x) = \sin 50x$ in four viewing rectangles



**FIGURE 7**  $f(x) = \sin 50x$ 

In order to explain the big differences in appearance of these graphs and to find an appropriate viewing rectangle, we need to find the period of the function  $y = \sin 50x$ . We know that the function  $y = \sin x$  has period  $2\pi$ , so the period of  $y = \sin 50x$  is

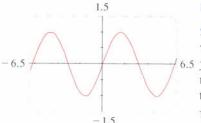
$$\frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126$$

This suggests that we should deal only with small values of x in order to show just a few oscillations of the graph. If we choose the viewing rectangle [-0.25, 0.25] by [-1.5, 1.5], we get the graph shown in Figure 7.

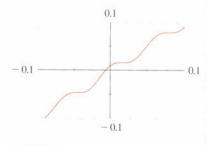
Now we see what went wrong in Figure 6. The oscillations of  $y = \sin 50x$  are so rapid that when the calculator plots points and joins them, it misses most of the maximum and minimum points and therefore gives a very misleading impression of the graph.

We have seen that the use of an inappropriate viewing rectangle can give a misleading impression of the graph of a function. In Examples 1 and 3 we solved the problem by changing to a larger viewing rectangle. In Example 4 we had to make the viewing rectangle smaller. In the next example we look at a function for which there is no single viewing rectangle that reveals the true shape of the graph.











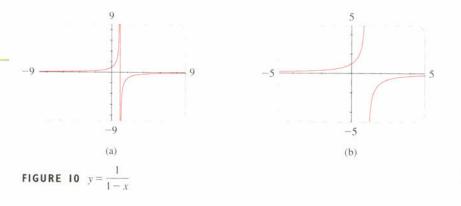
Another way to avoid the extraneous line is to change the graphing mode on the calculator so that the dots are not connected.

**EXAMPLE 5** Graph the function  $f(x) = \sin x + \frac{1}{100} \cos 100x$ .

**SOLUTION** Figure 8 shows the graph of *f* produced by a graphing calculator with viewing rectangle [-6.5, 6.5] by [-1.5, 1.5]. It looks much like the graph of 6.5  $y = \sin x$ , but perhaps with some bumps attached. If we zoom in to the viewing rectangle [-0.1, 0.1] by [-0.1, 0.1], we can see much more clearly the shape of these bumps in Figure 9. The reason for this behavior is that the second term,  $\frac{1}{100} \cos 100x$ , is very small in comparison with the first term,  $\sin x$ . Thus, we really need two graphs to see the true nature of this function.

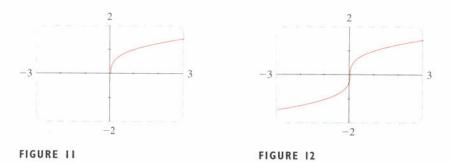
**EXAMPLE 6** Draw the graph of the function 
$$y = \frac{1}{1-x}$$

**SOLUTION** Figure 10(a) shows the graph produced by a graphing calculator with viewing rectangle [-9, 9] by [-9, 9]. In connecting successive points on the graph, the calculator produced a steep line segment from the top to the bottom of the screen. That line segment is not truly part of the graph. Notice that the domain of the function y = 1/(1 - x) is  $\{x \mid x \neq 1\}$ . We can eliminate the extraneous near-vertical line by experimenting with a change of scale. When we change to the smaller viewing rectangle [-5, 5] by [-5, 5], we obtain the much better graph in Figure 10(b).



**EXAMPLE 7** Graph the function  $y = \sqrt[3]{x}$ .

**SOLUTION** Some graphing devices display the graph shown in Figure 11, whereas others produce a graph like that in Figure 12. We know from Section 1.2 (Figure 5) that the graph in Figure 12 is correct, so what happened in Figure 11? The explanation is that, in some machines,  $x^{1/3}$  is computed as  $e^{(1/3) \ln x}$  and  $\ln x$  is not defined for x < 0, so only the right half of the graph is produced.



You should experiment with your own machine to see which of these two graphs is produced. If you get the graph in Figure 11, you can obtain the correct picture by graphing the function

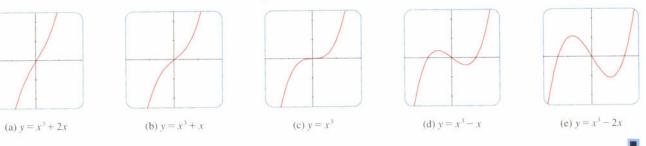
$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$$

Notice that this function is equal to  $\sqrt[3]{x}$  (except when x = 0).

To understand how the expression for a function relates to its graph, it is helpful to graph a **family of functions**, that is, a collection of functions whose equations are related. In the next example we graph members of a family of cubic polynomials.

**EXAMPLE 8** Graph the function  $y = x^3 + cx$  for various values of the number c. How does the graph change when c is changed?

**SOLUTION** Figure 13 shows the graphs of  $y = x^3 + cx$  for c = 2, 1, 0, -1, and -2. We see that, for positive values of c, the graph increases from left to right with no maximum or minimum points (peaks or valleys). When c = 0, the curve is flat at the origin. When c is negative, the curve has a maximum point and a minimum point. As c decreases, the maximum point becomes higher and the minimum point lower.

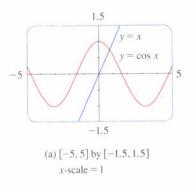


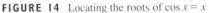
### FIGURE 13

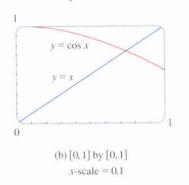
Several members of the family of functions  $y = x^3 + cx$ , all graphed in the viewing rectangle [-2, 2] by [-2.5, 2.5]

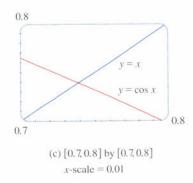
**EXAMPLE 9** Find the solution of the equation  $\cos x = x$  correct to two decimal places.

**SOLUTION** The solutions of the equation  $\cos x = x$  are the *x*-coordinates of the points of intersection of the curves  $y = \cos x$  and y = x. From Figure 14(a) we see that there is only one solution and it lies between 0 and 1. Zooming in to the









viewing rectangle [0, 1] by [0, 1], we see from Figure 14(b) that the root lies between 0.7 and 0.8. So we zoom in further to the viewing rectangle [0.7, 0.8]by [0.7, 0.8] in Figure 14(c). By moving the cursor to the intersection point of the two curves, or by inspection and the fact that the *x*-scale is 0.01, we see that the root of the equation is about 0.74.



1. Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function

Exercises · · · · · ·

$$f(x) = 10 + 25x - x^3$$

- (a) [-4, 4] by [-4, 4]
- (b) [−10, 10] by [−10, 10]
- (c) [-20, 20] by [-100, 100]
- (d) [-100, 100] by [-200, 200]
- **2.** Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function

 $f(x) = \sqrt{8x - x^2}$ 

(a) [-4,4] by [-4,4] (b) [-5,5] by [0,100] (c) [-10,10] by [-10,40] (d) [-2,10] by [-2,6]

**3–14** ■ Determine an appropriate viewing rectangle for the given function and use it to draw the graph.

- **3.**  $f(x) = \sqrt[4]{256 x^2}$  **4.**  $f(x) = 0.01x^3 - x^2 + 5$  **5.**  $y = \frac{1}{x^2 + 25}$  **6.**  $y = \frac{2x - 1}{x + 3}$  **7.**  $y = x^4 - 4x^3$  **8.**  $y = 2x - |x^2 - 5|$  **9.**  $f(x) = \cos 100x$  **10.**  $f(x) = 3 \sin 120x$  **11.**  $f(x) = \sin(x/40)$  **12.**  $y = \tan 25x$  **13.**  $y = 3^{\cos(x^2)}$  **14.**  $y = x^2 + 0.02 \sin 50x$ **15.** Graph the ellipse  $4x^2 + 2y^2 = 1$  by graphing the
- functions whose graphs are the upper and lower halves of the ellipse.
- 16. Graph the hyperbola  $y^2 9x^2 = 1$  by graphing the functions whose graphs are the upper and lower branches of the hyperbola.

17-19 = Find all solutions of the equation correct to two decimal places.

- 17.  $3x^3 + x^2 + x 2 = 0$
- **18.**  $x^4 + 8x + 16 = 2x^3 + 8x^2$
- **19.**  $2 \sin x = x$

- **20.** We saw in Example 9 that the equation  $\cos x = x$  has exactly one solution.
  - (a) Use a graph to show that the equation  $\cos x = 0.3x$  has three solutions and find their values correct to two decimal places.
  - (b) Find an approximate value of m such that the equation  $\cos x = mx$  has exactly two solutions.
- **21.** Use graphs to determine which of the functions  $f(x) = 10x^2$  and  $g(x) = x^3/10$  is eventually larger (that is, larger when x is very large).
- **22.** Use graphs to determine which of the functions  $f(x) = x^4 100x^3$  and  $g(x) = x^3$  is eventually larger.
- **23.** For what values of x is it true that  $|\sin x x| < 0.1$ ?
- 24. Graph the polynomials  $P(x) = 3x^5 5x^3 + 2x$  and  $Q(x) = 3x^5$  on the same screen, first using the viewing rectangle [-2, 2] by [-2, 2] and then changing to [-10, 10] by [-10,000, 10,000]. What do you observe from these graphs?
- **25.** In this exercise we consider the family of functions  $f(x) = \sqrt[n]{x}$ , where *n* is a positive integer.
  - (a) Graph the root functions  $y = \sqrt{x}$ ,  $y = \sqrt[4]{x}$ , and  $y = \sqrt[6]{x}$  on the same screen using the viewing rectangle [-1, 4] by [-1, 3].
  - (b) Graph the root functions  $y = x, y = \sqrt[3]{x}$ , and  $y = \sqrt[5]{x}$  on the same screen using the viewing rectangle [-3, 3] by [-2, 2]. (See Example 7.)
  - (c) Graph the root functions  $y = \sqrt{x}$ ,  $y = \sqrt[3]{x}$ ,  $y = \sqrt[4]{x}$ , and  $y = \sqrt[5]{x}$  on the same screen using the viewing rectangle [-1, 3] by [-1, 2].
  - (d) What conclusions can you make from these graphs?

**26.** In this exercise we consider the family of functions  $f(x) = 1/x^n$ , where *n* is a positive integer.

- (a) Graph the functions y = 1/x and y = 1/x<sup>3</sup> on the same screen using the viewing rectangle [-3, 3] by [-3, 3].
- (b) Graph the functions  $y = 1/x^2$  and  $y = 1/x^4$  on the same screen using the same viewing rectangle as in part (a).
- (c) Graph all of the functions in parts (a) and (b) on the same screen using the viewing rectangle [-1, 3] by [-1, 3].
- (d) What conclusions can you make from these graphs?

- **27.** Graph the function  $f(x) = x^4 + cx^2 + x$  for several values of *c*. How does the graph change when *c* changes?
- **28.** Graph the function  $f(x) = \sqrt{1 + cx^2}$  for various values of *c*. Describe how changing the value of *c* affects the graph.
- **29.** Graph the function  $y = x^n 2^{-x}$ ,  $x \ge 0$ , for n = 1, 2, 3, 4, 5, and 6. How does the graph change as *n* increases?
- 30. The curves with equations

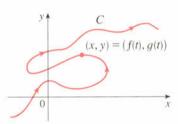
$$y = \frac{|x|}{\sqrt{c - x^2}}$$

are called **bullet-nose curves**. Graph some of these curves to see why. What happens as *c* increases?

- **31.** What happens to the graph of the equation  $y^2 = cx^3 + x^2$  as c varies?
- **32.** This exercise explores the effect of the inner function g on a composite function y = f(g(x)).
  - (a) Graph the function  $y = \sin(\sqrt{x})$  using the viewing rectangle [0, 400] by [-1.5, 1.5]. How does this graph differ from the graph of the sine function?
  - (b) Graph the function y = sin(x<sup>2</sup>) using the viewing rectangle [-5, 5] by [-1.5, 1.5]. How does this graph differ from the graph of the sine function?



Parametric Curves



Imagine that a particle moves along the curve *C* shown in Figure 1. It is impossible to describe *C* by an equation of the form y = f(x) because *C* fails the Vertical Line Test. But the *x*- and *y*-coordinates of the particle are functions of time and so we can write x = f(t) and y = g(t). Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as continuous functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point (x, y) = (f(t), g(t)) varies and traces out a curve C, which we call a **parametric curve**. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and we can interpret (x, y) = (f(t), g(t)) as the position of a particle at time t.

**EXAMPLE I** Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

**SOLUTION** Each value of t gives a point on the curve, as shown in the table. For instance, if t = 0, then x = 0, y = 1 and so the corresponding point is (0, 1). In

1	X	$\mathcal{N}$
-2	8	-1
-1	8	0
0	0	1
1	-1	2
2	0	3
3	0 3 8	4
4	8	5



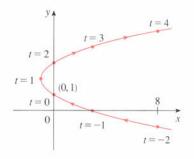


FIGURE 2

Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as tincreases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. This can be confirmed by eliminating the parameter t as follows. We obtain t = y - 1 from the second equation and substitute into the first equation. This gives

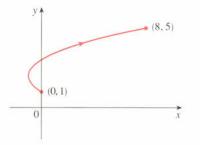
$$x = (y - 1)^{2} - 2(y - 1) = y^{2} - 4y + 3$$

and so the curve represented by the given parametric equations is the parabola  $x = y^2 - 4y + 3$ .

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number. But sometimes we restrict t to lie in a finite interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \le t \le 4$$

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point (0, 1) and ends at the point (8, 5). The arrowhead indicates the direction in which the curve is traced as *t* increases from 0 to 4.





In general, the curve with parametric equations

x = f(t) y = g(t)  $a \le t \le b$ 

has initial point (f(a), g(a)) and terminal point (f(b), g(b)).

**EXAMPLE 2** What curve is represented by the parametric equations  $x = \cos t$ ,  $y = \sin t$ ,  $0 \le t \le 2\pi$ ?

**SOLUTION** We can eliminate t by noting that

$$x^{2} + y^{2} = \cos^{2}t + \sin^{2}t = 1$$

Thus, the point (x, y) moves on the unit circle  $x^2 + y^2 = 1$ . Notice that in this example the parameter *t* can be interpreted as the angle shown in Figure 4. As *t* increases from 0 to  $2\pi$ , the point  $(x, y) = (\cos t, \sin t)$  moves once around the circle in the counterclockwise direction starting from the point (1, 0).

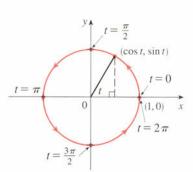
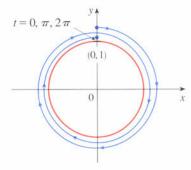


FIGURE 4





**EXAMPLE 3** What curve is represented by the parametric equations  $x = \sin 2t$ ,  $y = \cos 2t$ ,  $0 \le t \le 2\pi$ ?

**SOLUTION** Again we have

 $x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$ 

so the parametric equations again represent the unit circle  $x^2 + y^2 = 1$ . But as *t* increases from 0 to  $2\pi$ , the point  $(x, y) = (\sin 2t, \cos 2t)$  starts at (0, 1) and moves *twice* around the circle in the clockwise direction as indicated in Figure 5.

Examples 2 and 3 show that different sets of parametric equations can represent the same curve. Thus, we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced in a particular order.

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations. In fact, it is instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

**EXAMPLE 4** Use a graphing device to graph the curve  $x = y^4 - 3y^2$ .

**SOLUTION** If we let the parameter be t = y, then we have the equations

$$x = t^4 - 3t^2 \qquad y = 1$$

Using these parametric equations to graph the curve, we obtain Figure 6. It would be possible to solve the given equation  $(x = y^4 - 3y^2)$  for y as four functions of x and graph them individually, but the parametric equations provide a much easier method.

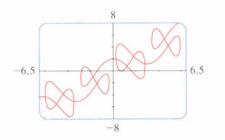
In general, if we need to graph an equation of the form x = g(y), we can use the parametric equations

$$x = g(t) \qquad y = t$$

Notice also that curves with equations y = f(x) (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

$$x = t$$
  $y = f(t)$ 

Graphing devices are particularly useful when sketching complicated curves. For instance, the curves shown in Figures 7 and 8 would be virtually impossible to produce by hand.



**FIGURE 7**  $x = t + 2 \sin 2t, y = t + 2 \cos 5t$ 

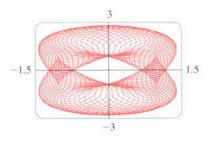


FIGURE 8  $x = \cos t - \cos 80t \sin t, y = 2 \sin t - \sin 80t$ 

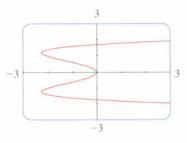


FIGURE 6

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Laboratory Project after Section 3.5 we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in laser printers.

**EXAMPLE 5** Sketch the curve with parametric equations  $x = \sin t$ ,  $y = \sin^2 t$ .

**SOLUTION** Observe that  $y = x^2$  and so the point (x, y) moves on the parabola  $y = x^2$ . But note also that, since  $-1 \le \sin t \le 1$ , we have  $-1 \le x \le 1$ , so the parametric equations represent only the part of the parabola for which  $-1 \le x \le 1$ . Since  $\sin t$  is periodic, the point  $(x, y) = (\sin t, \sin^2 t)$  moves back and forth infinitely often along the parabola from (-1, 1) to (1, 1) (see Figure 9).

**EXAMPLE 6** The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 10). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



**SOLUTION** We choose as parameter the angle of rotation  $\theta$  of the circle ( $\theta = 0$  when *P* is at the origin). When the circle has rotated through  $\theta$  radians, the distance it has rolled from the origin is

$$|OT| = \operatorname{arc} PT = r\theta$$

and so the center of the circle is  $C(r\theta, r)$ . Let the coordinates of P be (x, y). Then from Figure 11 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

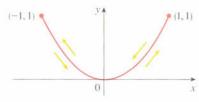
Therefore, parametric equations of the cycloid are

1 
$$x = r(\theta - \sin \theta)$$
  $y = r(1 - \cos \theta)$   $\theta \in \mathbb{R}$ 

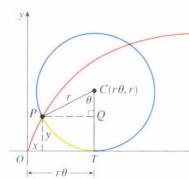
One arch of the cycloid comes from one rotation of the circle and so is described by  $0 \le \theta \le 2\pi$ . Although Equations 1 were derived from Figure 11, which illustrates the case where  $0 \le \theta \le \pi/2$ , it can be seen that these equations are still valid for other values of  $\theta$  (see Exercise 27).

Although it is possible to eliminate the parameter  $\theta$  from Equations 1, the resulting Cartesian equation in x and y is very complicated and not as convenient to work with as the parametric equations.

One of the first people to study the cycloid was Galileo, who proposed that bridges be built in the shape of cycloids and who tried to find the area under one arch of a cycloid. Later this curve arose in connection with the *Brachistochrone problem:* Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A.









#### 52 CHAPTER I FUNCTIONS AND MODELS

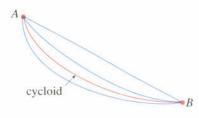


FIGURE 12



FIGURE 13

The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, as in Figure 12, the particle will take the least time sliding from A to B if the curve is an inverted arch of a cycloid.

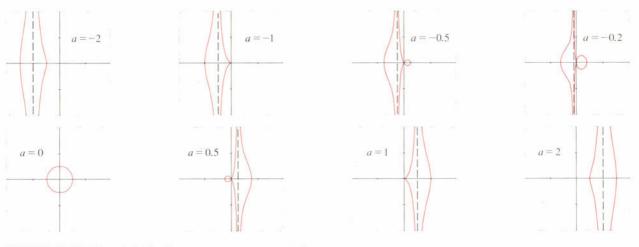
The Dutch physicist Huygens had already shown that the cycloid is also the solution to the Tautochrone problem; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 13). Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

**EXAMPLE 7** Investigate the family of curves with parametric equations

 $x = a + \cos t$   $y = a \tan t + \sin t$ 

What do these curves have in common? How does the shape change as a increases?

**SOLUTION** We use a graphing device to produce the graphs for the cases a = -2, -1, -0.5, -0.2, 0, 0.5, 1, and 2 shown in Figure 14. Notice that all of these curves (except the case a = 0) have two branches, and both branches approach the vertical asymptote x = a as x approaches a from the left or right.



**FIGURE 14** Members of the family  $x = a + \cos t$ ,  $y = a \tan t + \sin t$ , all graphed in the viewing rectangle [-4, 4] by [-4, 4]

> When a < -1, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When a = 0, both branches come together and form a circle (see Example 2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when a = 1. For a > 1, the branches become smooth again and, as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

> These curves are called conchoids of Nicomedes after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell.

1.4

#### 1-4 ■

(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as *t* increases.

Exercises · · · · · · · · · · ·

(b) Eliminate the parameter to find a Cartesian equation of the curve.

**1.** 
$$x = 2t - 1$$
,  $y = 2 - t$ ,  $-3 \le t \le 3$ 

**2.**  $x = 3t^2$ , y = 2 + 5t,  $0 \le t \le 2$ 

**3.** 
$$x = \sqrt{t}$$
,  $y = 1 - t$ 

4. 
$$x = t^2$$
,  $y = t^3$ 

- 5-10 =
- (a) Eliminate the parameter to find a Cartesian equation of the curve.
- (b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

5. 
$$x = \sin \theta$$
,  $y = \cos \theta$ ,  $0 \le \theta \le \pi$ 

- **6.**  $x = 3\cos\theta$ ,  $y = 2\sin\theta$ ,  $0 \le \theta \le 2\pi$
- 7.  $x = \sin^2 \theta$ ,  $y = \cos^2 \theta$
- 8.  $x = \sec \theta$ ,  $y = \tan \theta$ ,  $-\pi/2 < \theta < \pi/2$

9. 
$$x = \cos^2 \theta$$
,  $y = \sin \theta$ 

**10.**  $x = \cos t$ ,  $y = \cos 2t$ 

**11–15** Describe the motion of a particle with position (x, y) as t varies in the given interval.

- 11.  $x = \cos \pi t$ ,  $y = \sin \pi t$ ,  $1 \le t \le 2$
- **12.**  $x = 2 + \cos t$ ,  $y = 3 + \sin t$ ,  $0 \le t \le 2\pi$
- **13.**  $x = 2 \sin t, y = 3 \cos t, 0 \le t \le 2\pi$
- **14.**  $x = \cos^2 t$ ,  $y = \cos t$ ,  $0 \le t \le 4\pi$
- 15.  $x = \sin t$ ,  $y = \csc t$ ,  $\pi/6 \le t \le 1$
- I6-17 Graph x and y as functions of t and observe how x and y increase or decrease as t increases. Use these observations to make a rough sketch by hand of the parametric curve. Then use a graphing device to check your sketch.

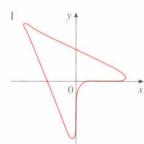
**16.** 
$$x = 3(t^2 - 3), y = t^3 - 3t$$

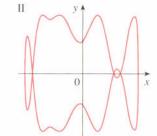
**17.** 
$$x = t^4 - 1$$
,  $y = t^3 + 1$ 

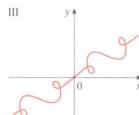
18. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices. (Do not use a graphing device.)
(a) x = t<sup>3</sup> - 2t, y = t<sup>2</sup> - t

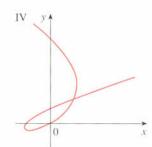
(b)  $x = t^3 - 1$ ,  $y = 2 - t^2$ (c)  $x = \sin 3t$ ,  $y = \sin 4t$ 

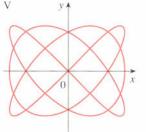
- (d)  $x = t + \sin 2t$ ,  $y = t + \sin 3t$
- (e)  $x = \sin(t + \sin t)$ ,  $y = \cos(t + \cos t)$
- (f)  $x = \cos t$ ,  $y = \sin(t + \sin 5t)$











- 19. Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t$$

$$y = y_1 + (y_2 - y_1)t$$

where  $0 \le t \le 1$ , describe the line segment that joins the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ .

- 20. Use a graphing device and the result of Exercise 19 to draw the triangle with vertices A(1, 1), B(4, 2), and C(1, 5).
- **21.** Graph the curve  $x = y 3y^3 + y^5$ .
- **22.** Graph the curves  $y = x^5$  and  $x = y(y 1)^2$  and find their points of intersection correct to one decimal place.

#### 54 CHAPTER I FUNCTIONS AND MODELS

- 23. Find parametric equations for the path of a particle that moves along the circle  $x^2 + (y 1)^2 = 4$  in the following manner:
  - (a) Once around clockwise, starting at (2, 1)
  - (b) Three times around counterclockwise, starting at (2, 1)
  - (c) Halfway around counterclockwise, starting at (0,3)
- **24.** Graph the semicircle traced by the particle in Exercise 23(c).
- 25. (a) Find parametric equations for the ellipse
   x<sup>2</sup>/a<sup>2</sup> + y<sup>2</sup>/b<sup>2</sup> = 1. [*Hint:* Modify the equations of a circle in Example 2.]
  - (b) Use these parametric equations to graph the ellipse when a = 3 and b = 1, 2, 4, and 8.
  - (c) How does the shape of the ellipse change as b varies?
  - **26.** If a projectile is fired with an initial velocity of  $v_0$  meters per second at an angle  $\alpha$  above the horizontal and air resistance is assumed to be negligible, then its position after *t* seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t$$
$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity (9.8 m/s<sup>2</sup>).

- (a) If a gun is fired with  $\alpha = 30^{\circ}$  and  $v_0 = 500$  m/s, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?
- (b) Use a graphing device to check your answers to part (a). Then graph the path of the projectile for several other values of the angle α to see where it hits the ground. Summarize your findings.
- (c) Show that the path is parabolic by eliminating the parameter.
- **27.** Derive Equations 1 for the case  $\pi/2 < \theta < \pi$ .

 $\wedge$ 

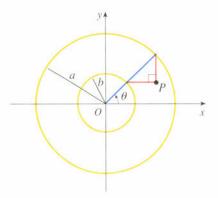
28. Let P be a point at a distance d from the center of a circle of radius r. The curve traced out by P as the circle rolls along a straight line is called a trochoid. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with d = r. Using the same parameter θ as for the cycloid and assuming the line is the x-axis and θ = 0 when P is at one of its lowest points, show that the parametric equations of the trochoid are

$$x = r\theta - d\sin\theta$$
$$y = r - d\cos\theta$$

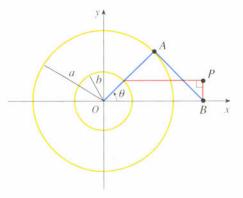
Sketch the trochoid for the cases d < r and d > r.

**29.** If *a* and *b* are fixed numbers, find parametric equations for the set of all points *P* determined as shown in the

figure, using the angle  $\theta$  as the parameter. Then eliminate the parameter and identify the curve.



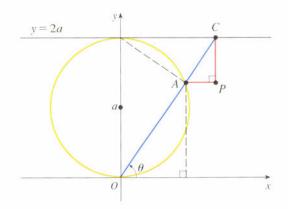
**30.** If *a* and *b* are fixed numbers, find parametric equations for the set of all points *P* determined as shown in the figure, using the angle  $\theta$  as the parameter. The line segment *AB* is tangent to the larger circle.



**31.** A curve, called a **witch of Maria Agnesi**, consists of all points *P* determined as shown in the figure. Show that parametric equations for this curve can be written as

$$c = 2a\cot\theta \qquad y = 2a\sin^2\theta$$

Sketch the curve.



**32.** Suppose that the position of one particle at time *t* is given by

 $x_1 = 3\sin t \qquad y_1 = 2\cos t \qquad 0 \le t \le 2\pi$ 

and the position of a second particle is given by

 $x_2 = -3 + \cos t$   $y_2 = 1 + \sin t$   $0 \le t \le 2\pi$ 

- (a) Graph the paths of both particles. How many points of intersection are there?
- (b) Are any of these points of intersection *collision points?* In other words, are the particles ever at the same place at the same time? If so, find the collision points.
- (c) Describe what happens if the path of the second particle is given by

 $x_2 = 3 + \cos t$   $y_2 = 1 + \sin t$   $0 \le t \le 2\pi$ 

**33.** Investigate the family of curves defined by the parametric equations  $x = t^2$ ,  $y = t^3 - ct$ . How does the

Laboratory Project

## Families of Hypocycloids

In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A hypocycloid is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. (This makes more sense physically if a > b, but the curves are just as pretty if a < b.) Show that if the initial position of P is (a, 0) and the parameter  $\theta$  is chosen as in the figure, then parametric equations of the hypocycloid are

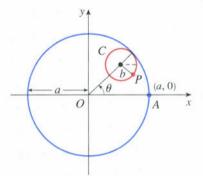
$$x = (a - b)\cos\theta + b\cos\left(\frac{a - b}{b}\theta\right)$$
  $y = (a - b)\sin\theta - b\sin\left(\frac{a - b}{b}\theta\right)$ 

**2.** Use a graphing device to draw the graphs of hypocycloids with a a positive integer and b = 1. How does the value of a affect the graph? Show that if we take a = 4, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a hypocycloid of four cusps, or an astroid.

- 3. Now try b = 1 and a = n/d, a fraction where n and d have no common factor. First let n = 1 and try to determine graphically the effect of the denominator d on the shape of the graph. Then let n vary while keeping d constant. What happens when n = d + 1? Try to show algebraically why these graphs look familiar. [*Hint:* Substitute φ = -θ/d and magnify the graph by a factor of d.]
- **4.** What happens if b = 1 and a is irrational? Experiment with an irrational number like  $\sqrt{2}$  or e 2. Take larger and larger values for  $\theta$  and speculate on what would happen if we were to graph the hypocycloid for all real values of  $\theta$ .
- 5. If the circle C rolls on the *outside* of the fixed circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.



curves have in common? How do they change when c increases?35. The curves with equations x = a sin nt, y = b cos t are

shape change as c increases? Illustrate by graphing

several members of the family.

- **35.** The curves with equations  $x = a \sin nt$ ,  $y = b \cos t$  are called **Lissajous figures**. Investigate how these curves vary when *a*, *b*, and *n* vary. (Take *n* to be a positive integer.)
- **36.** Investigate the family of curves defined by the parametric equations

$$x = \sin t (c - \sin t)$$
  $y = \cos t (c - \sin t)$ 

How does the shape change as c changes? In particular, you should identify the transitional values of c for which the basic shape of the curve changes.

- 6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2-4.
- 7. Let b = 1. Show algebraically that the epicycloid with a = n (any natural number) has the same shape as the hypocycloid with a = n/(n + 1). Show that the epicycloid with a = 1/n has the same shape as a hypocycloid. What is the value of a for the hypocycloid?



## Exponential Functions

The function  $f(x) = 2^x$  is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function  $g(x) = x^2$ , in which the variable is the base.

In general, an exponential function is a function of the form

$$f(x) = a^x$$

where *a* is a positive constant. Let's recall what this means. If x = n, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{u \text{ factors}}$$

If x = 0, then  $a^0 = 1$ , and if x = -n, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, x = p/q, where p and q are integers and q > 0, then

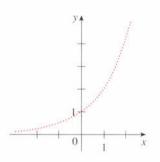
$$a^x = a^{p/q} = \sqrt[q]{a^p}$$

But what is the meaning of  $a^x$  if x is an irrational number? For instance, what is meant by  $2^{\sqrt{3}}$  or  $5^{\pi}$ ?

To help us answer this question we first look at the graph of the function  $y = 2^x$ , where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of  $y = 2^x$  to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining  $f(x) = 2^x$ , where  $x \in \mathbb{R}$ , so that f is an increasing function. In particular, since the irrational number  $\sqrt{3}$  satisfies

 $\begin{array}{l} 1.7 < \sqrt{3} \ < \ 1.8 \\ \\ 2^{1.7} < \ 2^{\sqrt{3}} \ < \ 2^{1.8} \end{array}$ 



**FIGURE I** Representation of  $y = 2^x$ , *x* rational

we must have

and we know what 2<sup>1.7</sup> and 2<sup>1.8</sup> mean because 1.7 and 1.8 are rational numbers.

Similarly, using better approximations for  $\sqrt{3}$ , we obtain better approximations for  $2^{\sqrt{3}}$ :

$$1.73 < \sqrt{3} < 1.74 \Rightarrow 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$$

$$1.732 < \sqrt{3} < 1.733 \Rightarrow 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}$$

$$1.7320 < \sqrt{3} < 1.7321 \Rightarrow 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}$$

$$1.73205 < \sqrt{3} < 1.73206 \Rightarrow 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

It can be shown that there is exactly one number that is greater than all of the numbers

 $2^{1.7}, 2^{1.73}, 2^{1.732}, 2^{1.7320}, 2^{1.73205}, \ldots$ 

and less than all of the numbers

 $2^{1.8}, 2^{1.74}, 2^{1.733}, 2^{1.7321}, 2^{1.73206}, \ldots$ 

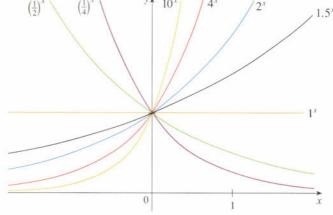
We define  $2^{\sqrt{3}}$  to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

Similarly, we can define  $2^x$  (or  $a^x$ , if a > 0) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function  $f(x) = 2^x$ ,  $x \in \mathbb{R}$ .

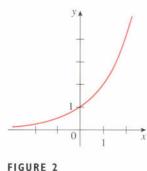
The graphs of members of the family of functions  $y = a^x$  are shown in Figure 3 for various values of the base a. Notice that all of these graphs pass through the same point (0, 1) because  $a^0 = 1$  for  $a \neq 0$ . Notice also that as the base a gets larger, the exponential function grows more rapidly (for x > 0).

If 0 < a < 1, then  $a^x$  approaches 0 as x becomes large. If a > 1, then  $a^x$ approaches 0 as x decreases through negative values. In both cases the x-axis is a horizontal asymptote. These matters are discussed in Section 2.5.





You can see from Figure 3 that there are basically three kinds of exponential functions  $y = a^x$ . If 0 < a < 1, the exponential function decreases; if a = 1, it is a constant; and if a > 1, it increases. These three cases are illustrated in Figure 4.





For a review of reflecting and shifting

graphs, see Section 1.2.

Observe that if  $a \neq 1$ , then the exponential function  $y = a^x$  has domain  $\mathbb{R}$  and range  $(0, \infty)$ . Notice also that, since  $(1/a)^x = 1/a^x = a^{-x}$ , the graph of  $y = (1/a)^x$  is just the reflection of the graph of  $y = a^x$  about the y-axis.

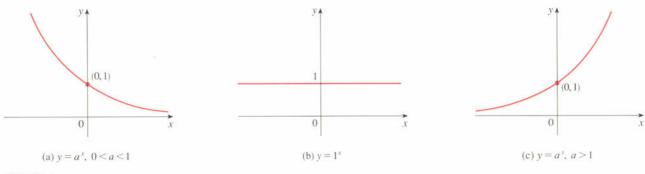


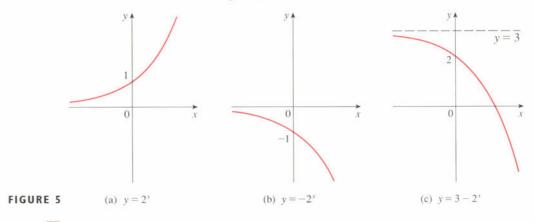
FIGURE 4

One reason for the importance of the exponential function lies in the following properties. If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y.

**Laws of Exponents** If a and b are positive numbers and x and y are any real numbers, then 1.  $a^{x+y} = a^x a^y$  2.  $a^{x-y} = \frac{a^x}{a^y}$  3.  $(a^x)^y = a^{xy}$  4.  $(ab)^x = a^x b^x$ 

**EXAMPLE I** Sketch the graph of the function  $y = 3 - 2^x$  and determine its domain and range.

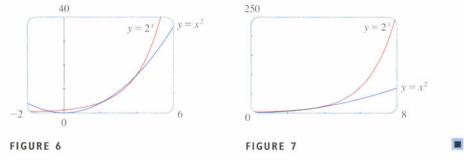
**SOLUTION** First we reflect the graph of  $y = 2^x$  (shown in Figure 2) about the *x*-axis to get the graph of  $y = -2^x$  in Figure 5(b). Then we shift the graph of  $y = -2^x$  upward three units to obtain the graph of  $y = 3 - 2^x$  in Figure 5(c). The domain is  $\mathbb{R}$  and the range is  $(-\infty, 3)$ .



**EXAMPLE 2** Use a graphing device to compare the exponential function  $f(x) = 2^x$  and the power function  $g(x) = x^2$ . Which function grows more quickly when x is large?

Example 2 shows that  $y = 2^x$  increases more quickly than  $y = x^2$ . To demonstrate just how quickly  $f(x) = 2^x$ increases, let us perform the following thought experiment. Suppose we start with a piece of paper a thousandth of an inch thick and we fold it in half 50 times. Each time we fold the paper in half, the thickness of the paper doubles, so the thickness of the resulting paper would be  $2^{50}/1000$  inches. How thick do you think that is? It works out to be more than 17 million miles!

**SOLUTION** Figure 6 shows both functions graphed in the viewing rectangle [-2, 6] by [0, 40]. We see that the graphs intersect three times, but for x > 4, the graph of  $f(x) = 2^x$  stays above the graph of  $g(x) = x^2$ . Figure 7 gives a more global view and shows that for large values of x, the exponential function  $y = 2^x$  grows far more rapidly than the power function  $y = x^2$ .



### Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth and radioactive decay. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is p(t), where t is measured in hours, and the initial population is p(0) = 1000, then we have

$$p(1) = 2p(0) = 2 \times 1000$$
  

$$p(2) = 2p(1) = 2^{2} \times 1000$$
  

$$p(3) = 2p(2) = 2^{3} \times 1000$$

It seems, from this pattern, that, in general,

$$p(t) = 2^{t} \times 1000 = (1000)2^{t}$$

This population function is a constant multiple of the exponential function  $y = 2^t$ , so it exhibits the rapid growth that we observed in Figures 2 and 7. Under ideal conditions (unlimited space and nutrition and freedom from disease) this exponential growth is typical of what actually occurs in nature. In Chapter 3 we will be able to compute rates of growth for such populations. In Chapter 7 we will modify this model to determine what happens when there are constraints to growth.

**EXAMPLE 3** The *half-life* of strontium-90, <sup>90</sup>Sr, is 25 years. This means that half of any given quantity of <sup>90</sup>Sr will disintegrate in 25 years.

(a) If a sample of <sup>90</sup>Sr has a mass of 24 mg, find an expression for the mass m(t) that remains after t years.

(b) Find the mass remaining after 40 years, correct to the nearest milligram.

(c) Use a graphing device to graph m(t) and use the graph to estimate the time required for the mass to be reduced to 5 mg.

#### SOLUTION

(a) The mass is initially 24 mg and is halved during each 25-year period, so

0.4

(0)

$$m(0) = 24$$

$$m(25) = \frac{1}{2}(24)$$

$$m(50) = \frac{1}{2} \cdot \frac{1}{2}(24) = \frac{1}{2^2}(24)$$

$$m(75) = \frac{1}{2} \cdot \frac{1}{2^2}(24) = \frac{1}{2^3}(24)$$

$$m(100) = \frac{1}{2} \cdot \frac{1}{2^3}(24) = \frac{1}{2^4}(24)$$

From this pattern, it appears that the mass remaining after t years is

$$m(t) = \frac{1}{2^{t/25}}(24) = 24 \cdot 2^{-t/25}$$

This is an exponential function with base  $a = 2^{-1/25} = 1/2^{1/25}$ .

(b) The mass that remains after 40 years is

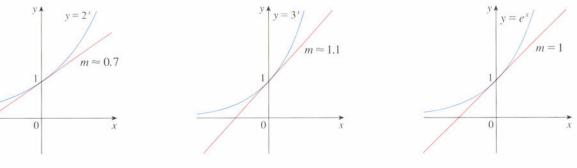
$$m(40) = 24 \cdot 2^{-40/25} \approx 7.9 \text{ mg}$$

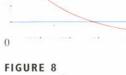
(c) We use a graphing calculator or computer to graph the function  $m(t) = 24 \cdot 2^{-t/25}$  in Figure 8. We also graph the line m = 5 and use the cursor to estimate that m(t) = 5 when  $t \approx 57$ . So the mass of the sample will be reduced to 5 mg after about 57 years.

## The Number e

100

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base *a* is influenced by the way the graph of  $y = a^x$  crosses the y-axis. Figures 9 and 10 show the tangent lines to the graphs of  $y = 2^x$  and  $y = 3^x$  at the point (0, 1). (Tangent lines will be defined precisely in Section 2.6. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines, we find that  $m \approx 0.7$  for  $y = 2^x$  and  $m \approx 1.1$  for  $y = 3^x$ .





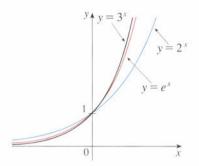


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FIGURE 9

FIGURE 10

FIGURE II



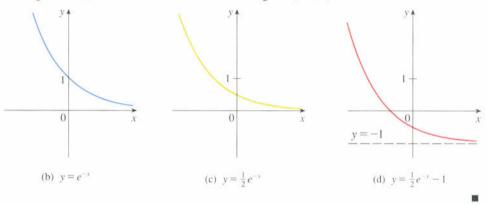
It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base *a* so that the slope of the tangent line to  $y = a^x$  at (0, 1) is *exactly* 1 (see Figure 11). In fact, there is such a number and it is denoted by the letter *e*. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.) In view of Figures 9 and 10, it comes as no surprise that the number *e* lies between 2 and 3 and the graph of  $y = e^x$  lies between the graphs of  $y = 2^x$  and  $y = 3^x$  (see Figure 12). In Chapter 3 we will see that the value of *e*, correct to five decimal places, is

$$e \approx 2.71828$$

FIGURE 12

**EXAMPLE 4** Graph the function  $y = \frac{1}{2}e^{-x} - 1$  and state the domain and range.

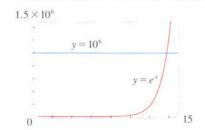
**SOLUTION** We start with the graph of  $y = e^x$  from Figures 11 and 13(a) and reflect about the y-axis to get the graph of  $y = e^{-x}$  in Figure 13(b). (Notice that the graph crosses the y-axis with a slope of -1). Then we compress the graph vertically by a factor of 2 to obtain the graph of  $y = \frac{1}{2}e^{-x}$  in Figure 13(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 13(d). The domain is  $\mathbb{R}$  and the range is  $(-1, \infty)$ .



How far to the right do you think we would have to go for the height of the graph of  $y = e^x$  to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

**EXAMPLE 5** Use a graphing device to find the values of x for which  $e^x > 1,000,000$ .

**SOLUTION** In Figure 14 we graph both the function  $y = e^x$  and the horizontal line y = 1,000,000. We see that these curves intersect when  $x \approx 13.8$ . Thus,  $e^x > 10^6$  when x > 13.8. It is perhaps surprising that the values of the exponential function have already surpassed a million when x is only 14.



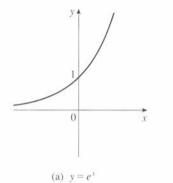




FIGURE 14

×.

Exercises • •

1.5

- 1. (a) Write an equation that defines the exponential function with base a > 0.
  - (b) What is the domain of this function?
  - (c) If  $a \neq 1$ , what is the range of this function?
  - (d) Sketch the general shape of the graph of the exponential function for each of the following cases.
    - (i) a > 1
    - (ii) a = 1
    - (iii) 0 < a < 1
- 2. (a) How is the number e defined?
  - (b) What is an approximate value for e?
  - (c) What is the natural exponential function?

**3–6** Graph the given functions on a common screen. How are these graphs related?

**3.** 
$$y = 2^{x}$$
,  $y = e^{x}$ ,  $y = 5^{x}$ ,  $y = 20^{x}$   
**4.**  $y = e^{x}$ ,  $y = e^{-x}$ ,  $y = 8^{x}$ ,  $y = 8^{-x}$   
**5.**  $y = 3^{x}$ ,  $y = 10^{x}$ ,  $y = (\frac{1}{3})^{x}$ ,  $y = (\frac{1}{10})^{x}$   
**6.**  $y = 0.9^{x}$ ,  $y = 0.6^{x}$ ,  $y = 0.3^{x}$ ,  $y = 0.1^{x}$ 

**7–14** Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 3 and 12 and, if necessary, the transformations of Section 1.2.

- **7.**  $y = 2^x + 1$  **8.**  $y = 2^{x+1}$
- **9.**  $y = 3^{-x}$  **10.**  $y = -3^{x}$
- **11.**  $y = -3^{-x}$  **12.**  $y = 2^{|x|}$

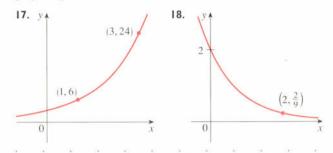
**13.**  $y = 3 - e^{x}$ 

**14.**  $y = 2 + 5(1 - e^{-x})$ 

- **15.** Starting with the graph of  $y = e^x$ , write the equation of the graph that results from
  - (a) shifting 2 units downward
  - (b) shifting 2 units to the right
  - (c) reflecting about the x-axis
  - (d) reflecting about the y-axis
  - (e) reflecting about the *x*-axis and then about the *y*-axis
- 16. Starting with the graph of  $y = e^x$ , find the equation of the graph that results from
  - (a) reflecting about the line y = 4
  - (b) reflecting about the line x = 2

**17–18** Find the exponential function  $f(x) = Ca^x$  whose graph is given.

. . . . . . . . . . . . . . . . . .



- 19. Show that if the graphs of  $f(x) = x^2$  and  $g(x) = 2^x$  are drawn on a coordinate grid where the unit of measurement is 1 inch, then at a distance 2 ft to the right of the origin the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.
- Compare the functions f(x) = x<sup>5</sup> and g(x) = 5<sup>x</sup> by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?
- Compare the functions f(x) = x<sup>10</sup> and g(x) = e<sup>x</sup> by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?
- **22.** Use a graph to estimate the values of x such that  $e^x > 1,000,000,000$ .
  - **23.** Under ideal conditions a certain bacteria population is known to double every three hours. Suppose that there are initially 100 bacteria.
    - (a) What is the size of the population after 15 hours?
    - (b) What is the size of the population after *t* hours?
    - (c) Estimate the size of the population after 20 hours.
    - (d) Graph the population function and estimate the time for the population to reach 50,000.
  - **24.** An isotope of sodium, <sup>24</sup>Na, has a half-life of 15 hours. A sample of this isotope has mass 2 g.
    - (a) Find the amount remaining after 60 hours.
    - (b) Find the amount remaining after t hours.
    - (c) Estimate the amount remaining after 4 days.
    - (d) Use a graph to estimate the time required for the mass to be reduced to 0.01 g.

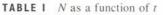
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## **Inverse Functions and Logarithms**

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: N = f(t).

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the *inverse* function of f, denoted by  $f^{-1}$ , and read "f inverse." Thus,  $t = f^{-1}(N)$  is the time required for the population level to reach N. The values of  $f^{-1}$  can be found by reading Table 1 backward or by consulting Table 2. For instance,  $f^{-1}(550) = 6$ because f(6) = 550.



1

2

3

4

5

6

7

8

100

168

259

358

445

509

550

573

586

**TABLE 2** t as a function of N N = f(t) $t = f^{-1}(N)$ = population at time t N = time to reach N bacteria 100 0 168 1 259 2 358 3 445 4 509 5 550 6 573 7 586 8

Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1.

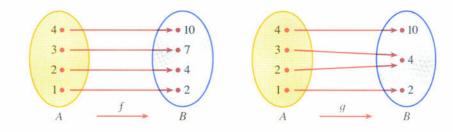


FIGURE I

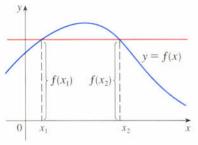
but

Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

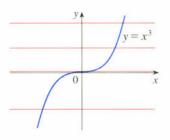
$$g(2) = g(3)$$

 $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$ 

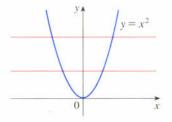
Functions that have this latter property are called *one-to-one functions*.



**FIGURE 2** This function is not one-to-one because  $f(x_1) = f(x_2)$ .



**FIGURE 3**  $f(x) = x^3$  is one-to-one.



**FIGURE 4**  $g(x) = x^2$  is not one-to-one.

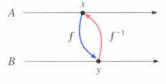


FIGURE 5

**Definition** A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

 $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$ 

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers  $x_1$  and  $x_2$  such that  $f(x_1) = f(x_2)$ . This means that f is not one-to-one. Therefore, we have the following geometric method for determining whether a function is one-to-one.

**Horizontal Line Test** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

**EXAMPLE I** Is the function  $f(x) = x^3$  one-to-one?

**SOLUTION 1** If  $x_1 \neq x_2$ , then  $x_1^3 \neq x_2^3$  (two different numbers cannot have the same cube). Therefore, by Definition 1,  $f(x) = x^3$  is one-to-one.

**SOLUTION 2** From Figure 3 we see that no horizontal line intersects the graph of  $f(x) = x^3$  more than once. Therefore, by the Horizontal Line Test, *f* is one-to-one.

**EXAMPLE 2** Is the function  $g(x) = x^2$  one-to-one?

**SOLUTION I** This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same image.

**SOLUTION 2** From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

**2** Definition Let f be a one-to-one function with domain A and range B. Then its inverse function  $f^{-1}$  has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then  $f^{-1}$  maps y back into x. (If f were not one-to-one, then  $f^{-1}$  would not be uniquely defined.) The arrow diagram in Figure 5 indicates that  $f^{-1}$  reverses the effect of f. Note that

domain of  $f^{-1}$  = range of frange of  $f^{-1}$  = domain of f

For example, the inverse function of  $f(x) = x^3$  is  $f^{-1}(x) = x^{1/3}$  because if  $y = x^3$ , then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

**Caution** • Do not mistake the -1 in  $f^{-1}$  for an exponent. Thus  $\oslash$ 

$$f^{-1}(x)$$
 does *not* mean  $\frac{1}{f(x)}$ 

The reciprocal 1/f(x) could, however, be written as  $[f(x)]^{-1}$ .

**EXAMPLE 3** If f(1) = 5, f(3) = 7, and f(8) = -10, find  $f^{-1}(7)$ ,  $f^{-1}(5)$ , and  $f^{-1}(-10).$ 

**SOLUTION** From the definition of  $f^{-1}$  we have

$$f^{-1}(7) = 3$$
 because  $f(3) = 7$   
 $f^{-1}(5) = 1$  because  $f(1) = 5$   
 $f^{-1}(-10) = 8$  because  $f(8) = -10$ 

The diagram in Figure 6 makes it clear how  $f^{-1}$  reverses the effect of f in this case.

The letter x is traditionally used as the independent variable, so when we concentrate on  $f^{-1}$  rather than on f, we usually reverse the roles of x and y in Definition 2 and write

3

4

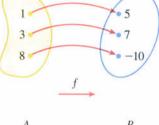
$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following cancellation equations:

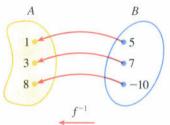
$$f^{-1}(f(x)) = x$$
 for every x in A  
 $f(f^{-1}(x)) = x$  for every x in B

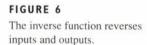
The first cancellation equation says that if we start with x, apply f, and then apply  $f^{-1}$ , we arrive back at x, where we started (see the machine diagram in Figure 7). Thus,  $f^{-1}$  undoes what f does. The second equation says that f undoes what  $f^{-1}$ does.

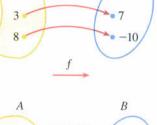
**FIGURE 7** 



A







В

$$x \longrightarrow f \longrightarrow f(x) \longrightarrow f^{-1} \longrightarrow x$$

For example, if  $f(x) = x^3$ , then  $f^{-1}(x) = x^{1/3}$  and the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$
$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function y = f(x) and are able to solve this equation for x in terms of y, then according to Definition 2 we must have  $x = f^{-1}(y)$ . If we want to call the independent variable x, we then interchange x and y and arrive at the equation  $y = f^{-1}(x)$ .

**Step I** Write y = f(x).

**Step 2** Solve this equation for x in terms of y (if possible).

**Step 3** To express  $f^{-1}$  as a function of x, interchange x and y.

The resulting equation is  $y = f^{-1}(x)$ .

**EXAMPLE 4** Find the inverse function of  $f(x) = x^3 + 2$ .

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for *x*:

$$x^{3} = y - 2$$
$$x = \sqrt[3]{y - 2}$$

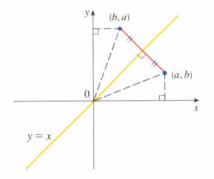
Finally, we interchange x and y:

In Example 4, notice how 
$$f'$$
 reverses  
the effect of  $f$ . The function  $f$  is the  
rule "Cube, then add 2";  $f^{-1}$  is the rule  
"subtract 2, then take the cube root."

 $y = \sqrt[3]{x - 2}$ 

Therefore, the inverse function is  $f^{-1}(x) = \sqrt[3]{x-2}$ .

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of  $f^{-1}$  from the graph of f. Since f(a) = b if and only if  $f^{-1}(b) = a$ , the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of  $f^{-1}$ . But we get the point (b, a) from (a, b) by reflecting about the line y = x (see Figure 8).



Therefore, as illustrated by Figure 9:

The graph of  $f^{-1}$  is obtained by reflecting the graph of *f* about the line y = x.

**EXAMPLE 5** Sketch the graphs of  $f(x) = \sqrt{-1 - x}$  and its inverse function using the same coordinate axes.

**SOLUTION** First we sketch the curve  $y = \sqrt{-1 - x}$  (the top half of the parabola  $y^2 = -1 - x$ , or  $x = -y^2 - 1$ ) and then we reflect about the line y = x to get the graph of  $f^{-1}$  (see Figure 10). As a check on our graph, notice that the expression for  $f^{-1}$  is  $f^{-1}(x) = -x^2 - 1$ ,  $x \ge 0$ . So the graph of  $f^{-1}$  is the right half of the parabola  $y = -x^2 - 1$  and this seems reasonable from Figure 10.

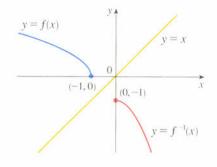


FIGURE IO

Many graphing devices won't plot the inverse of a given function directly, but we can obtain the desired graph by using the parametric graphing capability of such a device. We saw in Section 1.4 that the curve with equation y = f(x) can be written as a parametric curve with parametric equations

$$x = t$$
  $y = f(t)$ 

We know that the graph of the inverse function is obtained by interchanging the x- and y-coordinates of the points on the graph of f. Therefore, parametric equations for the graph of  $f^{-1}$  are

$$x = f(t)$$
  $y = t$ 

**EXAMPLE 6** Show that the function  $f(x) = \sqrt{x^3 + x^2 + x + 1}$  is one-to-one and graph both f and  $f^{-1}$ .

**SOLUTION** We plot the graph in Figure 11 and observe that f is one-to-one by the Horizontal Line Test.

To graph f and  $f^{-1}$  on the same screen we use parametric graphs. Parametric equations for the graph of f are

$$x = t$$
  $y = \sqrt{t^3 + t^2 + t + 1}$ 

and parametric equations for the graph of  $f^{-1}$  are

$$x = \sqrt{t^3 + t^2 + t + 1}$$
  $y = t$ 

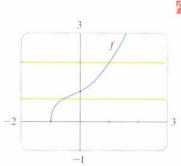


FIGURE II



y = x

f

y A

0

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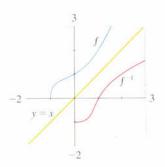


FIGURE 12

Let's also plot the line y = x:

$$x = t$$
  $y = t$ 

Figure 12 shows all three graphs and, indeed, it appears that the graph of  $f^{-1}$  is the reflection of the graph of f in the line y = x.

It's remarkable that we were able to graph  $f^{-1}$  in Example 6 even though we had no explicit formula for  $f^{-1}$ . In fact, it is possible to find an explicit, though very complicated, expression for  $f^{-1}(x)$ . (We ask you to do this in Exercise 55 with the help of a computer algebra system.) It turns out to be impossible to find an explicit formula for the inverse function of  $f(x) = x + \sin x$ , but we can still use the method of Example 6 to graph  $f^{-1}$ . (See Exercise 54.)

### Logarithmic Functions

If a > 0 and  $a \neq 1$ , the exponential function  $f(x) = a^x$  is either increasing or decreasing and so it is one-to-one. It therefore has an inverse function  $f^{-1}$ , which is called the **logarithmic function with base** a and is denoted by  $\log_a$ . If we use the formulation of an inverse function given by (3),

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

$$\log_a x = y \iff a^y = x$$

Thus, if x > 0, then  $\log_a x$  is the exponent to which the base *a* must be raised to give *x*. For example,  $\log_{10} 0.001 = -3$  because  $10^{-3} = 0.001$ .

The cancellation equations (4), when applied to  $f(x) = a^x$  and  $f^{-1}(x) = \log_a x$ , become

7

 $\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$  $a^{\log_a x} = x \quad \text{for every } x > 0$ 

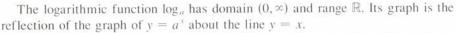
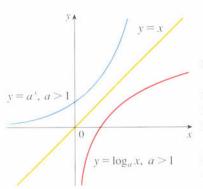


Figure 13 shows the case where a > 1. (The most important logarithmic functions have base a > 1.) The fact that  $y = a^x$  is a very rapidly increasing function for x > 0 is reflected in the fact that  $y = \log_a x$  is a very slowly increasing function for x > 1.

Figure 14 shows the graphs of  $y = \log_a x$  with various values of the base *a*. Since  $\log_a 1 = 0$ , the graphs of all logarithmic functions pass through the point (1,0).

The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.5.





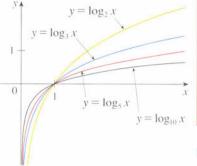


FIGURE 14

Laws of Logarithms If x and y are positive numbers, then 1.  $\log_a(xy) = \log_a x + \log_a y$ 2.  $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ 3.  $\log_a(x^r) = r \log_a x$  (where r is any real number)

**EXAMPLE 7** Is Use the laws of logarithms to evaluate  $\log_2 80 - \log_2 5$ .

**SOLUTION** Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2 \left(\frac{80}{5}\right) = \log_2 16 = 4$$

because  $2^4 = 16$ .

## Natural Logarithms

Of all possible bases a for logarithms, we will see in Chapter 3 that the most convenient choice of a base is the number e, which was defined in Section 1.5. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

If we put a = e and  $\log_e = \ln in$  (6) and (7), then the defining properties of the natural logarithm function become

8

 $\ln x = y \iff e^y = x$ 

9

 $\ln(e^x) = x \qquad x \in \mathbb{R}$  $e^{\ln x} = x \qquad x > 0$ 

In particular, if we set x = 1, we get

$$\ln e = 1$$

**EXAMPLE 8** Find x if  $\ln x = 5$ .

SOLUTION | From (8) we see that

 $\ln x = 5$  means  $e^5 = x$ 

Therefore,  $x = e^5$ .

(If you have trouble working with the "ln" notation, just replace it by  $\log_e$ . Then the equation becomes  $\log_e x = 5$ ; so, by the definition of logarithm,  $e^5 = x$ .)

## **Notation for Logarithms**

Most textbooks in calculus and the sciences, as well as calculators, use the notation  $\ln x$  for the natural logarithm and  $\log x$  for the "common logarithm,"  $\log_{10} x$ . In the more advanced mathematical and scientific literature and in computer languages, however, the notation  $\log x$  usually denotes the natural logarithm.

**SOLUTION 2** Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (9) says that  $e^{\ln x} = x$ . Therefore,  $x = e^5$ .

**EXAMPLE 9** Solve the equation  $e^{5-3x} = 10$ .

SOLUTION We take natural logarithms of both sides of the equation and use (9):

$$n(e^{3-3x}) = \ln 10$$
  

$$5 - 3x = \ln 10$$
  

$$3x = 5 - \ln 10$$
  

$$x = \frac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution to four decimal places:  $x \approx 0.8991$ .

**EXAMPLE 10** Express  $\ln a + \frac{1}{2} \ln b$  as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\ln a + \frac{1}{2} \ln b = \ln a + \ln b^{1/2}$$
$$= \ln a + \ln \sqrt{b}$$
$$= \ln(a\sqrt{b})$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

10 For any positive number  $a \ (a \neq 1)$ , we have

$$\log_a x = \frac{\ln x}{\ln a}$$

**Proof** Let  $y = \log_a x$ . Then, from (6), we have  $a^y = x$ . Taking natural logarithms of both sides of this equation, we get  $y \ln a = \ln x$ . Therefore

$$y = \frac{\ln x}{\ln a}$$

Scientific calculators have a key for natural logarithms, so Formula 10 enables us to use a calculator to compute a logarithm with any base (as shown in the next example). Similarly, Formula 10 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 43 and 44).

**EXAMPLE II** Evaluate log<sub>8</sub>5 correct to six decimal places.

**SOLUTION** Formula 10 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$

**EXAMPLE 12** In Example 3 in Section 1.5 we showed that the mass of <sup>90</sup>Sr that remains from a 24-mg sample after t years is  $m = f(t) = 24 \cdot 2^{-t/25}$ . Find the inverse of this function and interpret it.

**SOLUTION** We need to solve the equation  $m = 24 \cdot 2^{-t/25}$  for t. We start by taking natural logarithms of both sides:

$$\ln m = \ln(24 \cdot 2^{-t/25}) = \ln 24 + \ln(2^{-t/25})$$
$$\ln m = \ln 24 - \frac{t}{25} \ln 2$$
$$\frac{t}{25} \ln 2 = \ln 24 - \ln m$$
$$t = \frac{25}{\ln 2} (\ln 24 - \ln m)$$

So the inverse function is

$$f^{-1}(m) = \frac{25}{\ln 2} (\ln 24 - \ln m)$$

This function gives the time required for the mass to decay to m milligrams. In particular, the time required for the mass to be reduced to 5 mg is

$$t = f^{-1}(5) = \frac{25}{\ln 2} (\ln 24 - \ln 5) \approx 56.58$$
 years

This answer agrees with the graphical estimate that we made in Example 3 in Section 1.5.

The graphs of the exponential function  $y = e^x$  and its inverse function, the natural logarithm function, are shown in Figure 15. Because the curve  $y = e^x$  crosses the *y*-axis with a slope of 1, it follows that the curve  $\ln x$  crosses the *x*-axis with a slope of 1.

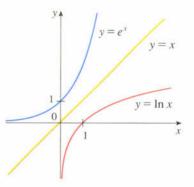


FIGURE 15

In common with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on  $(0, \infty)$  and the y-axis is a vertical asymptote. (This means that the values of  $\ln x$  become very large negative as x approaches 0. See Section 2.5.)

### **EXAMPLE 13** Sketch the graph of the function $y = \ln(x - 2) - 1$ .

**SOLUTION** We start with the graph of  $y = \ln x$  as given in Figure 15. Using the transformations of Section 1.2, we shift it two units to the right to get the graph of  $y = \ln(x - 2)$  and then we shift it one unit downward to get the graph of  $y = \ln(x - 2) - 1$  (see Figure 16).

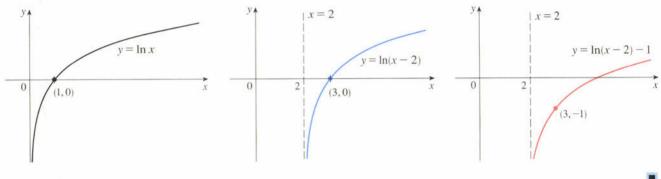


FIGURE 16

Although  $\ln x$  is an increasing function, it grows *very* slowly when x > 1. In fact,  $\ln x$  grows more slowly than any positive power of x. To illustrate this fact, we compare approximate values of the functions  $y = \ln x$  and  $y = x^{1/2} = \sqrt{x}$  in the following table and we graph them in Figures 17 and 18. You can see that initially the graphs of  $y = \sqrt{x}$  and  $y = \ln x$  grow at comparable rates, but eventually the root function far surpasses the logarithm.

X	1	2	5	10	50	100	500	1000	10,000	100,000
ln.v	0	0.69	1.61	2.30	3.91	4.6	6.2	6,9	9.2	11.5
XX.	I.	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\frac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

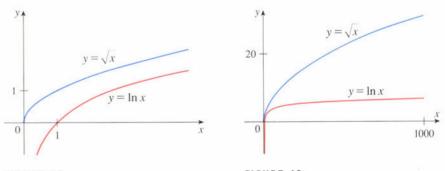


FIGURE 17

FIGURE 18

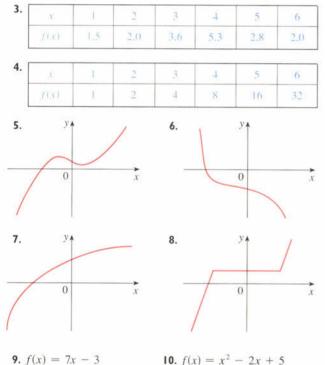


I. (a) What is a one-to-one function?

Exercises · · · · · · ·

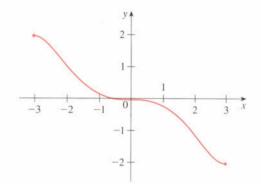
- (b) How can you tell from the graph of a function whether it is one-to-one?
- **2.** (a) Suppose f is a one-to-one function with domain Aand range *B*. How is the inverse function  $f^{-1}$ defined? What is the domain of  $f^{-1}$ ? What is the range of  $f^{-1}$ ?
  - (b) If you are given a formula for f, how do you find a formula for  $f^{-1}$ ?
  - (c) If you are given the graph of f, how do you find the graph of  $f^{-1}$ ?

3-14 = A function f is given by a table of values, a graph, a formula, or a verbal description. Determine whether f is one-to-one.



- **9.** f(x) = 7x 3**12.**  $q(x) = \sqrt{x}$
- 11. q(x) = |x|
- **13.** f(t) is the height of a football t seconds after kickoff
- 14. f(t) is your height at age t
- 15-16 Use a graph to decide whether f is one-to-one.
- 15.  $f(x) = x^3 x$ **16.**  $f(x) = x^3 + x$
- 17. If f is a one-to-one function such that f(2) = 9, what is  $f^{-1}(9)$ ?

- **18.** If  $f(x) = 3 + x^2 + \tan(\pi x/2)$ , where -1 < x < 1. find  $f^{-1}(3)$ .
- **19.** If  $q(x) = 3 + x + e^x$ , find  $q^{-1}(4)$ .
- **20.** The graph of f is given.
  - (a) Why is f one-to-one?
  - (b) State the domain and range of  $f^{-1}$ .
  - (c) Estimate the value of  $f^{-1}(1)$ .



- **21.** The formula  $C = \frac{5}{9}(F 32)$ , where  $F \ge -459.67$ , expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?
- 22. In the theory of relativity, the mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

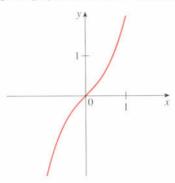
where  $m_0$  is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

**23–28** ■ Find a formula for the inverse of the function.

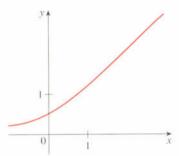
- **23.**  $f(x) = \frac{1+3x}{5-2x}$ **24.**  $f(x) = 5 - 4x^3$ **26.**  $y = 2^{10^x}$ **25.**  $f(x) = \sqrt{2 + 5x}$ **28.**  $y = \frac{1 + e^x}{1 - e^x}$ **27.**  $y = \ln(x + 3)$
- **29–30** Find an explicit formula for  $f^{-1}$  and use it to graph  $f^{-1}$ , f, and the line y = x on the same screen. To check your work, see whether the graphs of f and  $f^{-1}$  are reflections about the line.

**29.** 
$$f(x) = 1 - 2/x^2$$
,  $x > 0$   
**30.**  $f(x) = \sqrt{x^2 + 2x}$ ,  $x > 0$ 

**31.** Use the given graph of f to sketch the graph of  $f^{-1}$ .



**32.** Use the given graph of f to sketch the graphs of  $f^{-1}$  and 1/f.



- **33.** (a) How is the logarithmic function  $y = \log_a x$  defined?
  - (b) What is the domain of this function?
  - (c) What is the range of this function?
  - (d) Sketch the general shape of the graph of the function  $y = \log_a x$  if a > 1.
- 34. (a) What is the natural logarithm?
  - (b) What is the common logarithm?
  - (c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.
- **35–38** Find the exact value of each expression.

<b>35.</b> (a) $\log_2 64$ (b)	5)	$\log_{6} \frac{1}{36}$
--------------------------------	----	-------------------------

- **36.** (a)  $\log_8 2$  (b)  $\ln e^{\sqrt{2}}$
- **37.** (a)  $\log_{10} 1.25 + \log_{10} 80$ (b)  $\log_5 10 + \log_5 20 - 3\log_5 2$
- **38.** (a)  $2^{(\log_2 3 + \log_2 5)}$  (b)  $e^{3 \ln 2}$
- **39–40** Express the given quantity as a single logarithm.

**39.**  $2 \ln 4 - \ln 2$  **40.**  $\ln x + a \ln y - b \ln z$ 

- 41. Use Formula 10 to evaluate each logarithm correct to six decimal places.
  (a) log<sub>2</sub> 5 (b) log<sub>5</sub> 26.05
- **42.** Find the domain and range of the function  $g(x) = \ln(4 x^2)$ .

¥3-44 ■ Use Formula 10 to graph the given functions on a common screen. How are these graphs related?

**43.**  $y = \log_{1.5} x$ ,  $y = \ln x$ ,  $y = \log_{10} x$ ,  $y = \log_{50} x$ 

**44.**  $y = \ln x$ ,  $y = \log_{10} x$ ,  $y = e^x$ ,  $y = 10^x$ 

- **45.** Suppose that the graph of  $y = \log_2 x$  is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?
- 46. Compare the functions f(x) = x<sup>0.1</sup> and g(x) = ln x by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?

**47–48** ■ Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 14 and 15 and, if necessary, the transformations of Section 1.2.

- **47.** (a)  $y = \log_{10}(x + 5)$  (b)  $y = -\ln x$  **48.** (a)  $y = \ln(-x)$  (b)  $y = \ln |x|$ **49-52** Solve each equation for *x*.
- **49.** (a)  $e^x = 16$ (b)  $\ln x = -1$ **50.** (a)  $\ln(2x 1) = 3$ (b)  $e^{3x-4} = 2$ **51.** (a)  $2^{x-5} = 3$ (b)  $\ln x + \ln(x 1) = 1$ **52.** (a)  $\ln(\ln x) = 1$ (b)  $e^{ax} = Ce^{bx}$ , where  $a \neq b$
- **52.** (a)  $\ln(\ln x) = 1$  (b)  $e^{ax} = Ce^{bx}$ , where  $a \neq b$
- **53–54** Show that f is one-to-one. Then graph f,  $f^{-1}$ , and y = x on the same screen using parametric graphs.

**53.** 
$$f(x) = \sqrt{x^2 + 1} - x$$

**54.**  $f(x) = x + \sin x$ 

- **CAS 55.** Use a computer algebra system to find an explicit expression for the inverse of the function  $f(x) = \sqrt{x^3 + x^2 + x + 1}$  that we considered in Example 6. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)
- **CAS 56.** (a) If  $g(x) = x^6 + x^4$ ,  $x \ge 0$ , use a computer algebra system to find an expression for  $g^{-1}(x)$ .
  - (b) Use the expression in part (a) to graph y = g(x), y = x, and  $y = g^{-1}(x)$  on the same screen.
  - **57.** If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after *t* hours is  $n = f(t) = 100 \cdot 2^{t/3}$  (see Exercise 23 in Section 1.5).
    - (a) Find the inverse of this function and explain its meaning.
    - (b) When will the population reach 50,000?

58. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is  $Q_0$  and t is measured in seconds.)

- (a) Find the inverse of this function and explain its meaning.
- (b) How long does it take to recharge the capacitor to 90% of capacity if a = 2?
- **59.** Starting with the graph of  $y = \ln x$ , find the equation of the graph that results from
  - (a) shifting 3 units upward
  - (b) shifting 3 units to the left

- (c) reflecting about the x-axis
- (d) reflecting about the y-axis
- (e) reflecting about the line y = x
- (f) reflecting about the *x*-axis and then about the line y = x
- (g) reflecting about the y-axis and then about the line y = x
- (h) shifting 3 units to the left and then reflecting about the line y = x
- **60.** (a) If we shift a curve to the left, what happens to its reflection about the line y = x? In view of this geometric principle, find an expression for the inverse of g(x) = f(x + c), where f is a one-to-one function.
  - (b) Find an expression for the inverse of h(x) = f(cx), where  $c \neq 0$ .



# Models and Curve Fitting

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

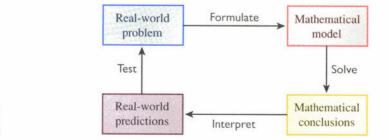


FIGURE I The modeling process

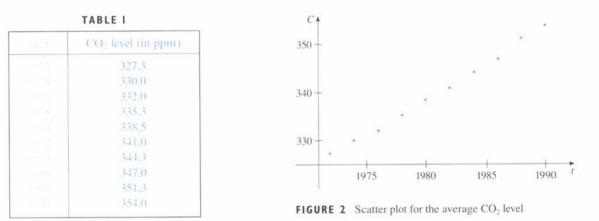
The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

### Empirical Models

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1972 to 1990. We use these data to make the scatter plot in Figure 2, where *t* represents time (in years) and *C* represents the  $CO_2$  level (in parts per million, ppm).



Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One possibility is to choose the line that passes through the first and last data points. The slope of this line is

$$\frac{354.0 - 327.3}{1990 - 1972} = \frac{26.7}{18} \approx 1.48333$$

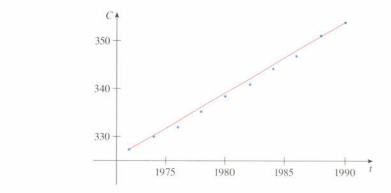
and its equation is

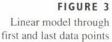
$$C - 327.3 = 1.48333(t - 1972)$$

or

$$C = 1.48333t - 2597.83$$

Equation 1 gives one possible linear model for the carbon dioxide level (see Figure 3) but there are other possibilities.





In general, suppose that we have data points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,... and we want to find a line y = mx + b that fits these points as well as possible, in some sense. Let

$$d_i = |y_i - (mx_i + b)|$$

be the vertical distance from the *i*th point to the line (see Figure 4).

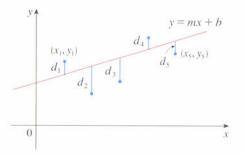


FIGURE 4

One method for choosing a line of best fit is to minimize the largest of the numbers  $d_1, d_2, d_3, \ldots$  so that the resulting line is not too far away from any given data point. Another method that is sometimes used is to minimize the sum of these distances, so that  $d_1 + d_2 + \cdots$  is as small as possible. The most popular curvefitting criterion, however, is the **least squares** method, in which we minimize the sum of the squares of these distances:  $d_1^2 + d_2^2 + \cdots$ . (Statisticians arrive at this criterion by assuming that possible errors in the data are randomly distributed.) With the help of ideas from multivariable calculus, it is possible to arrive at (complicated) formulas for the slope *m* and *y*-intercept *b* of the resulting line, which is called a **regression line**. Fortunately, most graphing calculators and computer algebra systems have statistics packages that calculate the values of *m* and *b* and plot the regression line together with the data points.

**EXAMPLE 1** Find a linear model for the carbon dioxide level by using the least squares regression line for the data of Table 1. Use the model to estimate the average  $CO_2$  level for 1987 and to predict the level for the year 2000. According to this model, when will the  $CO_2$  level exceed 400 parts per million?

**SOLUTION** If we use a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. (With Maple we use

the fit[leastsquare] command in the stats package; with Mathematica we use the Fit command.) The machine gives the slope and *y*-intercept of the regression line as

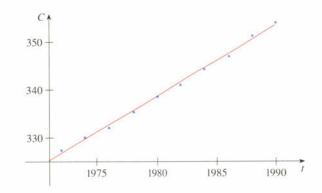
$$m = 1.496667$$
  $b = -2624.826667$ 

So our least squares model for the CO2 level is

2

C = 1.496667t - 2624.826667

In Figure 5 we graph the regression line as well as the data points. Comparing with Figure 3, we see that it gives a better fit than our previous linear model.





Using Equation 2 with t = 1987, we estimate that the average CO<sub>2</sub> level in 1987 was

$$C(1987) = (1.496667)(1987) - 2624.826667 \approx 349.05$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average  $CO_2$  level in 1987 was 348.8 ppm, so our estimate is quite accurate.)

With t = 2000, we get

$$C(2000) = (1.496667)(2000) - 2624.826667 \approx 368.51$$

So we predict that the average  $CO_2$  level in the year 2000 will be 368.5 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the region of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the  $CO_2$  level exceeds 400 ppm when

$$1.496667t - 2624.826667 > 400$$

Solving this inequality, we get

$$t > \frac{3024.826667}{1.496667} \approx 2021.04$$

We therefore predict that the CO<sub>2</sub> level will exceed 400 ppm by the year 2021. This prediction is somewhat risky because it involves a time quite remote from our observations.

Year	Population (millions)
1900	1650
910	1750
920	1860
1930	2070
1940	2300
1950	2520
1960	3020
1970	3700
1980	4450
1990	5300
1996	5770

**EXAMPLE 2** Find an appropriate model for the population of the world in the 20th century using the data in Table 2.

**SOLUTION** A scatter plot of the data is shown in Figure 6.

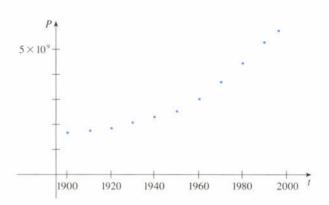


FIGURE 6 Scatter plot for world population growth

We see that the population is growing too quickly for a linear model to be appropriate, but the general trend is similar to the exponential functions that we studied in Section 1.5. So let's look for a member of the family of functions

$$y = Ce^{kt}$$

to serve as a model for population growth. Taking the natural logarithm of each side of the equation, we get

$$\ln y = \ln(Ce^{kt}) = \ln C + \ln e^{kt}$$
$$\ln y = \ln C + kt$$

This equation shows that if y is an exponential function of t, then  $\ln y$  is a linear function of t.

To see if an exponential model is appropriate, we calculate  $\ln P$  for each data point in Table 3 and plot  $\ln P$  against *t*. The resulting scatter plot is shown in Figure 7.

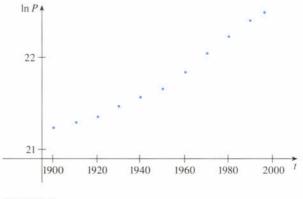


TABLE 3					
1	In P				
1900	21.224				
1910	21.283				
1920	21.344				
1930	21.451				
1940	21.556				
1950	21.648				
1960	21.829				
1970	22.032				
1980	22.216				

22.476

1990 1996

TADLES



Although the data points are not as clearly linear as in Example 1, a linear model for  $\ln P$  does seem to be a possibility. So we use a graphing calculator or computer algebra system to obtain the least squares regression line. The slope and y-intercept are

m = 0.013623 b = -4.7911

and so our model can be written in the form

 $\ln P = 0.013623t - 4.7911$ 

Figure 8 shows the regression line.

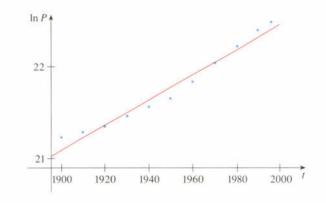


FIGURE 8

Solving Equation 3 for P, we get

$$P = e^{0.013623t} \cdot e^{-4.7911}$$

or

4

3

 $P = 0.008303e^{0.013623t}$ 

Equation 4 is the exponential model obtained by taking logarithms, applying linear regression to the resulting data, and then solving for P by applying the exponential function. This is the procedure to be used with most computer algebra systems. Most graphing calculators, on the other hand, are capable of applying the method of least squares *directly* to an exponential equation. One such calculator applies exponential regression to the data of Table 2 and obtains the function

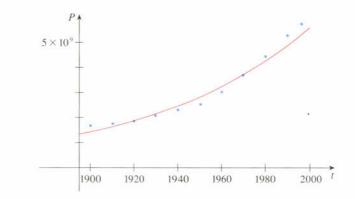
$$P = (0.008306312) \cdot (1.013716)'$$

For comparison with Equation 4 we convert to base *e* by writing  $1.013716 = e^k$  so that  $k = \ln 1.013716 \approx 0.013623$  and we have

## $P = 0.008306312e^{0.013623t}$

The slight discrepancy between Equations 4 and 5 is explained by the fact that applying least squares to the transformed data of Table 3 doesn't quite correspond to the least squares fit of the original data.

Figure 9 shows the exponential model for population growth, given by Equation 4 or 5, together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the depression of the 1930s.



**FIGURE 9** Exponential model for population growth

There are other possibilities for modeling the data of Table 2. Let's try, for instance, a cubic polynomial. A graphing calculator or computer algebra system uses the method of least squares and returns the cubic model

## 6

$$P = at^3 + bt^2 + ct + d$$

where

$$a = 2325.67 \qquad b = -1.306488 \times 10^{7}$$
  
$$c = 2.44631 \times 10^{10} \qquad d = -1.52658 \times 10^{13}$$

We graph this cubic function and the data points of Table 2 in Figure 10 and see that it models the world population of the 20th century very well. Perhaps surprisingly, it's substantially better than the exponential model and would be useful for estimating the world population in 1925 or 1985. For the purpose of predicting the population in the years 2050 and 2100, however, the cubic model would probably not be nearly so accurate. In Chapter 7 we will discuss possible models for population growth in greater detail.

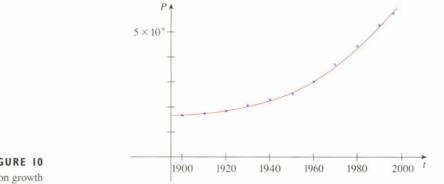


FIGURE 10 Cubic model for population growth

We have seen that graphing calculators and computers are capable of employing the method of least squares to model data by means of linear functions or exponential functions. Two other types of functions are also commonly found as models on such machines.

A power model is a function of the form  $y = ax^n$ , where a is a positive constant, n is any real number, and the domain is  $(0, \infty)$ . Notice that  $\ln y = \ln a + n \ln x$ , that is,  $\ln y$  is a linear function of  $\ln x$ . Thus, to determine if a power model is appropriate, we can make a scatter plot of  $\ln y$  against  $\ln x$  and see whether the points lie more or less along a straight line. If so, we could use linear regression to find a model for  $\ln y$  as a linear function of  $\ln x$  and then solve for y. Alternatively, most graphing calculators have a command for power regression that finds values for a and n directly.

Computer algebra systems can use least squares to fit a polynomial of any degree to a set of data. Indeed, we used a third-degree polynomial model in Example 2. Although polynomials of high degree could be used to mimic most fluctuations of data, that is usually not appropriate. It's best to use a simpler expression that encompasses the essential trend of the data. Graphing calculators are generally capable of curve-fitting with polynomials of degree 2, 3, and 4, that is, with quadratic, cubic, and quartic functions.



Exercises

## The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the 1989 National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9,4
\$60,000	8,2

- (a) Make a scatter plot of these data and decide whether a linear model is appropriate.
- (b) Find and graph a linear model using the first and last data points.
- (c) Find and graph the least squares regression line.
- (d) Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.
- (e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?
- (a) Use the data in Table 2, but only for the years 1950–1996, to model the population of the world in the second half of the 20th century. Use a linear model, an exponential model, and a cubic model.

From the graphs of these models and the data points, which appears to be the best model?

- (b) Use the models to estimate the world population in 1985 and to predict the world population for the years 2000 and 2100. Which of the models do you think gives the most reliable predictions?
- (c) Use the models to predict when the world population will reach 8 billion.
- **3.** The table gives lead emissions into the environment within the United States in millions of metric tons from 1970 to 1992.

Year	Emissions (in millions of metric tons)
1970	199,1
1975	143.8
1980	68.0
1985	18.3
1988	5.9
1989	5.5
1990	5.1
1991	4.5
1992	4.7

- (a) Fit both an exponential curve and a fourth-degree polynomial to these data. (Take t = 0 to correspond to 1970.)
- (b) Estimate the amount of lead emission in 1972 and in 1982.

**4.** A study by the U.S. Office of Science and Technology in 1972 estimated the cost (in 1972 dollars) to reduce automobile emissions by certain percentages:

Reduction in emissions (G)	Cost per car (in S)	Reduction in emissions (%)	Cost per car (in S)
50	45	75	9()
55	55	80	100
(51)	62	85	200
65	70	90	375
70	80	95	600

Find a model that captures the "diminishing returns" trend of these data.

 The table gives the percentage of high-school seniors who reported that they had used marijuana in the past 30 days.

Year	1980	1983	1985	1986	1987	1988
Percentage	33.7	27.0	25.7	23.4	21.0	18.0

Year	1989	1990	1991	1005	1993	1994
Percentage	16.7	14.0	13.8	11.9	15.5	19,0

Find a suitable model for these data and use your model to estimate the percentage for 1982 and to predict the percentage for 1995. How far into the future would you be willing to use your model to make predictions?

**6.** The table shows the mean (average) distances *d* of the planets from the Sun (taking the unit of measurement to be the distance from Earth to the Sun) and their periods *T* (time of revolution in years).

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19,190	84.008
Neptune	30.086	164.784
Pluto	39,507	248.350

- (a) Make a scatter plot of T against d and another scatter plot of ln T against ln d. Does a power model seem reasonable?
- (b) Fit a power model to the data.
- (c) Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model corroborate Kepler's Third Law?



# Chapter I Review

- Define each of the following in your own words. (Check by referring to the definition in the text.)
  - (a) Function
  - (b) Domain and range of a function
  - (c) Graph of a function
  - (d) Increasing function
  - (e) Composition of two functions
  - (f) Parametric curve
- **2.** (a) What is an even function? How can you tell if a function is even by looking at its graph?
  - (b) What is an odd function? How can you tell if a function is odd by looking at its graph?
- **3.** (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
  - (b) If f is a one-to-one function, how is its inverse function f<sup>-1</sup> defined? How do you obtain the graph of f<sup>-1</sup> from the graph of f?

- **4.** (a) State the Vertical Line Test.
  - (b) State the Horizontal Line Test.
  - 5. Give an example of each type of function.
    - (a) Constant function
    - (b) Power function
    - (c) Exponential function
    - (d) Linear function
    - (e) Quadratic function
    - (f) Polynomial of degree 5
    - (g) Rational function
  - **6.** Draw, by hand, a rough sketch of the graph of each function.
    - (a)  $y = x^{3}$ (b)  $y = x^{4}$ (c)  $y = \sin x$ (d)  $y = \tan x$ (e)  $y = e^{x}$ (f)  $y = \ln x$ (g) y = 1/x(h) y = |x|(i)  $y = \sqrt{x}$

### 🔺 TRUE-FALSE QUIZ 🔺

CONCEPT CHECK •

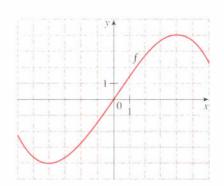
Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

- 1. If f is one-to-one, then  $f^{-1}(x) = \frac{1}{f(x)}$ .
- **2.** If  $x_1 < x_2$  and *f* is a decreasing function, then  $f(x_1) > f(x_2)$ .
- **3.** If f is a function, then f(s + t) = f(s) + f(t).

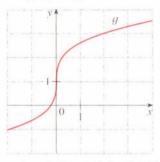
- **4.** If f(s) = f(t), then s = t.
- **5.** A vertical line intersects the graph of a function at most once.
- **6.** You can always divide by  $e^x$ .
- **7.** If 0 < a < b, then  $\ln a < \ln b$ .
- 8. If x > 0, then  $(\ln x)^6 = 6 \ln x$ .

### 🔶 EXERCISES 🤞

- Let f be the function whose graph is given.
   (a) Estimate the value of f(2).
  - (b) Estimate the values of x such that f(x) = 3.
  - (c) State the domain of f.
  - (d) State the range of f.
  - (e) On what interval is *f* increasing?
  - (f) Is f one-to-one? Explain.
  - (g) Is f even, odd, or neither even nor odd? Explain.



- 2. The graph of *q* is given.
  - (a) State the value of q(2).
  - (b) Why is *q* one-to-one?
  - (c) Estimate the value of  $q^{-1}(2)$ .
  - (d) Estimate the domain of  $q^{-1}$ .
  - (e) Sketch the graph of  $q^{-1}$ .



3. The distance traveled by a car is given by the values in the table.

t (seconds)	0	1	2	3	4	5
d (feet)	0	10	32	70	119	178

- (a) Use the data to sketch the graph of d as a function of t.
- (b) Use the graph to estimate the distance traveled after 4.5 seconds.
- 4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.
- **5–8** Find the domain and range of the function.

5.  $f(x) = \sqrt{4 - 3x^2}$ 6. q(x) = 1/(x + 1)**7.**  $h(t) = e^{-t^2}$  **8.**  $y = \ln(1 - x)$ 

**9.** Suppose that the graph of *f* is given. Describe how the graphs of the following functions can be obtained from the graph of f.

2

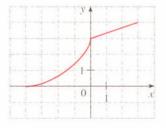
 $\bigwedge$ 

(a) 
$$y = f(x) + 8$$
  
(b)  $y = f(x + 8)$   
(c)  $y = 1 + 2f(x)$   
(d)  $y = f(x - 2) - (f)$   
(e)  $y = -f(x)$   
(f)  $y = f^{-1}(x)$ 

**10.** The graph of f is given. Draw the graphs of the following functions.

(a) y = f(x - 8)(b) y = -f(x)(d)  $y = \frac{1}{2} f(x) - 1$ (c) y = 2 - f(x)(x + 3)

(e) 
$$y = f^{-1}(x)$$
 (f)  $y = f^{-1}(x)$ 



**11–16** ■ Sketch the graph of the function.

- **12.**  $y = (x 1)^4 1$ 11.  $y = 1 + \sqrt{x+2}$ 14.  $y = 3 - 2 \sin x$ **13.**  $v = \cos 3x$ **16.**  $f(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ \ln x & \text{if } x \ge 1 \end{cases}$ **15.**  $f(x) = -e^x$
- 17. Determine whether f is even, odd, or neither even nor odd.

(a) 
$$f(x) = 2x^3 - 3x^2 + 2$$
  
(b)  $f(x) = x^3 - x'$   
(c)  $f(x) = e^{-x^2}$   
(d)  $f(x) = 1 + \sin x$ 

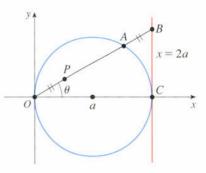
- 18. Find an expression for the function whose graph consists of the line segment from the point (-2, 2) to the point (-1, 0) together with the top half of the circle with center the origin and radius 1.
- 19. If  $f(x) = \ln x$  and  $g(x) = x^2 9$ , find the functions  $f \circ g, g \circ f, f \circ f, g \circ g$ , and their domains.
- **20.** Express the function  $F(x) = 1/\sqrt{x + \sqrt{x}}$  as a composition of three functions.
- **21.** If  $f(x) = 2x + \ln x$ , find  $f^{-1}(2)$ .
- **22.** Find the inverse function of  $f(x) = \frac{x+1}{2x+1}$ .
- 23. Find the exact value of each expression. (a)  $e^{2 \ln 3}$ (b)  $\log_{10} 25 + \log_{10} 4$
- **24.** Solve each equation for *x*. (b)  $\ln x = 2$  (c)  $e^{e^x} = 2$ (a)  $e^x = 5$
- 25. The half-life of palladium-100, <sup>100</sup>Pd, is four days. (So half of any given quantity of <sup>100</sup>Pd will disintegrate in four days.) The initial mass of a sample is one gram. (a) Find the mass that remains after 16 days.
  - (b) Find the mass m(t) that remains after t days.
  - (c) Find the inverse of this function and explain its meaning.
  - (d) When will the mass be reduced to 0.01 g?
- 26. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100,000}{100 + 900e^{-t}}$$

where t is measured in years.

- (a) Graph this function and estimate how long it takes for the population to reach 900.
- (b) Find the inverse of this function and explain its meaning.
- (c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).
- **27.** Graph members of the family of functions  $f(x) = \ln(x^2 - c)$  for several values of c. How does the graph change when c changes?

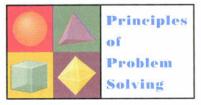
- 28. Graph the three functions y = x<sup>a</sup>, a<sup>x</sup>, and y = log<sub>a</sub>x on the same screen for two or three values of a > 1. For large values of x, which of these functions has the largest values and which has the smallest values?
  - **29.** (a) Sketch the curve represented by the parametric equations  $x = e^t$ ,  $y = \sqrt{t}$ ,  $0 \le t \le 1$ , and indicate with an arrow the direction in which the curve is traced as *t* increases.
    - (b) Eliminate the parameter to find a Cartesian equation of the curve.
  - **30.** (a) Find parametric equations for the path of a particle that moves counterclock wise halfway around the circle  $(x 2)^2 + y^2 = 4$ , from the top to the bottom.
- (b) Use the equations from part (a) to graph the semicircular path.
- **31.** Use parametric equations to graph the function  $f(x) = 2x + \ln x$  and its inverse function on the same screen.
  - **32.** (a) Find parametric equations for the set of all points P determined as shown in the figure so that |OP| = |AB|. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)
    - (b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.



**33.** Life expectancy has improved dramatically in this century. The table gives the life expectancy at birth (in years) of males born in the United States.

Birth year	Life expectancy
1900	48.3
1910	51.1
1920	55.2
1930	57.4
1940	62.5
1950	65.6
1960	66.6
1970	67.1
1980	70.0
1990	71.8

Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2000.



There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

## Understand the Problem

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

> What is the unknown? What are the given quantities? What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram. Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols, for instance, V for volume or t for time.



### Think of a Plan

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you do not see a connection immediately, the following ideas may be helpful in devising a plan.

**Try to Recognize Something Familiar** Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

**Try to Recognize Patterns** Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

**Use Analogy** Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

**Introduce Something Extra** It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

**Take Cases** We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation 3x - 5 = 7, we suppose that x is a number that satisfies 3x - 5 = 7 and work backward. We add 5 to each side of the equation and then divide each side by 3 to get x = 4. Since each of these steps can be reversed, we have solved the problem.

**Establish Subgoals** In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

**Indirect Reasoning** Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q we assume that P is true and Q is false and try to see why this cannot happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

**Mathematical Induction** In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

**Principle of Mathematical Induction** Let  $S_n$  be a statement about the positive integer n. Suppose that

1.  $S_1$  is true.

**2.**  $S_{k+1}$  is true whenever  $S_k$  is true.

Then  $S_n$  is true for all positive integers n.

This is reasonable because, since  $S_1$  is true, it follows from condition 2 (with k = 1) that  $S_2$  is true. Then, using condition 2 with k = 2, we see that  $S_3$  is true. Again using condition 2, this time with k = 3, we have that  $S_4$  is true. This procedure can be followed indefinitely.

### 3 Carry Out the Plan

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems." These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book. **Example 1** Express the hypotenuse *h* of a right triangle with area 25 m<sup>2</sup> as a function of its perimeter *P*.

quantity and the data:

Look Back

Solution Let us first sort out the information by identifying the unknown

Unknown: hypotenuse h

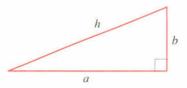
Given quantities: perimeter P, area 25 m<sup>2</sup>

Having completed our solution, it is wise to look back over it, partly to see if we

Draw a diagram

Understand the problem

It helps to draw a diagram and we do so in Figure 1.



#### Figure 1

Connect the given with the unknown
 Introduce something extra

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \frac{1}{2}ab \qquad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns a, b, and h:

h

	$h^2 = a^2 + b^2$
2	$25 = \frac{1}{2}ab$
3	P = a + b +

Relate to the familiar

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express  $(a + b)^2$  in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4(25)$$

 $(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$ 

From Equation 3 we have

Thus

$$h^{2} + 100 = P^{2} - 2Ph + h^{2}$$
$$2Ph = P^{2} - 100$$
$$h = \frac{P^{2} - 100}{2P}$$

This is the required expression for h as a function of P.

As the next example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

**Example 2** Solve the inequality |x - 3| + |x + 2| < 11.

Solution Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \ge 0\\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0\\ -(x-3) & \text{if } x-3 < 0 \end{cases}$$
$$= \begin{cases} x-3 & \text{if } x \ge 3\\ -x+3 & \text{if } x < 3 \end{cases}$$

Similarly

$$|x + 2| = \begin{cases} x + 2 & \text{if } x + 2 \ge 0\\ -(x + 2) & \text{if } x + 2 < 0 \end{cases}$$
$$= \begin{cases} x + 2 & \text{if } x \ge -2\\ -x - 2 & \text{if } x < -2 \end{cases}$$

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Take cases

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \le x < 3 \qquad x \ge 3$$

**CASE I** • If x < -2, we have

$$|x - 3| + |x + 2| < 11$$
  
 $-x + 3 - x - 2 < 11$   
 $-2x < 10$   
 $x > -5$ 

**CASE II** • If  $-2 \le x < 3$ , the given inequality becomes

-x + 3 + x + 2 < 11

5 < 11 (always true)

**CASE III** • If  $x \ge 3$ , the inequality becomes

$$x - 3 + x + 2 < 11$$
$$2x < 12$$
$$x < 6$$

Combining cases I, II, and III, we see that the inequality is satisfied when -5 < x < 6. So the solution is the interval (-5, 6).

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove it by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps.

- Step 1. Prove that  $S_n$  is true when n = 1.
- Step 2. Assume that  $S_n$  is true when n = k and deduce that  $S_n$  is true when n = k + 1.
- Step 3. Conclude that  $S_n$  is true for all n by the Principle of Mathematical Induction.

**Example 3** If  $f_0(x) = x/(x + 1)$  and  $f_{n+1} = f_0 \circ f_n$  for n = 0, 1, 2, ..., find a formula for  $f_n(x)$ .

Analogy: Try a similar, simpler problem Solution We start by finding formulas for  $f_n(x)$  for the special cases n = 1, 2, and 3.

$$f_{1}(x) = (f_{0} \circ f_{0})(x) = f_{0}(f_{0}(x)) = f_{0}\left(\frac{x}{x+1}\right)$$
$$= \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1}$$

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$$f_{2}(x) = (f_{0} \circ f_{1})(x) = f_{0}(f_{1}(x)) = f_{0}\left(\frac{x}{2x+1}\right)$$
$$= \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{\frac{x}{2x+1}}{\frac{3x+1}{2x+1}} = \frac{x}{3x+1}$$
$$f_{3}(x) = (f_{0} \circ f_{2})(x) = f_{0}(f_{2}(x)) = f_{0}\left(\frac{x}{3x+1}\right)$$
$$= \frac{\frac{x}{3x+1}}{\frac{x}{3x+1}+1} = \frac{\frac{x}{3x+1}}{\frac{4x+1}{3x+1}} = \frac{x}{4x+1}$$

We notice a pattern: The coefficient of x in the denominator of  $f_n(x)$  is n + 1 in the three cases we have computed. So we make the guess that, in general,

4 
$$f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (4) is true for n = 1. Assume that it is true for n = k; that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$
Then  $f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$ 

$$= \frac{\frac{x}{(k+1)x+1}}{\frac{x}{(k+1)x+1}+1} = \frac{\frac{x}{(k+2)x+1}}{\frac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (4) is true for n = k + 1. Therefore, by mathematical induction, it is true for all positive integers n.

- 1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.
- **2.** The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.
- **3.** Solve the equation |2x 1| |x + 5| = 3.
- 4. Solve the inequality  $|x 1| |x 3| \ge 5$ .

Look for a pattern

# **Problems**

- 5. Sketch the graph of the function  $f(x) = |x^2 4|x| + 3|$ .
- 6. Draw the graph of the equation |x| + |y| = 1 + |xy|.
- 7. Sketch the region in the plane consisting of all points (x, y) such that

$$|x| + |y| \le 1$$

8. Sketch the region in the plane consisting of all points (x, y) such that

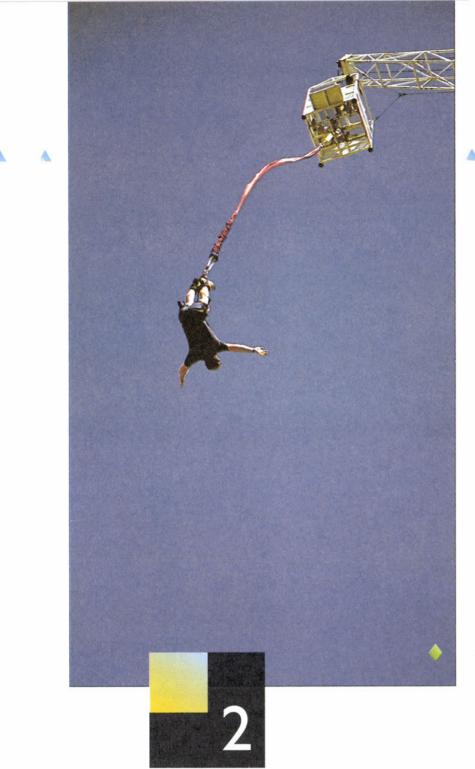
$$|x - y| + |x| - |y| \le 2$$

- **9.** Evaluate  $(\log_2 3) (\log_3 4) (\log_4 5) \cdots (\log_{31} 32)$ .
- 10. (a) Show that the function f(x) = ln(x + √x<sup>2</sup> + 1) is an odd function.
  (b) Find the inverse function of f.
- **II.** Solve the inequality  $\ln(x^2 2x 2) \le 0$ .
- 12. Use indirect reasoning to prove that  $\log_2 5$  is an irrational number.
- 13. Show that

$$\arcsin x + \arcsin y = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

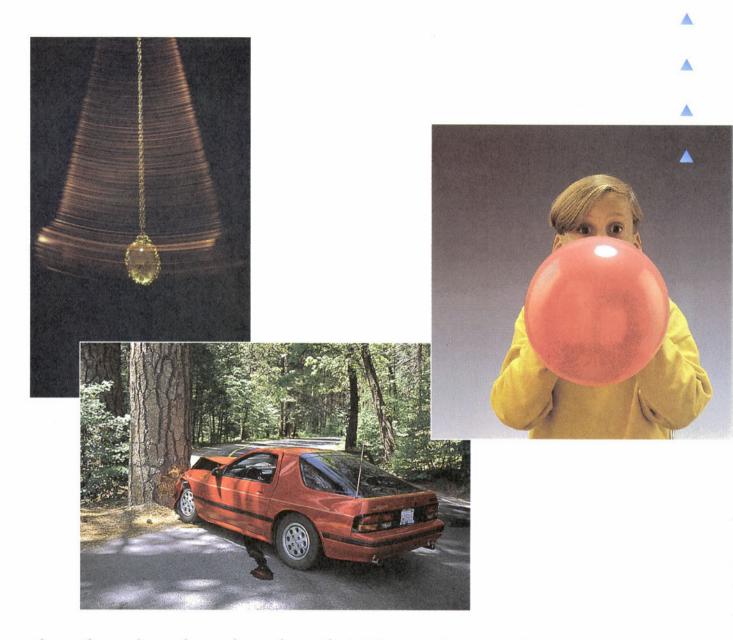
if the left side of this equation lies between  $-\pi/2$  and  $\pi/2$ .

- **14.** Is it true that  $f \circ (g + h) = f \circ g + f \circ h$ ?
- **15.** Prove that if n is a positive integer, then  $7^n 1$  is divisible by 6.
- **16.** Prove that  $1 + 3 + 5 + \dots + (2n 1) = n^2$ .
- **17.** If  $f_0(x) = x^2$  and  $f_{n+1}(x) = f_0(f_n(x))$  for n = 0, 1, 2, ..., find a formula for  $f_n(x)$ .
- **18.** (a) If  $f_0(x) = \frac{1}{2 x}$  and  $f_{n+1} = f_0 \circ f_n$  for n = 0, 1, 2, ..., find an expression for  $f_n(x)$  and use mathematical induction to prove it.
- (b) Graph  $f_0, f_1, f_2, f_3$  on the same screen and describe the effects of repeated composition.
- $\wedge$



• • • A limit is involved in each of these pictures. The bungee jumper should be glad that he's still above the ground when the cord is stretched to its limit. The tree is obviously the limiting position for the car. The balloon reaches its limit as it bursts. If there were no frictional forces, the pendulum would continue to oscillate and have no limiting behavior. But if left to run down, it comes to rest in its vertical limiting position.

# Limits and Derivatives



♦ In A Preview of Calculus (page 2) we saw how the idea of a limit underlies the various branches of calculus. Thus, it is appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative. We see how derivatives can be interpreted as rates of change in various situations and learn how the derivative of a function gives information about the original function.



# The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.



#### The Tangent Problem

The word tangent is derived from the Latin word tangens, which means "touching." Thus, a tangent to a curve is a line that touches the curve. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure 1(a). For more complicated curves this definition is inadequate. Figure l(b) shows two lines l and t passing through a point P on a curve C. The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t, on the other hand, looks like a tangent but it intersects C twice.

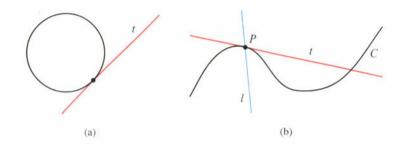


FIGURE I

To be specific, let's look at the problem of trying to find a tangent line t to the parabola  $y = x^2$  in the following example.

**EXAMPLE I** Find an equation of the tangent line to the parabola  $y = x^2$  at the point P(1, 1).

SOLUTION We will be able to find the equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point  $Q(x, x^2)$  on the parabola (as in Figure 2) and computing the slope  $m_{PQ}$  of the secant line PQ. We choose  $x \neq 1$  so that  $Q \neq P$ . Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point Q(1.5, 2.25) we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

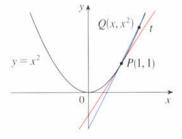


FIGURE 2

X	$m_{PQ}$
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The tables in the margin show the values of  $m_{PQ}$  for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer  $m_{PQ}$  is to 2. This suggests that the slope of the tangent line t should be m = 2.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m$$
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

and

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1)$$
 or  $y = 2x - 1$ 

Figure 3 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.

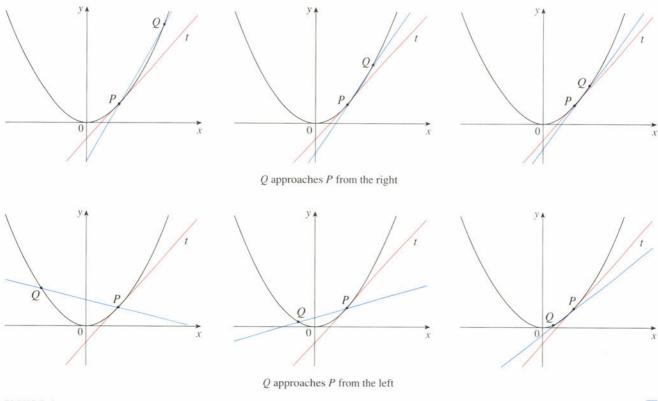


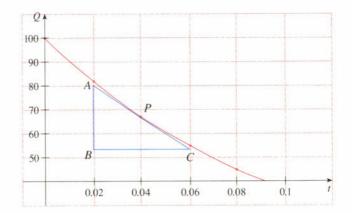
FIGURE 3

Many functions that occur in science are not described by an explicit equation; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

**EXAMPLE 2** The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data at the left describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where t = 0.04. [Note: The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microcompres).]

**SOLUTION** In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.



#### FIGURE 4

Given the points P(0.04, 67.03) and Q(0.00, 100.00) on the graph, we find that the slope of the secant line PQ is

$$m_{PQ} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

The table at the left shows the results of similar calculations for the slopes of
other secant lines. From this table we would expect the slope of the tangent line
at $t = 0.04$ to lie somewhere between $-742$ and $-607.5$ . In fact, the average of
the slopes of the two closest secant lines is

$$\frac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be -675.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 4. This gives an estimate of the slope of the tangent line as

$$\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$



### The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle does not stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

Q	$m_{PQ}$
(0.00, 100.00)	-824.25
(0.02, 81.87)	-742.00
(0.06, 54.88)	-607.50
(0.08, 44.93)	-552.50
(0.10, 36.76)	-504.50



The CN Tower in Toronto is currently the highest freestanding building in the world.

**EXAMPLE 3** Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

**SOLUTION** In trying to solve this problem we use the fact, discovered by Galileo almost four centuries ago, that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This neglects air resistance.) If the distance fallen after t seconds is denoted by s(t) and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time (t = 5) so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from t = 5 to t = 5.1:

0.1

average velocity = 
$$\frac{\text{distance traveled}}{\text{time elapsed}}$$
  
=  $\frac{s(5.1) - s(5)}{s(5)}$ 

$$=\frac{4.9(5.1)^2-4.9(5)^2}{0.1}=49.49 \text{ m/s}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (n) s
5 = 7 = 6	53.9
$5 \le i \le 5.1$	49,49
$5 \le i = 5.05$	49.245
$5 \le t \le 5.01$	49.049
$5 \le i \le 5.001$	49,0049

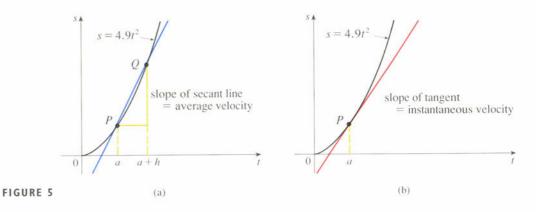
It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when t = 5 is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at t = 5. Thus, the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points  $P(a, 4.9a^2)$  and  $Q(a + h, 4.9(a + h)^2)$  on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a+h)^2 - 4.9a^2}{(a+h) - a}$$

which is the same as the average velocity over the time interval [a, a + h]. Therefore, the velocity at time t = a (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).



Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next four sections, we will return to the problems of finding tangents and velocities in Section 2.6.

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Exercises · ·

1. The experimental data in the table define *y* as a function of *x*.

X	0	1	2	3	4	5
<u>,</u> ¥	2.6	2.0	1.1	1.3	2.1	3.5

- (a) If *P* is the point (3, 1.3), find the slopes of the secant lines *PQ* when *Q* is the point on the graph with x = 0, 1, 2, 4, and 5.
- (b) Estimate the slope of the tangent line at *P* by averaging the slopes of two secant lines.
- (c) Use a graph of the function to estimate the slope of the tangent line at *P*.
- 2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heart-

beats after *t* minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

1 (min)	36	38	40	42	-44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between

(a) 
$$t = 36$$
 and  $t = 42$ 

- (b) t = 38 and t = 42
- (c) t = 40 and t = 42

(d) t = 42 and t = 44What are your conclusions?

- **3.** The point P(4, 2) lies on the curve  $y = \sqrt{x}$ .
  - (a) If Q is the point  $(x, \sqrt{x})$ , use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
    - (i) 5 (ii) 4.5
    - (iii) 4.1 (iv) 4.01
    - (v) 4.001 (vi) 3
    - (vii) 3.5 (viii) 3.9
    - (ix) 3.99 (x) 3.999
  - (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at *P*(4, 2).
  - (c) Using the slope from part (b), find an equation of the tangent line to the curve at P(4, 2).
- 4. The point P(0.5, 2) lies on the curve y = 1/x.
  - (a) If Q is the point (x, 1/x), use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
    - (i) 2 (ii) 1
    - (iii) 0.9 (iv) 0.8
    - (v) 0.7 (vi) 0.6
    - (vii) 0.55 (viii) 0.51
    - (ix) 0.45 (x) 0.49
  - (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at P(0.5, 2).
  - (c) Using the slope from part (b), find an equation of the tangent line to the curve at P(0.5, 2).
  - (d) Sketch the curve, two of the secant lines, and the tangent line.
- 5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet after t seconds is given by  $y = 40t 16t^2$ .
  - (a) Find the average velocity for the time period beginning when t = 2 and lasting
    - (i) 0.5 s (ii) 0.1 s
    - (iii) 0.05 s (iv) 0.01 s
  - (b) Find the instantaneous velocity when t = 2.
- 6. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by  $h = 58t 0.83t^2$ .

- (a) Find the average velocity over the given time intervals:
   (i) [1,2]
   (ii) [1,1.5]
   (iii) [1,1.1]
  - (i) [1, 2] (ii) [1, 1.5] (iii) [1, (iv) [1, 1.01] (v) [1, 1.001]

(b) Find the instantaneous velocity after 1 s.

- 7. The displacement (in feet) of a particle moving in a straight line is given by  $s = t^3/6$ , where t is measured in seconds.
  - (a) Find the average velocity over the following time periods:
    - (i) [1,3] (ii) [1,2]
    - (iii) [1,1.5] (iv) [1,1.1]
  - (b) Find the instantaneous velocity when t = 1.
  - (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities found in part (a).
  - (d) Draw the tangent line whose slope is the instantaneous velocity from part (b).
- 8. The position of a car is given by the values in the table.

t (seconds)	0	1	2	3	4	5
s (feet)	0	10	32	70	119	178

(a) Find the average velocity for the time period beginning when t = 2 and lasting

(i) 3 s (ii) 2 s (iii) 1 s

- (b) Use the graph of s as a function of t to estimate the instantaneous velocity when t = 2.
- 9. The point P(1,0) lies on the curve  $y = \sin(10\pi/x)$ .
  - (a) If Q is the point (x, sin(10π/x)), find the slope of the secant line PQ (correct to four decimal places) for x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8, and 0.9. Do the slopes appear to be approaching a limit?
  - (b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at *P*.
  - (c) By choosing appropriate secant lines, estimate the slope of the tangent line at *P*.



# The Limit of a Function

Æ

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and methods for computing them.

Let's investigate the behavior of the function f defined by  $f(x) = x^2 - x + 2$  for values of x near 2. The following table gives values of f(x) for values of x close to 2, but not equal to 2.

8	f(x)	<i>A</i> .	
0.0	2.000000	3.0	8,000000
1.5	2.750000	2.5	5,750000
1.8	3.440000	2.2	4,640000
1.9	3.710000	2.1	4,310000
1.95	3,852500	2.05	4,152500
1.99	3,970100	2.01	4,030100
1.995	3,985025	2.005	4.015025
1,000	3.097(10)	2.001	4,003001

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), f(x) is close to 4. In fact, it appears that we can make the values of f(x) as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function  $f(x) = x^2 - x + 2$  as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

**Definition** We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of f(x), as x approaches a, equals L"

if we can make the values of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a but not equal to a.

Roughly speaking, this says that the values of f(x) become closer and closer to the number L as x approaches the number a (from either side of a) but  $x \neq a$ .

An alternative notation for

is

$$\lim_{x \to a} f(x) = L$$
$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "f(x) approaches L as x approaches a."

Notice the phrase "but  $x \neq a$ " in the definition of limit. This means that in finding the limit of f(x) as x approaches a, we never consider x = a. In fact, f(x) need not even be defined when x = a. The only thing that matters is how f is defined *near a*.

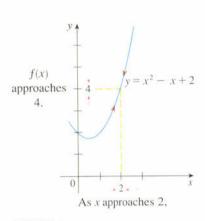
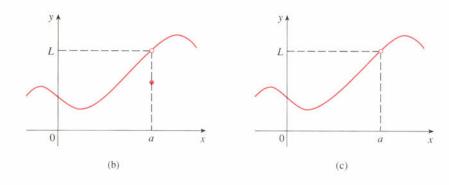


FIGURE I

Figure 2 shows the graphs of three functions. Note that in part (c), f(a) is not defined and in part (b),  $f(a) \neq L$ . But in each case, regardless of what happens at a,  $\lim_{x\to a} f(x) = L$ .



**EXAMPLE 1** Guess the value of  $\lim_{x \to 1} \frac{x-1}{x^2-1}$ .

**SOLUTION** Notice that the function  $f(x) = (x - 1)/(x^2 - 1)$  is not defined when x = 1, but that doesn't matter because the definition of  $\lim_{x\to a} f(x)$  says that we consider values of x that are close to a but not equal to a. The tables at the left give values of f(x) (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the table, we make the guess that

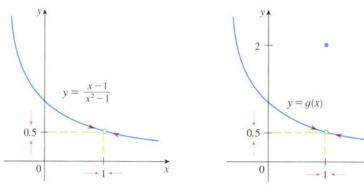
$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

х

Example 1 is illustrated by the graph of f in Figure 3. Now let's change f slightly by giving it the value 2 when x = 1 and calling the resulting function q:

$$g(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1\\ 2 & \text{if } x = 1 \end{cases}$$

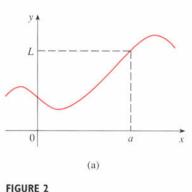
This new function g still has the same limit as x approaches 1 (see Figure 4).

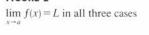


x



FIGURE 4





f(x)
0.666667
0.526316
0.502513
0.500250
0.500025

x > 1	f(x)
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

**EXAMPLE 2** Find 
$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$
.

**SOLUTION** The table lists values of the function for several values of t near 0.

t	$\frac{\sqrt{t^2+9}-3}{t^2}$
±1.0	0.16228
$\pm 0.5$	0.16553
$\pm 0.1$	0.16662
$\pm 0.05$	0.16666
$\pm 0.01$	0.16667

As t approaches 0, the values of the function seem to approach 0.1666666... and so we guess that

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6}$$

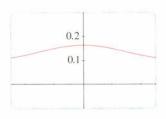
In Example 2 what would have happened if we had taken even smaller values of *t*? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of  $\frac{1}{6}$ ? No, the value of the limit is  $\frac{1}{6}$ , as we will show in the next section. The problem is that the calculator gave false values because  $\sqrt{t^2 + 9}$  is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for  $\sqrt{t^2 + 9}$  is 3.000...to as many digits as the calculator is capable of carrying.)

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

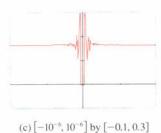
of Example 2 on a graphing calculator or computer. Parts (a) and (b) of Figure 5 show quite accurate graphs of f and when we use the trace mode (if available), we can estimate easily that the limit is about  $\frac{1}{6}$ . But if we zoom in too far, as in parts (c) and (d), then we get inaccurate graphs, again because of problems with subtraction.



<sup>(</sup>a) [-5, 5] by [-0.1, 0.3]



(b) [-0.1, 0.1] by [-0.1, 0.3]



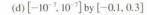


FIGURE 5

t	$\frac{\sqrt{t^2+9}-3}{t^2}$	
$\pm 0.0005$	0.16800	
$\pm 0.0001$	0.20000	
$\pm 0.00005$	0.00000	
$\pm 0.00001$	0.00000	

**EXAMPLE 3** Find  $\lim_{x \to 0} \frac{\sin x}{x}$ .

**SOLUTION** Again the function  $f(x) = (\sin x)/x$  is not defined when x = 0. Using a calculator (and remembering that, if  $x \in \mathbb{R}$ ,  $\sin x$  means the sine of the angle whose *radian* measure is x), we construct the following table of values correct to eight decimal places. From the table and the graph in Figure 6 we guess that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Section 3.4 using a geometric argument.

	$\frac{\sin x}{x}$	
21:0	0.84147098	$f(x) \uparrow 1$ $y = \frac{\sin x}{2}$
=0.5	0.95885108	$y = \frac{1}{x}$
* 0.4	0.97354586	
* ().3	0.98506736	
$\pm 0.2$	0.99334665	
	0.99833417	
10.05	0.99958339	-1 $0$ $1$ $x$
$\pm 0.01$	(1.99998333	FIGURE 6
$\pm (0.005)$	0.99999583	FIGURE 0
$\pm (0.001$	(1.99999983	

**EXAMPLE 4** Find 
$$\lim_{x \to 0} \sin \frac{\pi}{x}$$

**SOLUTION** Once again the function  $f(x) = \sin(\pi/x)$  is undefined at 0. Evaluating the function for some small values of x, we get

#### **Computer Algebra Systems**

Computer algebra systems have commands that compute limits. Because of the types of pitfalls demonstrated in Examples 2, 4, and 5, they do not find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. If you have access to a CAS, use the limit command to compute the limits in the examples of this section and to check your answers in the exercises of this chapter.

$f(1) = \sin \pi = 0$	$f(\frac{1}{2}) = \sin 2\pi = 0$
$f(\frac{1}{3}) = \sin 3\pi = 0$	$f(\frac{1}{4}) = \sin 4\pi = 0$
$f(0.1) = \sin 10\pi = 0$	$f(0.01) = \sin 100\pi = 0$

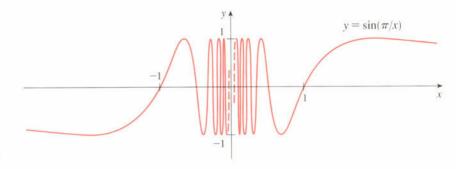
Similarly, f(0.001) = f(0.0001) = 0. On the basis of this information we might be tempted to guess that

$$\lim_{x \to 0} \sin \frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although  $f(1/n) = \sin n\pi = 0$  for any integer *n*, it is also true that f(x) = 1 for infinitely many values of *x* that approach 0. [In fact,  $\sin(\pi/x) = 1$  when

$$\frac{\pi}{x} = \frac{\pi}{2} + 2n\pi$$

and, solving for x, we get x = 2/(4n + 1).] The graph of f is given in Figure 7. The broken lines indicate that the values of  $\sin(\pi/x)$  oscillate between 1 and -1 infinitely often as x approaches 0. (Use a graphing device to graph f and zoom in toward the origin several times. What do you observe?)





Since the values of f(x) do not approach a fixed number as x approaches 0,

 $\lim_{x \to 0} \sin \frac{\pi}{x} \text{ does not exist}$ 

$$\begin{array}{c} x \\ 1 \\ 0.000028 \\ 0.124920 \\ 0.1 \\ 0.001088 \\ 0.05 \\ 0.000222 \\ 0.01 \\ 0.000101 \end{array}$$

N	$x^{+} + \frac{\cos 5x}{10,000}$	
0.001	0.00010000	

**EXAMPLE 5** Find  $\lim_{x \to 0} \left( x^3 + \frac{\cos 5x}{10,000} \right)$ . **SOLUTION** As before, we construct a table of values. From the table in the margin it appears that

$$\lim_{x \to 0} \left( x^3 + \frac{\cos 5x}{10,000} \right) = 0$$

But if we persevere with smaller values of x, the second table suggests that

$$\lim_{x \to 0} \left( x^3 + \frac{\cos 5x}{10,000} \right) = 0.000100 = \frac{1}{10,000}$$

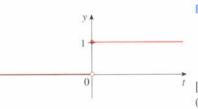
Later we will see that  $\lim_{x\to 0} \cos 5x = 1$  and then it follows that the limit is 0.0001.

Examples 4 and 5 illustrate some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give the wrong values. Later, however, we will develop foolproof methods for calculating limits.

**EXAMPLE 6** The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0\\ 1 & \text{if } t \ge 0 \end{cases}$$

<sup>t</sup> [This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time t = 0.] Its graph is shown in Figure 8.





As t approaches 0 from the left, H(t) approaches 0. As t approaches 0 from the right, H(t) approaches 1. There is no single number that H(t) approaches as t approaches 0. Therefore,  $\lim_{t\to 0} H(t)$  does not exist.

# **One-Sided Limits**

We noticed in Example 6 that H(t) approaches 0 as t approaches 0 from the left and H(t) approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

> $\lim H(t) = 0$ and  $\lim H(t) = 1$

The symbol " $t \rightarrow 0^{-1}$ " indicates that we consider only values of t that are less than 0. Likewise, " $t \rightarrow 0^+$ " indicates that we consider only values of t that are greater than 0.

**2** Definition We write

$$\lim_{x \to w^-} f(x) = L$$

and say the left-hand limit of f(x) as x approaches a [or the limit of f(x) as x approaches a from the left] is equal to L if we can make the values of f(x) as close to L as we like by taking x to be sufficiently close to a and x less than a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the righthand limit of f(x) as x approaches a is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus, the symbol " $x \to a^+$ " means that we consider only x > a. These definitions are illustrated in Figure 9.

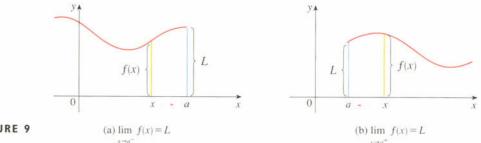


FIGURE 9

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

 $\lim f(x) = L$  $\lim f(x) = L$  and  $\lim f(x) = L$ if and only if  $x \rightarrow a$  $x \rightarrow a$ 

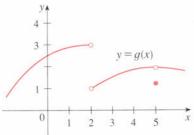


FIGURE 10

**EXAMPLE 7** The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

	(a)	$\lim_{x\to 2^-}g(x)$	(b)	$\lim_{x\to 2^+} g(x)$
	(c)	$\lim_{x \to 2} g(x)$	(d)	$\lim_{x\to 5^-}g(x)$
► X	(e)	$\lim_{x \to 5^+} g(x)$	(f)	$\lim_{x \to 5} g(x)$

**SOLUTION** From the graph we see that the values of g(x) approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

(a) 
$$\lim_{x \to 2^-} g(x) = 3$$
 and (b)  $\lim_{x \to 2^+} g(x) = 1$ 

(c) Since the left and right limits are different, we conclude from (3) that  $\lim_{x\to 2} g(x)$  does not exist.

The graph also shows that

(d) 
$$\lim_{x \to 5^{-}} g(x) = 2$$
 and (e)  $\lim_{x \to 5^{+}} g(x) = 2$ 

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x\to 5}g(x)=2$$

Despite this fact, notice that  $g(5) \neq 2$ .

**EXAMPLE 8** Find  $\lim_{x \to 0} \frac{1}{x^2}$  if it exists.

X	$1/x^2$	
34 P	I	
$\pm 0.5$	4	
0.2	25	
+(0, 1)	100	
+ 0.05	400	
$\pm 0.01$	10,000	
(),()()	1.000,000	

**SOLUTION** As x becomes close to 0,  $x^2$  also becomes close to 0, and  $1/x^2$  becomes very large. (See the table at the left.) In fact, it appears from the graph of the function  $f(x) = 1/x^2$  shown in Figure 11 that the values of f(x) can be made arbitrarily large by taking x close enough to 0. Thus, the values of f(x) do not approach a number, so  $\lim_{x\to 0} (1/x^2)$  does not exist.

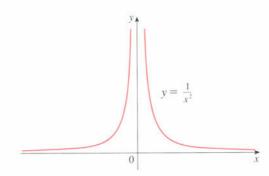


FIGURE II

At the beginning of this section we considered the function  $f(x) = x^2 - x + 2$ and, based on numerical and graphical evidence, we saw that

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

According to Definition 1, this means that the values of f(x) can be made as close to 4 as we like, provided that we take x sufficiently close to 2. In the following example we use graphical methods to determine just how close is sufficiently close.

**EXAMPLE 9** If  $f(x) = x^2 - x + 2$ , how close to 2 does x have to be to ensure that f(x) is within a distance 0.1 of the number 4?

**SOLUTION** If the distance from f(x) to 4 is less than 0.1, then f(x) lies between 3.9 and 4.1, so the requirement is that

$$3.9 < x^2 - x + 2 < 4.1$$

Thus, we need to determine the values of x such that the curve  $y = x^2 - x + 2$ lies between the horizontal lines y = 3.9 and y = 4.1. We graph the curve and lines near the point (2, 4) in Figure 12. With the cursor, we estimate that the x-coordinate of the point of intersection of the line y = 3.9 and the curve  $y = x^2 - x + 2$  is about 1.966. Similarly, the curve intersects the line y = 4.1when  $x \approx 2.033$ . So, rounding to be safe, we conclude that

$$3.9 < x^2 - x + 2 < 4.1$$
 when  $1.97 < x < 2.03$ 

Therefore, f(x) is within a distance 0.1 of 4 when x is within a distance 0.03 of 2.

The idea behind Example 9 can be used to formulate the precise definition of a limit that is discussed in Appendix D.

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet f(2) = 3? Explain.

2. Explain what it means to say that

 $\lim_{x \to 1^{-}} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^{+}} f(x) = 7$ 

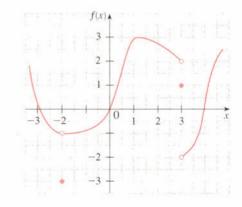
In this situation is it possible that  $\lim_{x\to 1} f(x)$  exists? Explain.

**3.** For the function *f* whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a)  $\lim_{x \to 1} f(x)$  (b)  $\lim_{x \to 3^{-}} f(x)$ (c)  $\lim_{x \to 3^{+}} f(x)$  (d)  $\lim_{x \to 3} f(x)$ 

(e) <i>f</i> (3)	(f) $\lim_{x \to -2^-} f(x)$	
(g) $\lim_{x \to -2^+} f(x)$	(h) $\lim_{x \to -2} f(x)$	

(i) 
$$f(-2)$$



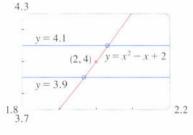
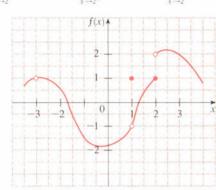


FIGURE 12

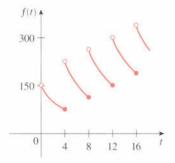
- **4.** State the value of the limit, if it exists, from the given graph. If it does not exist, explain why.
  - (a)  $\lim_{x \to 3} f(x)$  (b)  $\lim_{x \to 1} f(x)$  (c)  $\lim_{x \to -3} f(x)$ (d)  $\lim_{x \to 2^{-}} f(x)$  (e)  $\lim_{x \to 2^{+}} f(x)$  (f)  $\lim_{x \to 2} f(x)$



- 5. Use the graph of the function f(x) = 1/(1 + e<sup>1/x</sup>) to state the value of each limit, if it exists. If it does not exist, explain why.
  - (a)  $\lim_{x \to 0^{-}} f(x)$  (b)  $\lim_{x \to 0^{+}} f(x)$  (c)  $\lim_{x \to 0} f(x)$
  - 6. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount f(t) of the drug in the bloodstream after *t* hours. (Later we will be able to compute the dosage and time interval to ensure that the concentration of a drug does not reach a harmful level.) Find

$$\lim_{t \to 12^-} f(t) \quad \text{and} \quad \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits.



**7–8** Sketch the graph of an example of a function f that satisfies all of the given conditions.

7.  $\lim_{x \to 3^+} f(x) = 4$ ,  $\lim_{x \to 3^-} f(x) = 2$ ,  $\lim_{x \to -2} f(x) = 2$ , f(3) = 3, f(-2) = 1

8. 
$$\lim_{x \to 0^{-}} f(x) = 1$$
,  $\lim_{x \to 0^{+}} f(x) = -1$ ,  $\lim_{x \to 2^{-}} f(x) = 0$ ,  
 $\lim_{x \to 2^{+}} f(x) = 1$ ,  $f(2) = 1$ ,  $f(0)$  is undefined

9-12 ■ Evaluate the function at the given numbers (correct to six decimal places). Use the results to guess the value of the limit, or explain why it does not exist.

9. 
$$g(x) = \frac{x-1}{x^3-1}$$
;  
 $x = 0.2, 0.4, 0.6, 0.8, 0.9, 0.99, 1.8, 1.6, 1.4, 1.2, 1.1, 1.01;$   
 $\lim_{x \to 1} \frac{x-1}{x^3-1}$   
10.  $F(t) = \frac{\sqrt[3]{t}-1}{\sqrt{t}-1}$ ;  $t = 1.5, 1.2, 1.1, 1.01, 1.001;$   
 $\lim_{t \to 1} \frac{\sqrt[3]{t}-1}{\sqrt{t}-1}$   
11.  $f(x) = \frac{1-\cos x}{x^2};$   
 $x = 1, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01;$   
 $\lim_{x \to 0} \frac{1-\cos x}{x^2}$   
12.  $g(x) = \sqrt{x} \ln x;$ 

- **12.**  $g(x) = \sqrt{x} \ln x;$  x = 1, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001; $\lim_{x \to 0^+} \sqrt{x} \ln x$
- I3. (a) By graphing the function f(x) = (tan 4x)/x and zooming in toward the point where the graph crosses the y-axis, estimate the value of lim<sub>x→0</sub> f(x).
  - (b) Check your answer in part (a) by evaluating f(x) for values of x that approach 0.
- 14. (a) Estimate the value of

$$\lim_{x \to 0} \frac{6^x - 2^x}{x}$$

by graphing the function  $y = (6^x - 2^x)/x$ . State your answer correct to two decimal places.

- (b) Check your answer in part (a) by evaluating f(x) for values of x that approach 0.
- 15. (a) Estimate the value of the limit

$$\lim_{x \to 0} (1 + x)^{1/x}$$

to five decimal places. Does this number look familiar?

(b) Illustrate part (a) by graphing the function

$$y = (1 + x)^{1/2}$$

16. The slope of the tangent line to the graph of the exponential function  $y = 2^x$  at the point (0, 1) is  $\lim_{x\to 0} (2^x - 1)/x$ . Estimate the slope to three decimal places.

17. (a) Evaluate the function  $f(x) = x^2 - (2^x/1000)$  for x = 1, 0.8, 0.6, 0.4, 0.2, 0.1, and 0.05, and guess the value of

$$\lim_{x \to 0} \left( x^2 - \frac{2^x}{1000} \right)$$

- (b) Evaluate f(x) for x = 0.04, 0.02, 0.01, 0.005, 0.003, and 0.001. Guess again.
- **18.** (a) Evaluate  $h(x) = (\tan x x)/x^3$  for x = 1, 0.5, 0.1, 0.05, 0.01, and 0.005.
  - (b) Guess the value of  $\lim_{x \to 0} \frac{\tan x x}{x^3}$ .
  - (c) Evaluate h(x) for successively smaller values of x until you finally reach 0 values for h(x). Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.5 a method for evaluating the limit will be explained.)

- (d) Graph the function h in the viewing rectangle [-1, 1] by [0, 1]. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of h(x) as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).
- 19. Use a graph to determine how close to 0 we have to take x to ensure that e<sup>x</sup> is within a distance 0.2 of the number 1. What if we insist that e<sup>x</sup> be within 0.1 of 1?
- **20.** (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \to 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

(b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?



# Calculating Limits Using the Limit Laws

 $\mathcal{A}$ 

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

**Limit Laws** Suppose that c is a constant and the limits  $\lim f(x)$ and  $\lim q(x)$  $x \rightarrow a$  $x \rightarrow a$ exist. Then 1.  $\lim [f(x) + g(x)] = \lim f(x) + \lim g(x)$  $x \rightarrow a$  $x \rightarrow a$ **2.**  $\lim [f(x) - g(x)] = \lim f(x) - \lim g(x)$  $x \rightarrow a$  $x \rightarrow a$ **3.**  $\lim [cf(x)] = c \lim f(x)$ **4.**  $\lim [f(x)g(x)] = \lim f(x) \cdot \lim g(x)$  $x \rightarrow a$  $x \rightarrow a$  $x \rightarrow a$ 5.  $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if  $\lim g(x) \neq 0$  $x \rightarrow a$ 

These five laws can be stated verbally as follows:

1. The limit of a sum is the sum of the limits.

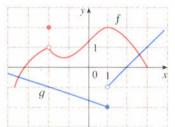
2. The limit of a difference is the difference of the limits.

**3.** The limit of a constant times a function is the constant times the limit of the function.

Sum Law Difference Law

Constant Multiple Law

Product Law Quotient Law



- **4.** The limit of a product is the product of the limits.
- **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if f(x) is close to L and g(x) is close to M, it is reasonable to conclude that f(x) + g(x) is close to L + M. This gives us an intuitive basis for believing that Law 1 is true. All of these laws can be proved using the precise definition of a limit. In Appendix E we give the proof of Law 1.

**EXAMPLE I** • Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) 
$$\lim_{x \to -2} [f(x) + 5g(x)]$$
 (b)  $\lim_{x \to 1} [f(x)g(x)]$  (c)  $\lim_{x \to 2} \frac{f(x)}{g(x)}$ 

### SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1$$
 and  $\lim_{x \to -2} g(x) = -1$ 

Therefore, we have

$$\lim_{x \to -2} \left[ f(x) + 5g(x) \right] = \lim_{x \to -2} f(x) + \lim_{x \to -2} \left[ 5g(x) \right] \quad \text{(by Law 1)}$$
$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \quad \text{(by Law 3)}$$
$$= 1 + 5(-1) = -4$$

(b) We see that  $\lim_{x\to 1} f(x) = 2$ . But  $\lim_{x\to 1} g(x)$  doesn't exist because the left and right limits are different:

$$\lim_{x \to 1^{-}} g(x) = -2 \qquad \lim_{x \to 1^{+}} g(x) = -1$$

So we can't use Law 4. The given limit doesn't exist.

(c) The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit doesn't exist.

If we use the Product Law repeatedly with g(x) = f(x), we obtain the following law.

Power Law

**6.**  $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \quad \text{where } n \text{ is a positive integer}$ 

In applying these six limit laws we need to use two special limits:

**7.** 
$$\lim_{x \to a} c = c$$
 **8.**  $\lim_{x \to a} x = a$ 



These limits are obvious from an intuitive point of view (state them in words or draw graphs of y = c and y = x).

If we now put f(x) = x in Law 6 and use Law 8, we get another useful special limit.

**9.**  $\lim_{x \to a} x^n = a^n$  where *n* is a positive integer

A similar limit holds for roots as follows.

**10.**  $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$  where *n* is a positive integer (If *n* is even, we assume that a > 0.)

More generally, we have the following law.

Root Law

11. 
$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$
 where *n* is a positive integer  
[If *n* is even, we assume that  $\lim_{x \to a} f(x) > 0$ .]

**EXAMPLE 2** Evaluate the following limits and justify each step.

(a) 
$$\lim_{x \to 5} (2x^2 - 3x + 4)$$
 (b)  $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$ 

SOLUTION

(a) 
$$\lim_{x \to 5} (2x^2 - 3x + 4) = \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4$$
 (by Laws 2 and 1)  
$$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4$$
 (by 3)  
$$= 2(5^2) - 3(5) + 4$$
 (by 9, 8, and 7)  
$$= 39$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)}$$
 (by Law 5)  
$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x}$$
 (by 1, 2, and 3)  
$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$
 (by 9, 8, and 7)  
$$= -\frac{1}{11}$$

#### **Newton and Limits**

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published Principia Mathematica. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

**NOTE** • If we let  $f(x) = 2x^2 - 3x + 4$ , then f(5) = 39. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 35 and 36). We state this fact as follows.

If f is a polynomial or a rational function and a is in the domain of f, then

 $\lim_{x \to a} f(x) = f(a)$ 

Functions with this direct substitution property are called *continuous at a* and will be studied in Section 2.4. However, not all limits can be evaluated by direct substitution, as the following examples show.

**EXAMPLE 3** Find 
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$
.

**SOLUTION** Let  $f(x) = (x^2 - 1)/(x - 1)$ . We can't find the limit by substituting x = 1 because f(1) is not defined. Nor can we apply the Quotient Law because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

The numerator and denominator have a common factor of x - 1. When we take the limit as x approaches 1, we have  $x \neq 1$  and so  $x - 1 \neq 0$ . Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$
$$= \lim_{x \to 1} (x + 1)$$
$$= 1 + 1 = 2$$

The limit in this example arose in Section 2.1 when we were trying to find the tangent to the parabola  $y = x^2$  at the point (1, 1).

**EXAMPLE 4** Find  $\lim_{x \to 1} g(x)$  where

 $g(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$ 

**SOLUTION** Here g is defined at x = 1 and  $g(1) = \pi$ , but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since g(x) = x + 1 for  $x \neq 1$ , we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 2$$

Note that the values of the functions in Examples 3 and 4 are identical except when x = 1 (see Figure 2) and so they have the same limit as x approaches 1.

**EXAMPLE 5** Evaluate 
$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$
.

**SOLUTION** If we define

$$F(h) = \frac{(3+h)^2 - 9}{h}$$

then, as in Example 3, we cannot compute  $\lim_{h\to 0} F(h)$  by letting h = 0 since F(0) is undefined. But if we simplify F(h) algebraically, we find

$$F(h) = \frac{(9+6h+h^2)-9}{h} = \frac{6h+h^2}{h} = 6+h$$

(Recall that we consider only  $h \neq 0$  when letting h approach 0.) Thus

 $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} (6+h) = 6$ EXAMPLE 6 Find  $\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}.$ 

**SOLUTION** We cannot apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

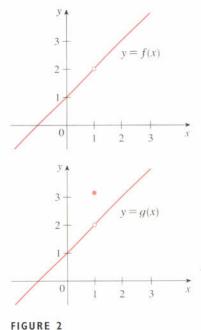
$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3}$$
$$= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2(\sqrt{t^2 + 9} + 3)} = \lim_{t \to 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)}$$
$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$

This calculation confirms the guess that we made in Example 2 in Section 2.2.

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

**Theorem** 
$$\lim_{x \to a} f(x) = L$$
 if and only if  $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$ 

When computing one-sided limits we use the fact that the Limit Laws also hold for one-sided limits.





**EXAMPLE 7** Show that  $\lim_{x \to 0} |x| = 0$ .

**SOLUTION** Recall that

 $|x| = \begin{cases} x & \text{if } x \ge 0\\ -x & \text{if } x < 0 \end{cases}$ 

Since |x| = x for x > 0, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For x < 0 we have |x| = -x and so

 $\lim_{x \to 0^{-}} |x| = \lim_{x \to 0^{-}} (-x) = 0$ 

Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0$$

**EXAMPLE 8** Prove that 
$$\lim_{x \to 0} \frac{|x|}{x}$$
 does not exist.

SOLUTION

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^{-}} \frac{|x|}{x} = \lim_{x \to 0^{-}} \frac{-x}{x} = \lim_{x \to 0^{-}} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that  $\lim_{x\to 0} |x|/x$  does not exist. The graph of the function f(x) = |x|/x is shown in Figure 4 and supports the limits that we found.

**EXAMPLE 9** The greatest integer function is defined by  $[\![x]\!]$  = the largest integer that is less than or equal to x. (For instance,  $[\![4]\!]$  = 4,  $[\![4.8]\!]$  = 4,  $[\![\pi]\!]$  = 3,  $[\![\sqrt{2}]\!]$  = 1,  $[\![-\frac{1}{2}]\!]$  = -1.) Show that  $\lim_{x\to 3} [\![x]\!]$  does not exist.

**SOLUTION** The graph of the greatest integer function is shown in Figure 5. Since [x] = 3 for  $3 \le x < 4$ , we have

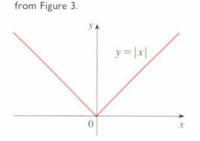
$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since  $\llbracket x \rrbracket = 2$  for  $2 \le x < 3$ , we have

$$\lim_{x \to 3^{-}} [\![x]\!] = \lim_{x \to 3^{-}} 2 = 2$$

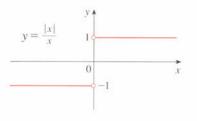
Because these one-sided limits are not equal,  $\lim_{x\to 3} [x]$  does not exist by Theorem 1.

The next two theorems give two additional properties of limits.



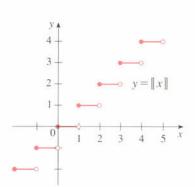
The result of Example 7 looks plausible

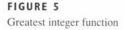
FIGURE 3



Other notations for [x] are [x] and  $\lfloor x \rfloor$ .







**2** Theorem If  $f(x) \le g(x)$  when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

**E** The Squeeze Theorem If  $f(x) \le g(x) \le h(x)$  when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$
$$\lim_{x \to a} g(x) = L$$

then

The Squeeze Theorem, sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 6. It says that if g(x) is squeezed between f(x) and h(x) near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

**EXAMPLE 10** Show that  $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$ .

**SOLUTION** First note that we *cannot* use

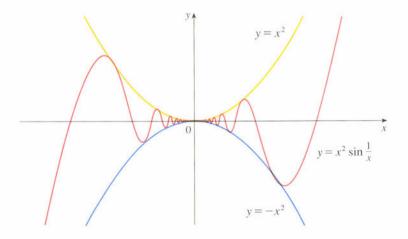
$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because  $\lim_{x\to 0} \sin(1/x)$  does not exist. (See Example 4 in Section 2.2.) However, since

$$-1 \leq \sin \frac{1}{x} \leq 1$$

we have, as illustrated by Figure 7,

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2$$



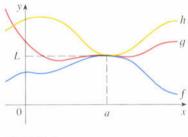


FIGURE 7



We know that

10 C 12 C 12 C

$$\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} -x^2 = 0$$

Taking  $f(x) = -x^2$ ,  $g(x) = x^2 \sin(1/x)$ , and  $h(x) = x^2$  in the Squeeze Theorem, we obtain

. . . . . . . . . . . . .

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

2.3

I. Given that

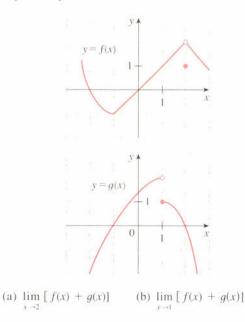
Exercises

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

(a)	$\lim_{x \to a} \left[ f(x) + h(x) \right]$	(b) $\lim_{x \to a} [f(x)]^2$
(c)	$\lim_{x \to a} \sqrt[3]{h(x)}$	(d) $\lim_{x \to a} \frac{1}{f(x)}$
(e)	$\lim_{x \to a} \frac{f(x)}{h(x)}$	(f) $\lim_{x \to a} \frac{g(x)}{f(x)}$
(g)	$\lim_{x \to a} \frac{f(x)}{g(x)}$	(h) $\lim_{x \to a} \frac{2f(x)}{h(x) - f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(c) $\lim_{x \to 0} [f(x)g(x)]$	(d) $\lim_{x \to -1} \frac{f(x)}{g(x)}$
(e) $\lim_{x \to 2} x^3 f(x)$	(f) $\lim_{x \to 1} \sqrt{3 + f(x)}$

**3–7** Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

- **3.**  $\lim_{x \to 4} (5x^2 2x + 3)$  **4.**  $\lim_{x \to -1} \frac{x 2}{x^2 + 4x 3}$
- 5.  $\lim_{t \to -2} (t + 1)^9 (t^2 1)$ 6.  $\lim_{x \to -1} \sqrt{x^3 + 2x + 7}$ 7.  $\lim \sqrt{16 - x^2}$

7.  $\lim_{x \to 4^-} \sqrt{10} = x^2$ 

8. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

9–18 ■ Evaluate the limit, if it exists.

9. 
$$\lim_{x \to -3} \frac{x^2 - x + 12}{x + 3}$$
10. 
$$\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3}$$
11. 
$$\lim_{h \to 0} \frac{(h - 5)^2 - 25}{h}$$
12. 
$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$$
13. 
$$\lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}}$$
14. 
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$$
15. 
$$\lim_{t \to 0} \frac{\sqrt{2 - t} - \sqrt{2}}{t}$$
16. 
$$\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$$
17. 
$$\lim_{x \to 1} \left[\frac{1}{x - 1} - \frac{2}{x^2 - 1}\right]$$
18. 
$$\lim_{h \to 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$$

- **19.** Use the Squeeze Theorem to show that  $\lim_{x\to 0} x^2 \cos 20\pi x = 0$ . Illustrate by graphing the functions  $f(x) = -x^2$ ,  $g(x) = x^2 \cos 20\pi x$ , and  $h(x) = x^2$  on the same screen.
- **20.** Use the Squeeze Theorem to show that  $\lim_{x\to 0} \sqrt{x^3 + x^2} \sin(\pi/x) = 0$ . Illustrate by graphing the functions *f*, *g*, and *h* (in the notation of the Squeeze Theorem) on the same screen.
  - **21.** If  $1 \le f(x) \le x^2 + 2x + 2$  for all x, find  $\lim_{x \to -1} f(x)$ .
  - **22.** If  $3x \le f(x) \le x^3 + 2$  for  $0 \le x \le 2$ , evaluate  $\lim_{x \to 1} f(x)$ .
  - 23. Prove that  $\lim_{x \to 0} x^4 \cos \frac{2}{x} = 0.$
  - 24. Prove that  $\lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0.$

**25–28** ■ Find the limit, if it exists. If the limit does not exist, explain why.

25. 
$$\lim_{x \to -4} |x + 4|$$
  
26.  $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$   
27.  $\lim_{x \to 0^{-}} \left(\frac{1}{x} - \frac{1}{|x|}\right)$   
28.  $\lim_{x \to 0^{+}} \left(\frac{1}{x} - \frac{1}{|x|}\right)$ 

29. Let

$$h(x) = \begin{cases} x & \text{if } x < 0\\ x^2 & \text{if } 0 < x \le 2\\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.
(i) lim h(x)
(ii) lim h(x)
(iii) lim h(x)

(iv) 
$$\lim_{x \to 2^-} h(x)$$
 (v)  $\lim_{x \to 2^+} h(x)$  (vi)  $\lim_{x \to 2^+} h(x)$  (vi)  $\lim_{x \to 2} h(x)$ 

(b) Sketch the graph of h.

**30.** Let 
$$F(x) = \frac{x^2 - 1}{|x - 1|}$$
.  
(a) Find  
(i)  $\lim_{x \to 1^+} F(x)$  (ii)  $\lim_{x \to 1^-} F(x)$ 

- (b) Does  $\lim_{x\to 1} F(x)$  exist?
- (c) Sketch the graph of F.
- 31. (a) If the symbol [[]] denotes the greatest integer function defined in Example 9, evaluate
  (i) lim [[x]]
  (ii) lim [[x]]
  (iii) lim [[x]]
  (iii) lim x→-2.4
  - (b) If *n* is an integer, evaluate (i)  $\lim_{x \to n^{-}} [x]$  (ii)  $\lim_{x \to n^{+}} [x]$
  - (c) For what values of a does  $\lim_{x\to a} [x]$  exist?

- **32.** Let f(x) = x [[x]].
  - (a) Sketch the graph of f.
  - (b) If *n* is an integer, evaluate (i)  $\lim_{x \to n^-} f(x)$  (ii)  $\lim_{x \to n^+} f(x)$

(c) For what values of a does  $\lim_{x\to a} f(x)$  exist?

- **33.** If  $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$ , show that  $\lim_{x \to 2} f(x)$  exists but is not equal to f(2).
- **34.** In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

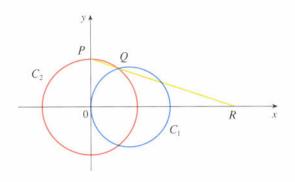
expresses the length L of an object as a function of its velocity v with respect to an observer, where  $L_0$  is the length of the object at rest and c is the speed of light. Find  $\lim_{v\to c^-} L$  and interpret the result. Why is a left-hand limit necessary?

- **35.** If p is a polynomial, show that  $\lim_{x\to a} p(x) = p(a)$ .
- **36.** If r is a rational function, use Exercise 35 to show that  $\lim_{x\to a} r(x) = r(a)$  for every number a in the domain of r.
- 37. Show by means of an example that lim<sub>x→a</sub> [f(x) + g(x)] may exist even though neither lim<sub>x→a</sub> f(x) nor lim<sub>x→a</sub> g(x) exists.
- 38. Show by means of an example that lim<sub>x→a</sub> [ f(x)g(x)] may exist even though neither lim<sub>x→a</sub> f(x) nor lim<sub>x→a</sub> g(x) exists.
- **39.** Is there a number *a* such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

**40.** The figure shows a fixed circle  $C_1$  with equation  $(x - 1)^2 + y^2 = 1$  and a shrinking circle  $C_2$  with radius r and center the origin. P is the point (0, r), Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as  $C_2$  shrinks, that is, as  $r \to 0^+$ ?



Continuity

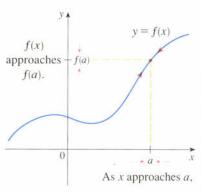


We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

**Definition** A function f is **continuous at a number** a if

 $\lim f(x) = f(a)$ 

As illustrated in Figure I, if f is continuous, then the points (x, f(x)) on the graph of f approach the point (a, f(a)) on the graph. So there is no gap in the curve.





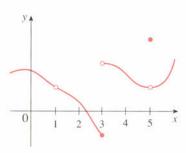


FIGURE 2

If f is not continuous at a, we say f is **discontinuous at** a, or f has a **discontinu**ity at a. Notice that Definition 1 implicitly requires three things if f is continuous at a:

- **1.** f(a) is defined (that is, a is in the domain of f).
- **2.**  $\lim f(x)$  exists (so f must be defined on an open interval that contains a).
- **3.**  $\lim f(x) = f(a)$ .

The definition says that f is continuous at a if f(x) approaches f(a) as x approaches a. Thus, a continuous function f has the property that a small change in x produces only a small change in f(x). In fact, the change in f(x) can be kept as small as we please by keeping the change in x sufficiently small.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But x discontinuities do occur in such situations as electric currents. [See Example 6 in Lectron 2.2, where the Heaviside function is discontinuous at 0 because  $\lim_{t\to 0} H(t)$  does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

**EXAMPLE I** Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

**SOLUTION** It looks as if there is a discontinuity when a = 1 because the graph has a break there. The official reason that f is discontinuous at 1 is that f(1) is not defined.

The graph also has a break when a = 3, but the reason for the discontinuity is different. Here, f(3) is defined, but  $\lim_{x\to 3} f(x)$  does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about a = 5? Here, f(5) is defined and  $\lim_{x\to 5} f(x)$  exists (because the left and right limits are the same). But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Now let's see how to detect discontinuities when a function is defined by a formula.

**EXAMPLE 2** Where are each of the following functions discontinuous?

(a) 
$$f(x) = \frac{x^2 - x - 2}{x - 2}$$
 (b)  $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0\\ 1 & \text{if } x = 0 \end{cases}$   
(c)  $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2\\ 1 & \text{if } x = 2 \end{cases}$  (d)  $f(x) = [x]$ 

# SOLUTION

(a) Notice that f(2) is not defined, so f is discontinuous at 2.

(b) Here f(0) = 1 is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0. (c) Here f(2) = 1 is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function f(x) = [x] has discontinuities at all of the integers because  $\lim_{x\to n} [x]$  does not exist if *n* is an integer. (See Example 9 and Exercise 31 in Section 2.3.)

Figure 3 shows the graphs of the functions in Example 2. In each case the graph cannot be drawn without lifting the pen from the paper because a hole or break or

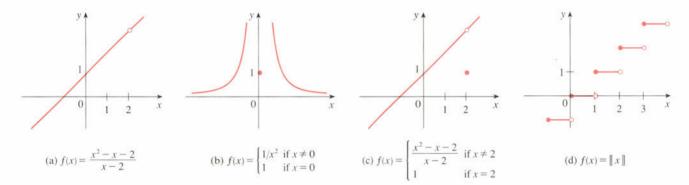


FIGURE 3 Graphs of the functions in Example 2

jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at 2. [The function g(x) = x + 1 is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.

**2** Definition A function f is continuous from the right at a number a if  $\lim_{x \to a^+} f(x) = f(a)$ 

and f is continuous from the left at a if

 $\lim f(x) = f(a)$ 

**EXAMPLE 3** At each integer *n*, the function f(x) = [x] [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but

 $\lim_{x \to n^{-}} f(x) = \lim_{x \to n^{-}} [\![x]\!] = n - 1 \neq f(n)$ 

Definition A function *f* is continuous on an interval if it is continuous at every number in the interval. (At an endpoint of the interval we understand *continuous* to mean *continuous from the right* or *continuous from the left*.)

**EXAMPLE 4** Show that the function  $f(x) = 1 - \sqrt{1 - x^2}$  is continuous on the interval [-1, 1].

**SOLUTION** If -1 < a < 1, then using the Limit Laws, we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left( 1 - \sqrt{1 - x^2} \right)$$
  
=  $1 - \lim_{x \to a} \sqrt{1 - x^2}$  (by Laws 2 and 7)  
=  $1 - \sqrt{\lim_{x \to a} (1 - x^2)}$  (by 11)  
=  $1 - \sqrt{1 - a^2}$  (by 2.7, and 9)  
=  $f(a)$ 

Thus, by Definition 1, f is continuous at a if -1 < a < 1. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on [-1, 1].

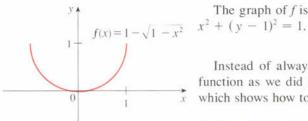


FIGURE 4

The graph of f is sketched in Figure 4. It is the lower half of the circle  $x^2 + (y - 1)^2 = 1$ .

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

**4** Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

**1.** 
$$f + g$$
 **2.**  $f - g$  **3.**  $cf$   
**4.**  $fg$  **5.**  $\frac{f}{g}$  if  $g(a) \neq 0$ 

**Proof** Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

Therefore

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} [f(x) + g(x)]$$
$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law I)}$$
$$= f(a) + g(a)$$
$$= (f + g)(a)$$

This shows that f + g is continuous at a.

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions f + g, f - g, cf, fg, and (if g is never 0) f/g. The following theorem was stated in Section 2.3.

#### 5 Theorem

- (a) Any polynomial is continuous everywhere; that is, it is continuous on  $\mathbb{R} = (-\infty, \infty)$ .
- (b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

#### Proof

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$$

where  $c_0, c_1, \ldots, c_n$  are constants. We know that

$$\lim_{x \to a} c_0 = c_0 \qquad (by \text{ Law 7})$$

and

$$\lim x^m = a^m$$
  $m = 1, 2, ..., n$  (by 9)

This equation is precisely the statement that the function  $f(x) = x^m$  is a continuous function. Thus, by part 3 of Theorem 4, the function  $g(x) = cx^m$  is continuous. Since *P* is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that *P* is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. The domain of *f* is  $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$ . We know from part (a) that *P* and *Q* are continuous everywhere. Thus, by part 5 of Theorem 4, *f* is continuous at every number in *D*.

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula  $V(r) = \frac{4}{3}\pi r^3$  shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet after t seconds is given by the formula  $h = 50t - 16t^2$ . Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2(b) in Section 2.3.

**EXAMPLE 5** Find 
$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$
.

**SOLUTION** The function

6

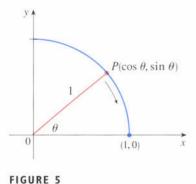
$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is  $\{x \mid x \neq \frac{5}{3}\}$ . Therefore

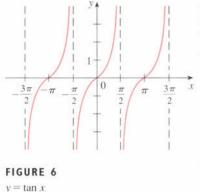
$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$
$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 113) is exactly the statement that root functions are continuous.

From the appearance of the graphs of the sine and cosine functions (Figure 10 in Section 1.2), we would certainly guess that they are continuous. We know from the definition of  $\sin \theta$  and  $\cos \theta$  that the coordinates of the point *P* in Figure 5 are  $(\cos \theta, \sin \theta)$ . As  $\theta \to 0$ , we see that *P* approaches the point (1, 0) and so  $\cos \theta \to 1$  and  $\sin \theta \to 0$ . Therefore



$$\lim_{\theta \to 0} \cos \theta = 1 \qquad \lim_{\theta \to 0} \sin \theta = 0$$



The inverse trigonometric functions are reviewed in Appendix C.

Since  $\cos 0 = 1$  and  $\sin 0 = 0$ , the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 41 and 42).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where  $\cos x = 0$ . This happens when x is an odd integer multiple of  $\pi/2$ , so  $y = \tan x$  has infinite discontinuities when  $x = \pm \pi/2$ ,  $\pm 3\pi/2$ ,  $\pm 5\pi/2$ , and so on (see Figure 6).

The inverse function of any continuous function is also continuous. (The graph of  $f^{-1}$  is obtained by reflecting the graph of f about the line y = x. So if the graph of f has no break in it, neither does the graph of  $f^{-1}$ .) Thus, the inverse trigonometric functions are continuous.

In Section 1.5 we defined the exponential function  $y = a^x$  so as to fill in the holes in the graph of  $y = a^x$  where x is rational. In other words, the very definition of  $y = a^x$  makes it a continuous function on  $\mathbb{R}$ . Therefore, its inverse function  $y = \log_a x$  is continuous on  $(0, \infty)$ .

**Theorem** The following types of functions are continuous at every number in their domains:

polynomials	rational	functions	root functions
trigonometric	c functions	inverse tri	gonometric functions
exponential f	functions	logarithm	ic functions

**EXAMPLE 6** Where is the function  $f(x) = \frac{\ln x + \tan^{-1}x}{x^2 - 1}$  continuous?

**SOLUTION** We know from Theorem 7 that the function  $y = \ln x$  is continuous for x > 0 and  $y = \tan^{-1}x$  is continuous on  $\mathbb{R}$ . Thus, by part 1 of Theorem 4,  $y = \ln x + \tan^{-1}x$  is continuous on  $(0, \infty)$ . The denominator,  $y = x^2 - 1$ , is a polynomial, so it is continuous everywhere. Therefore, by part 5 of Theorem 4, *f* is continuous at all positive numbers *x* except where  $x^2 - 1 = 0$ . So *f* is continuous on the intervals (0, 1) and  $(1, \infty)$ .

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function  $f \circ g$ . This fact is a consequence of the following theorem.

**B Theorem** If f is continuous at b and  $\lim_{x\to a} g(x) = b$ , then  $\lim_{x\to a} f(g(x)) = f(b)$ . In other words,

 $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$ 

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed. Intuitively, this theorem is reasonable because if x is close to a, then g(x) is close to b, and since f is continuous at b, if g(x) is close to b, then f(g(x)) is close to f(b).

**EXAMPLE 7** Evaluate 
$$\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$$
.

**SOLUTION** Because  $\sin^{-1}$  is a continuous function, we can apply Theorem 8:

$$\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}\right)$$
$$= \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)$$
$$= \arcsin\left(\lim_{x \to 1} \frac{1}{1+\sqrt{x}}\right)$$
$$= \arcsin\left(\frac{1}{2} = \frac{\pi}{6}$$

**2** Theorem If g is continuous at a and f is continuous at g(a), then  $(f \circ g)(x) = f(g(x))$  is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

**Proof** Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at b = g(a), we can apply Theorem 8 to obtain

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function h(x) = f(g(x)) is continuous at *a*; that is,  $f \circ g$  is continuous at *a*.

**EXAMPLE 8** Where are the following functions continuous? (a) h(x) = |x| (b)  $F(x) = \ln(1 + \cos x)$ 

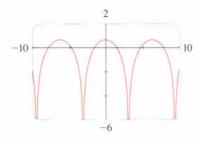
SOLUTION

(a) Since  $|x| = \sqrt{x^2}$  for all x, we have h(x) = f(g(x)), where

 $g(x) = x^2$  and  $f(x) = \sqrt{x}$ 

Now g is continuous on  $\mathbb{R}$  since it is a polynomial and f is continuous on the range of g,  $[0, \infty)$ , because f is a root function. Thus,  $h = f \circ g$  is continuous on  $\mathbb{R}$  by Theorem 9.

(b) We know from Theorem 7 that  $f(x) = \ln x$  is continuous and  $g(x) = 1 + \cos x$  is continuous (because both y = 1 and  $y = \cos x$  are continuous). Therefore, by Theorem 9, F(x) = f(g(x)) is continuous wherever it is defined. Now  $\ln(1 + \cos x)$  is defined when  $1 + \cos x > 0$ . So it is undefined



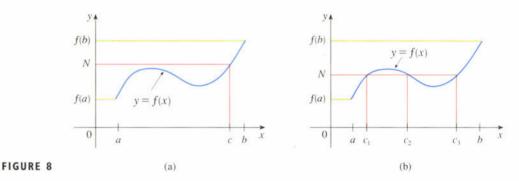


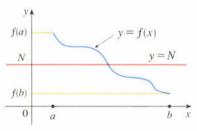
when  $\cos x = -1$ , and this happens when  $x = \pm \pi, \pm 3\pi, \ldots$ . Thus, *F* has discontinuities when *x* is an odd multiple of  $\pi$  and is continuous on the intervals between these values. (See Figure 7.)

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

**[10]** The Intermediate Value Theorem Suppose that f is continuous on the closed interval [a, b] and let N be any number strictly between f(a) and f(b). Then there exists a number c in (a, b) such that f(c) = N.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values f(a) and f(b). It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].





If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line y = N is given between y = f(a) and y = f(b) as in Figure 9, then the graph of f cannot jump over the line. It must intersect y = N somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 30).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

**EXAMPLE 9** Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

**SOLUTION** Let  $f(x) = 4x^3 - 6x^2 + 3x - 2$ . We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that f(c) = 0. Therefore we take a = 1, b = 2, and N = 0 in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$



and

FIGURE 10

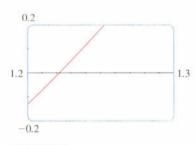
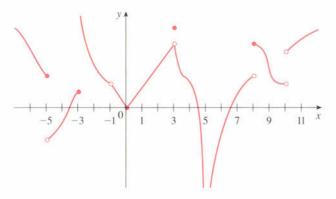


FIGURE II



- **1.** Write an equation that expresses the fact that a function *f* is continuous at the number 4.
- If f is continuous on (-∞,∞), what can you say about its graph?
- **3.** (a) From the graph of *f*, state the numbers at which *f* is discontinuous and explain why.
  - (b) For each of the numbers stated in part (a), determine whether *f* is continuous from the right, or from the left, or neither.



Thus f(1) < 0 < f(2), that is, N = 0 is a number between f(1) and f(2). Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that f(c) = 0. In other words, the equation  $4x^3 - 6x^2 + 3x - 2 = 0$  has at least one root c in the interval (1, 2).

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

f(1.2) = -0.128 < 0 and f(1.3) = 0.548 > 0

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

f(1.22) = -0.007008 < 0 and f(1.23) = 0.056068 > 0

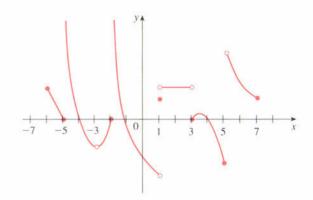
so a root lies in the interval (1.22, 1.23).

We use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 9. Figure 10 shows the graph of f in the viewing rectangle [-1,3] by [-3,3] and you can see the graph crossing the *x*-axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle [1.2, 1.3] by [-0.2, 0.2].

In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

and a remaining of

**4.** From the graph of *g*, state the intervals on which *g* is continuous.



- 5. Sketch the graph of a function that is continuous everywhere except at x = 3 and is continuous from the left at 3.
- Sketch the graph of a function that has a jump discontinuity at x = 2 and a removable discontinuity at x = 4, but is continuous elsewhere.

- **7.** A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
  - (a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.
  - (b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.
- Explain why each function is continuous or discontinuous.
  - (a) The temperature at a specific location as a function of time
  - (b) The temperature at a specific time as a function of the distance due west from New York City
  - (c) The altitude above sea level as a function of the distance due west from New York City
  - (d) The cost of a taxi ride as a function of the distance traveled
  - (e) The current in the circuit for the lights in a room as a function of time
- **9.** Use the definition of continuity and the properties of limits to show that the function

$$g(x) = (x + 1)/(2x^2 - 1)$$

is continuous at the number a = 4.

10. Use the definition of continuity and the properties of limits to show that the function  $f(x) = x\sqrt{16 - x^2}$  is continuous on the interval [-4, 4].

**II-14** Explain why the function is discontinuous at the given point. Sketch the graph of the function.

$$\begin{array}{ll} \mathbf{II.} f(x) = \frac{x^2 - 1}{x + 1} & a = -1 \\ \mathbf{I2.} f(x) = \begin{cases} \frac{x^2 - 1}{x + 1} & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases} & a = -1 \\ \mathbf{I3.} f(x) = \begin{cases} \frac{x^2 - 2x - 8}{x - 4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases} & a = 4 \\ \mathbf{I4.} f(x) = \begin{cases} 1 - x & \text{if } x \leq 2 \\ x^2 - 2x & \text{if } x > 2 \end{cases} & a = 2 \end{array}$$

. . . . . . . . . . .

**15–20** Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

**15.**  $G(x) = \frac{x^4 + 17}{6x^2 + x - 1}$  **16.**  $f(t) = 2t + \sqrt{25 - t^2}$  **17.**  $f(x) = e^x \sin 5x$  **18.**  $F(x) = \sin^{-1}(x^2 - 1)$  **19.**  $G(t) = \ln(t^4 - 1)$ **20.**  $H(x) = \cos(e^{\sqrt{x}})$  21-22 Locate the discontinuities of the function and illustrate by graphing.

**21.** 
$$y = \frac{1}{1 + e^{1/x}}$$
 **22.**  $y = \ln(\tan^2 x)$ 

. . . .

**23–26** ■ Use continuity to evaluate the limit.

**23.** 
$$\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$$
**24.** 
$$\lim_{x \to \pi} \sin(x + \sin x)$$
**25.** 
$$\lim_{x \to 1} e^{x^2 - x}$$
**26.** 
$$\lim_{x \to 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$$

**27.** Find the numbers at which the function

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \le -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x - 1 & \text{if } x \ge 1 \end{cases}$$

is discontinuous. At which of these points is *f* continuous from the right, from the left, or neither? Sketch the graph of *f*.

**28.** The gravitational force exerted by Earth on a unit mass at a distance *r* from the center of the planet is

$$F(r) = \begin{cases} \frac{GMr}{R^3} & \text{if } r < R\\ \frac{GM}{r^2} & \text{if } r \ge R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

**29.** For what value of the constant *c* is the function *f* continuous on  $(-\infty, \infty)$ ?

$$f(x) = \begin{cases} cx+1 & \text{if } x \le 3\\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

- 30. Suppose that a function f is continuous on [0, 1] except at 0.25 and that f(0) = 1 and f(1) = 3. Let N = 2. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).
- **31.** If  $f(x) = x^3 x^2 + x$ , show that there is a number c such that f(c) = 10.
- **32.** Use the Intermediate Value Theorem to prove that there is a positive number *c* such that  $c^2 = 2$ . (This proves the existence of the number  $\sqrt{2}$ .)

**33–36** Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

**33.**  $x^3 - 3x + 1 = 0$ , (0,1)

**34.**  $x^2 = \sqrt{x+1}$ , (1,2) **35.**  $\cos x = x$ , (0,1)

**36.**  $\ln x = e^{-x}$ , (1, 2)

**37–38** (a) Prove that the equation has at least one real root. (b) Use your calculator to find an interval of length 0.01 that contains a root.

**37.** 
$$e^x = 2 - x$$

**38.**  $x^5 - x^2 + 2x + 3 = 0$ 

39-40 (a) Prove that the equation has at least one real root. (b) Use your graphing device to find the root correct to three decimal places.

**39.** 
$$\sqrt{x-5} = \frac{1}{x+3}$$

**40.**  $\arctan x = 1 - x$ 



# **41.** To prove that sin is continuous we need to show that $\lim_{x\to a} \sin x = \sin a$ for every real number *a*. If we let h = x - a, then x = a + h and $x \to a \iff h \to 0$ . So an equivalent statement is that

 $\lim_{h \to 0} \sin(a + h) = \sin a$ 

Use (6) to show that this is true.

- 42. Prove that cosine is a continuous function.
- 43. Is there a number that is exactly one more than its cube?
- **44.** A Tibetan monk leaves the monastery at 7:00 A.M. and takes his usual path to the top of the mountain, arriving at 7:00 P.M. The following morning, he starts at 7:00 A.M. at the top and takes the same path back, arriving at the monastery at 7:00 P.M. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

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Limits Involving Infinity

In this section we investigate the global behavior of functions and, in particular, whether their graphs approach asymptotes, vertical or horizontal.

# Infinite Limits

In Example 8 in Section 2.2 we concluded that

.V	$1/x^{2}$	
±1	1	
-0.5	-1	
±0.2	25	
$\pm 0.1$	100	
±0.05	400	
$\pm (0.01$	10.000	
±0.001	1.000.000	

$$\lim_{x \to 0} \frac{1}{x^2}$$
 does not exist

by observing, from the table of values and the graph of  $y = 1/x^2$  in Figure 1, that the values of  $1/x^2$  can be made arbitrarily large by taking x close enough to 0. Thus, the values of f(x) do not approach a number, so  $\lim_{x\to 0} (1/x^2)$  does not exist.

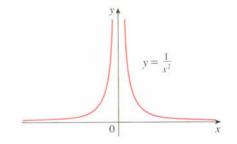


FIGURE I

To indicate this kind of behavior we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding  $\infty$  as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist:  $1/x^2$  can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of f(x) become larger and larger (or "increase without bound") as x approaches a.

**Definition** Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of f(x) can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (but not equal to a).

Another notation for  $\lim_{x\to a} f(x) = \infty$  is

$$f(x) \to \infty$$
 as  $x \to a$ 

Again, the symbol  $\infty$  is not a number, but the expression  $\lim_{x\to a} f(x) = \infty$  is often read as

"the limit of f(x), as x approaches a, is infinity"

"f(x) approaches infinity as x approaches a"

"f(x) increases without bound as x approaches a"

This definition is illustrated graphically in Figure 2. Similarly, as shown in Figure 3,

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of f(x) are as large negative as we like for all values of x that are sufficiently close to a, but not equal to a.

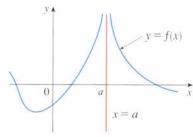
The symbol  $\lim_{x\to a} f(x) = -\infty$  can be read as "the limit of f(x), as x approaches a, is negative infinity" or "f(x) decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left( -\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^{-}} f(x) = \infty \qquad \lim_{x \to a^{+}} f(x) = \infty$$
$$\lim_{x \to a^{-}} f(x) = -\infty \qquad \lim_{x \to a^{+}} f(x) = -\infty$$

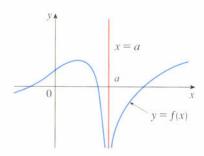
A more precise version of Definition I is given in Appendix D.



or

OF

**FIGURE 2**  $\lim_{x \to \infty} f(x) = \infty$ 





remembering that " $x \to a^{-}$ " means that we consider only values of x that are less than a, and similarly " $x \to a^{+}$ " means that we consider only x > a. Illustrations of these four cases are given in Figure 4.

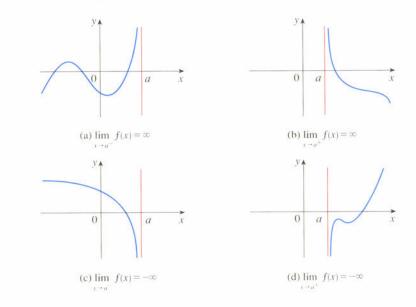


FIGURE 4

**2** Definition The line x = a is called a vertical asymptote of the curve y = f(x) if at least one of the following statements is true:

$\lim_{x\to a}f(x)=\infty$	$\lim_{x \to a^-} f(x) = \infty$	$\lim_{x\to a^+} f(x) = \infty$
$\lim_{x\to a} f(x) = -\infty$	$\lim_{x \to a^{-1}} f(x) = -\infty$	$\lim_{x \to a^+} f(x) = -\infty$

For instance, the y-axis is a vertical asymptote of the curve  $y = 1/x^2$  because  $\lim_{x\to 0} (1/x^2) = \infty$ . In Figure 4 the line x = a is a vertical asymptote in each of the four cases shown.

**EXAMPLE 1** Find 
$$\lim_{x \to 3^+} \frac{2}{x-3}$$
 and  $\lim_{x \to 3^-} \frac{2}{x-3}$ 

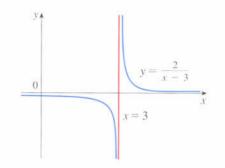
**SOLUTION** If x is close to 3 but larger than 3, then the denominator x - 3 is a small positive number and so 2/(x - 3) is a large positive number. Thus, intuitively we see that

$$\lim_{x \to 3^+} \frac{2}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then x - 3 is a small negative number and so 2/(x - 3) is a numerically large negative number. Thus

$$\lim_{x \to 3^-} \frac{2}{x-3} = -\infty$$

The graph of the curve y = 2/(x - 3) is given in Figure 5. The line x = 3 is a vertical asymptote.





1

 $y = \ln x$ 

Two familar functions whose graphs have vertical asymptotes are  $y = \tan x$ and  $y = \ln x$ . From Figure 6 we see that

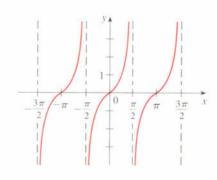


$$\lim_{x \to 0^+} \ln x = -\infty$$

and so the line x = 0 (the y-axis) is a vertical asymptote. In fact, the same is true for  $y = \log_a x$  provided that a > 1. (See Figures 13 and 14 in Section 1.6.) Figure 7 shows that

$$\lim_{x \to (\pi/2)^{-}} \tan x = \infty$$

and so the line  $x = \pi/2$  is a vertical asymptote. In fact, the lines  $x = (2n + 1)\pi/2$ , *n* an integer, are all vertical asymptotes of  $y = \tan x$ .





 $y = \tan x$ 

**EXAMPLE 2** Find  $\lim_{x\to 0} \ln(\tan^2 x)$ .

**SOLUTION** We introduce a new variable,  $t = \tan^2 x$ . Then  $t \ge 0$  and  $t = \tan^2 x \rightarrow \tan^2 0 = 0$  as  $x \rightarrow 0$  because tan is a continuous function. So, by (3), we have

The problem-solving strategy for Example 2 is *Introduce Something Extra* (see page 88). Here, the something extra, the auxiliary aid, is the new variable *t*.

$$\lim_{x \to 0} \ln(\tan^2 x) = \lim_{t \to 0^+} \ln t = -\infty$$

FIGURE 6

34

0

# Limits at Infinity

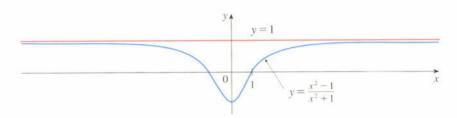
In computing infinite limits, we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). Here we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behavior of the function f defined by

f

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table in the margin gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 8.



#### FIGURE 8

As x grows larger and larger you can see that the values of f(x) get closer and closer to 1. In fact, it seems that we can make the values of f(x) as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the symbolism

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of f(x) approach L as x becomes larger and larger.

**4** Definition Let f be a function defined on some interval  $(a, \infty)$ . Then

$$\lim f(x) = L$$

means that the values of f(x) can be made as close to L as we like by taking x sufficiently large.

Another notation for  $\lim_{x\to\infty} f(x) = L$  is

$$f(x) \to L$$
 as  $x \to \infty$ 

The symbol  $\infty$  does not represent a number. Nonetheless, the expression  $\lim_{x\to\infty} f(x) = L$  is often read as

"the limit of f(x), as x approaches infinity, is L"

X	f(x)	
0	-1	
±1	0	
±2	0.600000	
+.3	0,800000	
+ 4	0.882353	
±5	0.923077	
$\pm 10$	0.980198	
$\pm 50$	0.999200	
$\pm 100$	0.999800	
$\pm 1000$	0.999998	

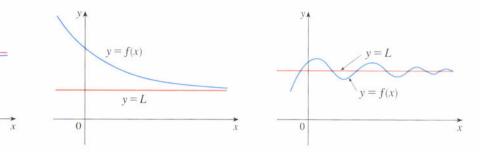
A more precise version of Definition 4 is given in Appendix D.

or "the limit of f(x), as x becomes infinite, is L"

or "the limit of f(x), as x increases without bound, is L"

The meaning of such phrases is given by Definition 4.

Geometric illustrations of Definition 4 are shown in Figure 9. Notice that there are many ways for the graph of f to approach the line y = L (which is called a *horizontal asymptote*).





v = L

y = f(x)

y

0

Referring back to Figure 8, we see that for numerically large negative values of x, the values of f(x) are close to 1. By letting x decrease through negative values without bound, we can make f(x) as close to 1 as we like. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, as shown in Figure 10, the notation

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of f(x) are as close to L as we like by taking x sufficiently large negative.

Again, the symbol  $-\infty$  does not represent a number, but the expression  $\lim_{x\to-\infty} f(x) = L$  is often read as

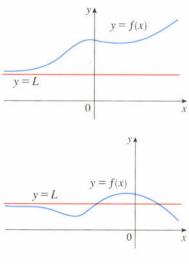
"the limit of f(x), as x approaches negative infinity, is L"

**5** Definition The line y = L is called a horizontal asymptote of the curve y = f(x) if either

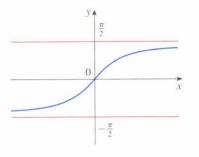
$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

For instance, the curve illustrated in Figure 8 has the line y = 1 as a horizontal asymptote because

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$



**FIGURE 10** Examples illustrating  $\lim f(x) = L$ 



5



An example of a curve with two horizontal asymptotes is  $y = \tan^{-1}x$  (see Figure 11). In fact,

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

so both of the lines  $y = -\pi/2$  and  $y = \pi/2$  are horizontal asymptotes. (This follows from the fact that the lines  $x = \pm \pi/2$  are vertical asymptotes of the graph of tan.)

**EXAMPLE 3** Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 12.

**SOLUTION** We see that the values of f(x) become large as  $x \to -1$  from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that f(x) becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^{-}} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^{+}} f(x) = \infty$$

Thus, both of the lines x = -1 and x = 2 are vertical asymptotes.

As x becomes large, we see that f(x) approaches 4. But as x decreases through negative values, f(x) approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 2$$

This means that both y = 4 and y = 2 are horizontal asymptotes.

**EXAMPLE 4** Find 
$$\lim_{x \to \infty} \frac{1}{x}$$
 and  $\lim_{x \to -\infty} \frac{1}{x}$ .

**SOLUTION** Observe that when x is large, 1/x is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10,000} = 0.0001 \qquad \frac{1}{1,000,000} = 0.000001$$

In fact, by taking x large enough, we can make 1/x as close to 0 as we please. Therefore, according to Definition 4, we have

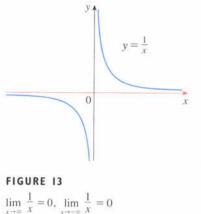
$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, 1/x is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$



2



It follows that the line y = 0 (the x-axis) is a horizontal asymptote of the curve y = 1/x. (This is an equilateral hyperbola; see Figure 13.)

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the *Limit Laws listed in Section 2.3 (with the exception of Laws 8, 9, and 10) are also valid if* " $x \rightarrow a$ " is replaced by " $x \rightarrow \infty$ " or " $x \rightarrow -\infty$ ." In particular, if we combine Law 6 with the results of Example 4 we obtain the following important rule for calculating limits.

**7** If *n* is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \qquad \lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

**EXAMPLE 5** Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

**SOLUTION** To evaluate the limit at infinity of a rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that  $x \neq 0$ , since we are interested only in large values of x.) In this case the highest power of x is  $x^2$  and so, using the Limit Laws, we have

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\frac{3x^2 - x - 2}{x^2}}{\frac{5x^2 + 4x + 1}{x^2}} = \lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$
$$= \frac{\lim_{x \to \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2\lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 5 + 4\lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}}{\frac{1}{x^2}}$$
$$= \frac{3 - 0 - 0}{5 + 0 + 0} \quad \text{(by (6))}$$
$$= \frac{3}{5}$$

A similar calculation shows that the limit as  $x \to -\infty$  is also  $\frac{3}{5}$ . Figure 14 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote  $y = \frac{3}{5}$ .

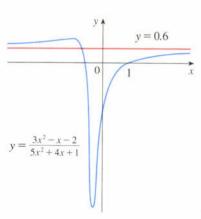


FIGURE 14

**EXAMPLE 6** Compute  $\lim_{x \to \infty} (\sqrt{x^2 + 1} - x)$ .

SOLUTION We first multiply numerator and denominator by the conjugate radical:

$$\lim_{x \to \infty} \left( \sqrt{x^2 + 1} - x \right) = \lim_{x \to \infty} \left( \sqrt{x^2 + 1} - x \right) \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x}$$
$$= \lim_{x \to \infty} \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x}$$

The Squeeze Theorem could be used to show that this limit is 0. But an easier method is to divide numerator and denominator by x. Doing this and remembering that  $x = \sqrt{x^2}$  for x > 0, we obtain

$$\lim_{x \to \infty} \left( \sqrt{x^2 + 1} - x \right) = \lim_{x \to \infty} \frac{\frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}} + 1}$$
$$= \frac{0}{\sqrt{1 + 0} + 1} = 0$$

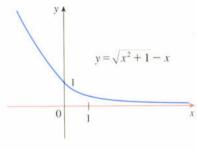
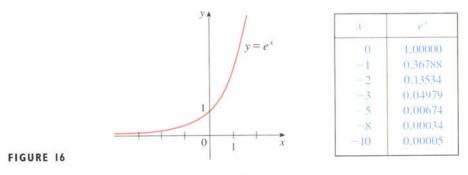


Figure 15 illustrates this result.

The graph of the natural exponential function  $y = e^x$  has the line y = 0 (the *x*-axis) as a horizontal asymptote. (The same is true of any exponential function with base a > 1.) In fact, from the graph in Figure 16 and the corresponding table of values, we see that

$$\lim_{x \to -\infty} e^x = 0$$

Notice that the values of  $e^x$  approach 0 very rapidly.



**EXAMPLE 7** Evaluate  $\lim_{x \to 0^-} e^{1/x}$ .

**SOLUTION** If we let t = 1/x, we know from Example 4 that  $t \to -\infty$  as  $x \to 0^-$ . Therefore, by (8),

$$\lim_{x \to 0^{-}} e^{1/x} = \lim_{t \to -\infty} e^{t} = 0$$



**EXAMPLE 8** Evaluate  $\lim \sin x$ .

**SOLUTION** As x increases, the values of sin x oscillate between 1 and -1 infinitely often. Thus,  $\lim_{x\to\infty} \sin x$  does not exist.

#### Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of f(x) become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

From Figures 16 and 17 we see that

$$\lim_{x \to \infty} e^x = \infty \qquad \lim_{x \to \infty} x^3 = \infty \qquad \lim_{x \to -\infty} x^3 = -\infty$$

but, as Figure 18 demonstrates,  $y = e^x$  becomes large as  $x \to \infty$  at a much faster rate than  $y = x^3$ .

**EXAMPLE 9** Find  $\lim_{x \to \infty} (x^2 - x)$ .

**SOLUTION** Note that we *cannot* write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x^2$$

The Limit Laws cannot be applied to infinite limits because  $\infty$  is not a number  $(\infty - \infty$  cannot be defined). However, we can write

 $\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$ 

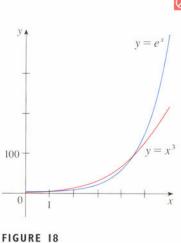
because both x and x - 1 become arbitrarily large.

**EXAMPLE 10** Find 
$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x}$$
.

**SOLUTION** We divide numerator and denominator by x (the highest power of xthat occurs in the denominator):

$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x} = \lim_{x \to \infty} \frac{x + 1}{\frac{3}{x} - 1} = -\infty$$

because  $x + 1 \rightarrow \infty$  and  $3/x - 1 \rightarrow -1$  as  $x \rightarrow \infty$ .



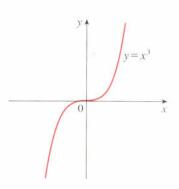
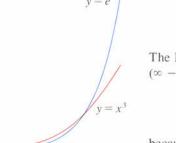


FIGURE 17





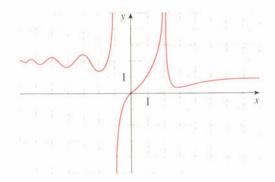
Exercises •

2.5

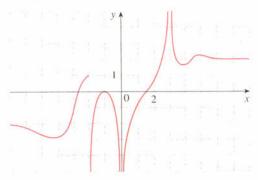
 Explain in your own words the meaning of each of the following.

(a) 
$$\lim_{x \to 2} f(x) = \infty$$
 (b)  $\lim_{x \to 1^+} f(x) = -\infty$   
(c)  $\lim_{x \to \infty} f(x) = 5$  (d)  $\lim_{x \to -\infty} f(x) = 3$ 

- (a) Can the graph of y = f(x) intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
  - (b) How many horizontal asymptotes can the graph of y = f(x) have? Sketch graphs to illustrate the possibilities.
- For the function f whose graph is given, state the following.
  - (a)  $\lim_{x \to 2} f(x)$  (b)  $\lim_{x \to -1^-} f(x)$ (c)  $\lim_{x \to -1^+} f(x)$  (d)  $\lim_{x \to \infty} f(x)$
  - (e)  $\lim_{x \to -\infty} f(x)$  (f) The equations of the asymptotes



- **4.** For the function *g* whose graph is given, state the following.
  - (a)  $\lim_{x \to \infty} g(x)$  (b)  $\lim_{x \to -\infty} g(x)$
  - (c)  $\lim g(x)$  (d)  $\lim g(x)$
  - $x \rightarrow 3$   $(x \rightarrow 0)$   $(x \rightarrow 0)$
  - (e)  $\lim_{x \to -2^+} g(x)$  (f) The equations of the asymptotes



**5–8** Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. f(0) = 0, f(1) = 1,  $\lim f(x) = 0$ , f is odd

and a second second second

- 6.  $\lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = 1,$  $\lim_{x \to -\infty} f(x) = 1$
- 7.  $\lim_{x \to 2} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = 0,$  $\lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = -\infty$
- **8.**  $\lim_{x \to -2} f(x) = \infty$ ,  $\lim_{x \to -\infty} f(x) = 3$ ,  $\lim_{x \to \infty} f(x) = -3$
- 9. Guess the value of the limit

$$\lim_{x\to\infty}\frac{x^2}{2^x}$$

by evaluating the function  $f(x) = x^2/2^x$  for x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50, and 100.

- **10.** Determine  $\lim_{x \to 1^{-}} \frac{1}{x^3 1}$  and  $\lim_{x \to 1^{+}} \frac{1}{x^3 1}$ 
  - (a) by evaluating  $f(x) = 1/(x^3 1)$  for values of x that approach 1 from the left and from the right,
  - (b) by reasoning as in Example 1,

(c) from a graph of f.

11-12 Use a graph to estimate the value of the limit.

**11.**  $\lim_{x \to \infty} \frac{3x^2}{2x^2 + 25\sin x}$  **12.**  $\lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^x$ 

I3-14 ■ Use a graph to find all the vertical and horizontal asymptotes of the curve.

**13.** 
$$y = \frac{x^3}{x^3 - 2x + 1}$$

14.  $y = \tan(2\sin x), \quad -\pi \le x \le \pi$ 

**15–29** ■ Find the limit.

- **15.**  $\lim_{x \to 3} \frac{1}{(x-3)^8}$  **16.**  $\lim_{x \to \pi^-} \csc x$  **17.**  $\lim_{x \to -2^+} \frac{x-1}{x^2(x+2)}$  **18.**  $\lim_{x \to 5^+} \ln(x-5)$  **19.**  $\lim_{x \to \infty} \frac{x+4}{x^2-2x+5}$  **20.**  $\lim_{t \to \infty} \frac{7t^3+4t}{2t^3-t^2+3}$
- **21.**  $\lim_{t \to -\infty} \frac{6t^2 + 5t}{(1-t)(2t-3)}$  **22.**  $\lim_{x \to \infty} \frac{\sqrt{1+4x^2}}{4+x}$

- **23.**  $\lim_{x \to \infty} (\sqrt{x^2 + 3x + 1} x)$  **24.**  $\lim_{x \to \infty} \frac{\sin^2 x}{x^2}$
- **25.**  $\lim_{x \to \infty} \cos x$
- **27.**  $\lim_{x \to \infty} \frac{x^7 1}{x^6 + 1}$
- **29.**  $\lim_{x \to \infty} \sin \frac{1}{x}$
- **30.** (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

**26.**  $\lim \tan^{-1}(x^4 - x^2)$ 

 $\chi \rightarrow \infty$ 

**28.** lim  $e^{\tan x}$ 

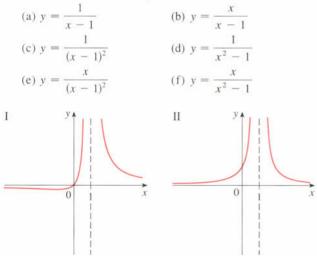
 $x \rightarrow (\pi/2)^+$ 

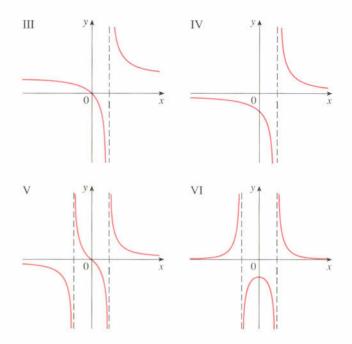
$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

- (b) By calculating values of f(x), give numerical estimates of the limits in part (a).
- (c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]
- 31-32 Find the horizontal and vertical asymptotes of each curve. Check your work by graphing the curve and estimating the asymptotes.

**31.** 
$$y = \frac{x^2 + 4}{x^2 - 1}$$
 **32.**  $y = \frac{x - 9}{\sqrt{4x^2 + 3x + 2}}$ 

**33.** Match each function in (a)–(f) with its graph (labeled I–VI). Give reasons for your choices.





- **34.** Find a formula for a function that has vertical asymptotes x = 1 and x = 3 and horizontal asymptote y = 1.
- **35.** Find a formula for a function *f* that satisfies the following conditions:  $\lim_{x \to \infty} f(x) = 0 \quad \lim_{x \to \infty} f(x) = -\infty \quad f(2) = 0$

$$\lim_{x \to \pm \infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$$
$$\lim_{x \to 3^{-}} f(x) = \infty, \quad \lim_{x \to 3^{+}} f(x) = -\infty$$

- 36. By the *end behavior* of a function we mean a description of what happens to its values as x → ∞ and as x → -∞.
  - (a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x$$
  $Q(x) = 3x^5$ 

by graphing both functions in the viewing rectangles [-2, 2] by [-2, 2] and [-10, 10] by [-10,000, 10,000].

- (b) Two functions are said to have the same end behavior if their ratio approaches 1 as x → ∞. Show that P and Q have the same end behavior.
- **37.** Let P and Q be polynomials. Find

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

**38.** Make a rough sketch of the curve  $y = x^n$  (*n* an integer) for the following five cases:

(i) 
$$n = 0$$
 (ii)  $n > 0, n$  odd

- (iii) n > 0, n even (iv) n < 0, n odd
- (v) n < 0, n even

Then use these sketches to find the following limits.

- (a)  $\lim_{x\to 0^+} x^n$  (b)  $\lim_{x\to 0^-} x^n$ (c)  $\lim_{x\to\infty} x^n$  (d)  $\lim_{x\to-\infty} x^n$
- **39.** Find  $\lim_{x\to\infty} f(x)$  if

$$\frac{4x-1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all x > 5.

**40.** In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where  $m_0$  is the rest mass of the particle and c is the speed of light. What happens as  $v \rightarrow c^{-2}$ ?

**41.** (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after *t* minutes (in grams per liter) is

$$C(t) = \frac{30t}{200+t}$$

(b) What happens to the concentration as  $t \to \infty$ ?

**42.** In Chapter 7 we will be able to show, under certain assumptions, that the velocity v(t) of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and  $v^*$  is the terminal velocity of the raindrop.

(a) Find  $\lim_{t\to\infty} v(t)$ .

(b) Graph v(t) if  $v^* = 1$  m/s and g = 9.8 m/s<sup>2</sup>. How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

**43.** (a) Show that  $\lim_{x\to\infty} e^{-x/10} = 0$ .

- (b) By graphing  $y = e^{-x/10}$  and y = 0.1 on a common screen, discover how large you need to make x so that  $e^{-x/10} < 0.1$ .
  - (c) Can you solve part (b) without using a graphing device?

**44.** (a) Show that 
$$\lim_{x \to \infty} \frac{4x^2 - 5x}{2x^2 + 1} = 2.$$

(b) By graphing the function in part (a) and the line y = 1.9 on a common screen, find a number N such that

$$\frac{4x^2 - 5x}{2x^2 + 1} > 1.9 \quad \text{when} \quad x > N$$

What if 1.9 is replaced by 1.99?



# Tangents, Velocities, and Other Rates of Change

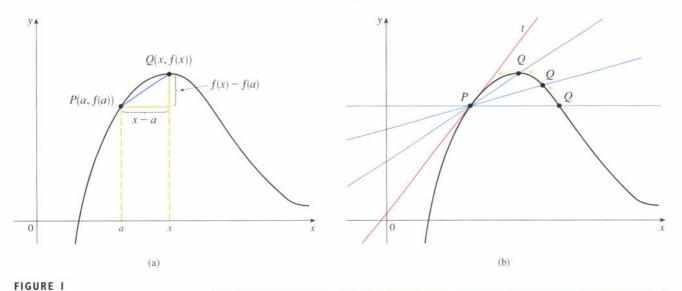
In Section 2.1 we guessed the values of slopes of tangent lines and velocities on the basis of numerical evidence. Now that we have defined limits and have learned techniques for computing them, we return to the tangent and velocity problems with the ability to calculate slopes of tangents, velocities, and other rates of change.

# Tangents

If a curve C has equation y = f(x) and we want to find the tangent to C at the point P(a, f(a)), then we consider a nearby point Q(x, f(x)), where  $x \neq a$ , and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If  $m_{PQ}$  approaches a number m, then we define the *tangent t* to be the line through P with



slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

**Definition** The **tangent line** to the curve y = f(x) at the point P(a, f(a)) is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 1 in Section 2.1.

**EXAMPLE I** Find an equation of the tangent line to the parabola  $y = x^2$  at the point P(1, 1).

**SOLUTION** Here we have a = 1 and  $f(x) = x^2$ , so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$
$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$
$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) is

$$y - 1 = 2(x - 1)$$
 or  $y = 2x - 1$ 

We sometimes refer to the slope of the tangent line to a curve at a point as the *slope of the curve* at the point. The idea is that if we zoom in far enough toward the

Point-slope form for a line through the point  $(x_1, y_1)$  with slope *m*:

 $y - y_1 = m(x - x_1)$ 

point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve  $y = x^2$  in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.



# **FIGURE 2** Zooming in toward the point (1, 1) on the parabola $y = x^2$

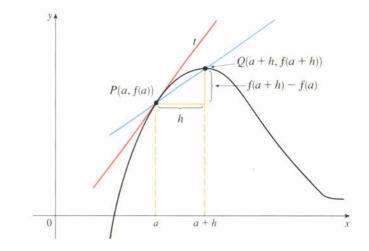
There is another expression for the slope of a tangent line that is sometimes easier to use. Let

$$h = x - a$$
  
Then  $x = a + h$ 

so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case h > 0 is illustrated and Q is to the right of P. If h < 0, however, Q would be to the left of P.)



#### FIGURE 3

Notice that as x approaches a, h approaches 0 and so the expression for the slope of the tangent line in Definition 1 becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

**EXAMPLE 2** Find an equation of the tangent line to the hyperbola y = 3/x at the point (3, 1).

**SOLUTION** Let f(x) = 3/x. Then the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{3 - (3+h)}{3+h}}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h}$$
$$= -\frac{1}{3}$$

Therefore, an equation of the tangent at the point (3, 1) is

$$y - 1 = -\frac{1}{3}(x - 3)$$

which simplifies to x + 3y - 6 = 0

The hyperbola and its tangent are shown in Figure 4.

# Velocities

3

In Section 2.1 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion s = f(t), where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from t = a to t = a + h the change in position is f(a + h) - f(a) (see Figure 5). The average velocity over this time interval is

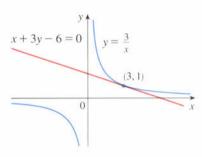
average velocity = 
$$\frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in Figure 6.

Now suppose we compute the average velocities over shorter and shorter time intervals [a, a + h]. In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) v(a) at time t = a to be the limit of these average velocities:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This means that the velocity at time t = a is equal to the slope of the tangent line at *P*. (Compare Equations 2 and 3.)



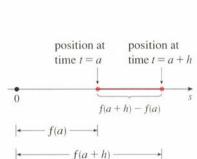
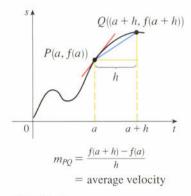




FIGURE 4





Now that we know how to compute limits, let's reconsider the problem of the falling ball.

**EXAMPLE 3** Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

- (a) What is the velocity of the ball after 5 seconds?
- (b) How fast is the ball traveling when it hits the ground?

**SOLUTION** We first use the equation of motion  $s = f(t) = 4.9t^2$  to find the velocity v(a) after a seconds:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h}$$
$$= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}$$
$$= \lim_{h \to 0} 4.9(2a+h) = 9.8a$$

(a) The velocity after 5 s is v(5) = (9.8)(5) = 49 m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time  $t_1$  when  $s(t_1) = 450$ , that is,

$$4.9t_1^2 = 450$$

This gives

$$t_1^2 = \frac{450}{4.9}$$
 and  $t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$ 

The velocity of the ball as it hits the ground is therefore

$$v(t_1) = 9.8t_1 = 9.8 \sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$

#### **N** 0

# Other Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus, y is a function of x and we write y = f(x). If x changes from  $x_1$  to  $x_2$ , then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

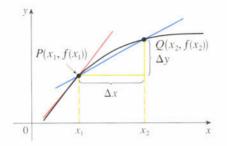
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Recall from Section 2.1: The distance (in meters) fallen after t seconds is  $4.9t^2$ .

is called the **average rate of change of** y **with respect to** x **over the interval**  $[x_1, x_2]$  and can be interpreted as the slope of the secant line PQ in Figure 7.



average rate of change =  $m_{PQ}$ instantaneous rate of change = slope of tangent at *P* 

#### FIGURE 7

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting  $x_2$  approach  $x_1$  and therefore letting  $\Delta x$  approach 0. The limit of these average rates of change is called the (**instantaneous**) rate of change of y with respect to x at  $x = x_1$ , which is interpreted as the slope of the tangent to the curve y = f(x) at  $P(x_1, f(x_1))$ :

4 instantaneous rate of change = 
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
  
=  $\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ 

x (h)	$T(^{\circ}C)$	<i>x</i> (h)	$T(^{\circ}C)$
0	6.5	13	16.0
1	6.1	14	17.3
1 2 3	5.6	15	18.2
3	4.9	16	18.8
4	4.2	17	17.6
5	4.0	18	16.0
6	4.0	19	14.1
7	4.8	20	11.5
8	6.1	21	10.2
9	8.3	22	9.0
10	10.0	23	7.9
11	12.1	24	7.0
12	14.3		

#### A Note on Units

The units for the average rate of change  $\Delta T/\Delta x$  are the units for  $\Delta T$  divided by the units for  $\Delta x$ , namely, degrees Celsius per hour. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: degrees Celsius per hour.

**EXAMPLE 4** Temperature readings T (in degrees Celsius) were recorded every hour starting at midnight on a day in April in Whitefish, Montana. The time x is measured in hours from midnight. The data are given in the table at the left. (a) Find the average rate of change of temperature with respect to time

- (i) from noon to 3 P.M. (ii) from noon to 2 P.M.
- (iii) from noon to 1 P.M.

(b) Estimate the instantaneous rate of change at noon.

#### SOLUTION

(a) (i) From noon to 3 P.M. the temperature changes from 14.3 °C to 18.2 °C, so

$$\Delta T = T(15) - T(12) = 18.2 - 14.3 = 3.9 \,^{\circ}\text{C}$$

while the change in time is  $\Delta x = 3$  h. Therefore, the average rate of change of temperature with respect to time is

$$\frac{\Delta T}{\Delta x} = \frac{3.9}{3} = 1.3 \,^{\circ}\text{C/h}$$

(ii) From noon to 2 P.M. the average rate of change is

$$\frac{\Delta T}{\Delta x} = \frac{T(14) - T(12)}{14 - 12} = \frac{17.3 - 14.3}{2} = 1.5 \,^{\circ}\text{C/h}$$

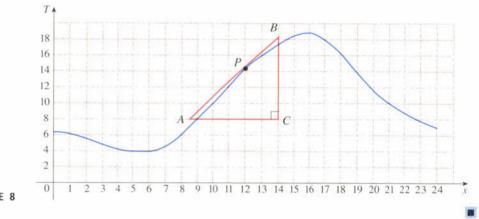
(iii) From noon to 1 P.M. the average rate of change is

$$\frac{\Delta T}{\Delta x} = \frac{T(13) - T(12)}{13 - 12} = \frac{16.0 - 14.3}{1} = 1.7 \,^{\circ}\text{C/h}$$

(b) We plot the given data in Figure 8 and use them to sketch a smooth curve that approximates the graph of the temperature function. Then we draw the tangent at the point P where x = 12 and, after measuring the sides of triangle *ABC*, we estimate that the slope of the tangent line is

$$\frac{|BC|}{|AC|} = \frac{10.3}{5.5} \approx 1.9$$

Therefore, the instantaneous rate of change of temperature with respect to time at noon is about 1.9 °C/h.



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FIGURE 8
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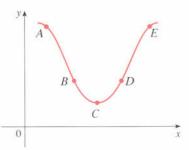
The velocity of a particle is the rate of change of displacement with respect to time. Physicists are interested in other rates of change as well—for instance, the rate of change of work with respect to time (which is called power). Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the rate of reaction). A steel manufacturer is interested in the rate of change of the cost of producing x tons of steel per day with respect to x (called the marginal cost). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.3.

All these rates of change can be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering. 2.6

**I.** A curve has equation y = f(x).

Exercises · · · ·

- (a) Write an expression for the slope of the secant line through the points P(3, f(3)) and Q(x, f(x)).
- (b) Write an expression for the slope of the tangent line at *P*.
- **2.** Suppose an object moves with position function s = f(t).
  - (a) Write an expression for the average velocity of the object in the time interval from t = a to t = a + h.
  - (b) Write an expression for the instantaneous velocity at time t = a.
- **3.** Consider the slope of the given curve at each of the five points shown. List these five slopes in decreasing order and explain your reasoning.



- Graph the curve y = e<sup>x</sup> in the viewing rectangles
   [-1,1] by [0,2], [-0.5,0.5] by [0.5,1.5], and [-0.1,0.1] by [0.9,1.1]. What do you notice about the curve as you zoom in toward the point (0,1)?
  - 5. (a) Find the slope of the tangent line to the parabola  $y = x^2 + 2x$  at the point (-3, 3)
    - (i) using Definition 1 (ii) using Equation 2
    - (b) Find the equation of the tangent line in part (a).
  - (c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point (-3, 3) until the parabola and the tangent line are indistinguishable.
  - 6. (a) Find the slope of the tangent line to the curve y = x<sup>3</sup> at the point (-1, -1)
    (i) using Definition 1 (ii) using Equation 2
    - (b) Find the equation of the tangent line in part (a).
  - (c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at (-1, -1) until the curve and the line appear to coincide.

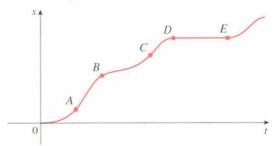
**7–9** Find the equation of the tangent line to the curve at the given point.

**7.** 
$$y = \sqrt{x}$$
, (1,1)  
**8.**  $y = x/(1-x)$ , (0,0)

9.  $y = 1/x^2$ ,  $(-2, \frac{1}{4})$ 

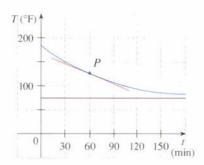
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- **10.** (a) Find the slope of the tangent to the parabola  $y = 1 + x + x^2$  at the point where x = a.
  - (b) Find the slopes of the tangent lines at the points whose *x*-coordinates are (i) -1, (ii)  $-\frac{1}{2}$ , and (iii) 1.
  - (c) Graph the curve and the three tangents on a common screen.
- (a) Find the slope of the tangent to the curve  $y = x^3 4x + 1$  at the point where x = a.
  - (b) Find the equations of the tangent lines at the points (1, −2) and (2, 1).
  - (c) Graph the curve and both tangents on a common screen.
- **12.** (a) Find the slope of the tangent to the curve  $y = 1/\sqrt{5-2x}$  at the point where x = a.
  - (b) Find the equations of the tangent lines at the points (2, 1) and (-2, <sup>1</sup>/<sub>3</sub>).
  - (c) Graph the curve and both tangents on a common screen.
  - **13.** The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.
    - (a) What was the initial velocity of the car?
    - (b) Was the car going faster at B or at C?
    - (c) Was the car slowing down or speeding up at *A*, *B*, and *C*?
    - (d) What happened between D and E?



14. Valerie is driving along a highway. Sketch the graph of the position function of her car if she drives in the following manner: At time t = 0 min, the car is at mile marker 15 and is traveling at a constant speed of 55 mi/h. She travels at this speed for exactly an hour. Then the car slows gradually over a two-minute period as Valerie comes to a stop for dinner. Dinner lasts 26 min; then she restarts the car, gradually speeding up to 65 mi/h over a two-minute period. She drives at a constant 65 mi/h for two hours and then over a threeminute period gradually slows to a complete stop.

- **15.** If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by  $y = 40t 16t^2$ . Find the velocity when t = 2.
- 16. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by  $H = 58t 0.83t^2$ .
  - (a) Find the velocity of the arrow after 1 s.
  - (b) Find the velocity of the arrow when t = a.
  - (c) When will the arrow hit the moon?
  - (d) With what velocity will the arrow hit the moon?
- 17. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion  $s = 4t^3 + 6t + 2$ , where t is measured in seconds. Find the velocity of the particle at times t = a, t = 1, t = 2, and t = 3.
- 18. The displacement (in meters) of a particle moving in a straight line is given by  $s = t^2 8t + 18$ , where t is measured in seconds.
  - (a) Find the average velocities over the following time intervals:
    - (i) [3,4] (ii) [3.5,4]
    - (iii) [4,5] (iv) [4,4.5]
  - (b) Find the instantaneous velocity when t = 4.
  - (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).
- **19.** A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?
- **20.** A roast turkey is taken from an oven when its temperature has reached 185 °F and is placed on a table in a room where the temperature is 75 °F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. (In Section 7.5 we will be able to use Newton's Law of Cooling to find an equation for *T* as a function of time.) By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



- **21.** (a) Use the data in Example 4 to find the average rate of change of temperature with respect to time
  - (i) from 8 p.m. to 11 p.m.
  - (ii) from 8 p.m. to 10 p.m.
  - (iii) from 8 p.m. to 9 p.m.
  - (b) Estimate the instantaneous rate of change of *T* with respect to time at 8 P.M. by measuring the slope of a tangent.
- **22.** The population *P* (in thousands) of the city of San Jose, California, from 1984 to 1994 is given in the table.

Year	1984	1986	1988	1990	1992	1994
P	695	716	733	782	800	817

- (a) Find the average rate of growth
  - (i) from 1986 to 1992
  - (ii) from 1988 to 1992
  - (iii) from 1990 to 1992
  - (iv) from 1992 to 1994
  - In each case, include the units.
- (b) Estimate the instantaneous rate of growth in 1992 by taking the average of two average rates of change. What are its units?
- (c) Estimate the instantaneous rate of growth in 1992 by measuring the slope of a tangent.
- **23.** The cost (in dollars) of producing x units of a certain commodity is  $C(x) = 5000 + 10x + 0.05x^2$ .
  - (a) Find the average rate of change of *C* with respect to *x* when the production level is changed
    - (i) from x = 100 to x = 105
  - (ii) from x = 100 to x = 101
    (b) Find the instantaneous rate of change of C with respect to x when x = 100. (This is called the *marginal cost.* Its significance will be explained in Section 3.3.)
- 24. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in 1 h, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000 \left(1 - \frac{t}{60}\right)^2 \qquad 0 \le t \le 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times t = 0, 10, 20, 30, 40, 50, and 60, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?



# Derivatives

In Section 2.6 we defined the slope of the tangent to a curve with equation y = f(x) at the point where x = a to be

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We also saw that the velocity of an object with position function s = f(t) at time t = a is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In fact, limits of the form

П

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

**2** Definition The derivative of a function f at a number a, denoted by f'(a), is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we write x = a + h, then h = x - a and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

**EXAMPLE 1** Find the derivative of the function  $f(x) = x^2 - 8x + 9$  at the number *a*.

**SOLUTION** From Definition 2 we have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
  
= 
$$\lim_{h \to 0} \frac{\left[(a+h)^2 - 8(a+h) + 9\right] - \left[a^2 - 8a + 9\right]}{h}$$
  
= 
$$\lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$
  
= 
$$\lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8)$$
  
= 
$$2a - 8$$

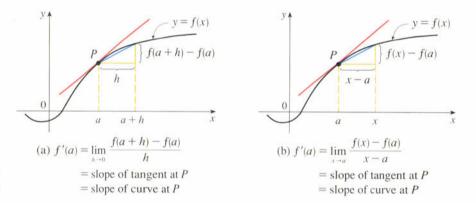
f'(a) is read "f prime of a."

# Interpretation of the Derivative As the Slope of a Tangent

In Section 2.6 we defined the tangent line to the curve y = f(x) at the point P(a, f(a)) to be the line that passes through P and has slope m given by Equation 1. Since, by Definition 2, this is the same as the derivative f'(a), we can now say that

The tangent line to y = f(x) at (a, f(a)) is the line through (a, f(a)) whose slope is equal to f'(a), the derivative of f at a.

Thus, the geometric interpretation of a derivative [as defined by either (2) or (3)] is as shown in Figure 1.



**FIGURE I** Geometric interpretation of the derivative

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve y = f(x) at the point (a, f(a)):

$$y - f(a) = f'(a)(x - a)$$

**EXAMPLE 2** Find an equation of the tangent line to the parabola  $y = x^2 - 8x + 9$  at the point (3, -6).

**SOLUTION** From Example 1 we know that the derivative of  $f(x) = x^2 - 8x + 9$  at the number *a* is f'(a) = 2a - 8. Therefore, the slope of the tangent line at (3, -6) is f'(3) = 2(3) - 8 = -2. Thus, an equation of the tangent line, shown in Figure 2, is

$$y - (-6) = (-2)(x - 3)$$
 or  $y = -2x$ 

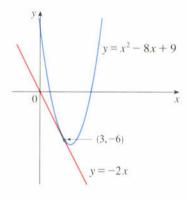
**EXAMPLE 3** Let  $f(x) = 2^x$ . Estimate the value of f'(0) in two ways: (a) By using Definition 2 and taking successively smaller values of h.

(b) By interpreting f'(0) as the slope of a tangent and using a graphing calculator to zoom in on the graph of  $y = 2^x$ .

#### SOLUTION

(a) From Definition 2 we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}$$





h	$\frac{2^n-1}{h}$
0.1	0,718
0.01	0,696
0.001	0.693
0.0001	0.693
0.1	0.670
0.01	0.691
0.001	0.693
-(1,000)1	0.693

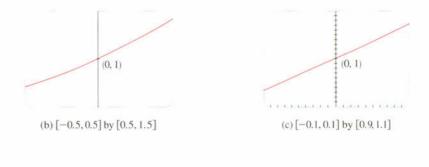
Since we are not yet able to evaluate this limit exactly, we use a calculator to approximate the values of  $(2^h - 1)/h$ . From the numerical evidence in the table we see that as *h* approaches 0, these values appear to approach a number near 0.69. So our estimate is

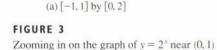
$$f'(0) \approx 0.69$$

(b) In Figure 3 we graph the curve  $y = 2^x$  and zoom in toward the point (0, 1). We see that the closer we get to (0, 1), the more the curve looks like a straight line. In fact, in Figure 3(c) the curve is practically indistinguishable from its tangent line at (0, 1). Since the *x*-scale and the *y*-scale are both 0.01, we estimate that the slope of this line is

$$\frac{0.14}{0.20} = 0.7$$

So our estimate of the derivative is  $f'(0) \approx 0.7$ . In Section 3.5 we will show that, correct to six decimal places,  $f'(0) \approx 0.693147$ .





(0, 1)

# Interpretation of the Derivative As a Rate of Change

In Section 2.6 we defined the instantaneous rate of change of y = f(x) with respect to x at  $x = x_1$  as the limit of the average rates of change over smaller and smaller intervals. If the interval is  $[x_1, x_2]$ , then the change in x is  $\Delta x = x_2 - x_1$ , the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

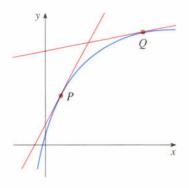
and

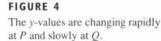
1

instantaneous rate of change = 
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

From Equation 3 we recognize this limit as being the derivative of f at  $x_1$ , that is,  $f'(x_1)$ . This gives a second interpretation of the derivative:

The derivative f'(a) is the instantaneous rate of change of y = f(x) with respect to x when x = a.





The connection with the first interpretation is that if we sketch the curve y = f(x), then the instantaneous rate of change is the slope of the tangent to this curve at the point where x = a. This means that when the derivative is large (and therefore the curve is steep, as at the point *P* in Figure 4), the *y*-values change rapidly. When the derivative is small, the curve is relatively flat and the *y*-values change slowly.

In particular, if s = f(t) is the position function of a particle that moves along a straight line, then f'(a) is the rate of change of the displacement s with respect to the time t. In other words, f'(a) is the velocity of the particle at time t = a. (See Section 2.6.) The speed of the particle is the absolute value of the velocity, that is, |f'(a)|.

**EXAMPLE 4** The position of a particle is given by the equation of motion s = f(t) = 1/(1 + t), where t is measured in seconds and s in meters. Find the velocity and the speed after 2 seconds.

**SOLUTION** The derivative of f when t = 2 is

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{1+(2+h)} - \frac{1}{1+2}}{h}$$
$$= \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{\frac{3 - (3+h)}{3(3+h)}}{h}$$
$$= \lim_{h \to 0} \frac{-h}{3(3+h)h} = \lim_{h \to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$$

Thus, the velocity after 2 s is  $f'(2) = -\frac{1}{9}$  m/s, and the speed is  $|f'(2)| = |-\frac{1}{9}| = \frac{1}{9}$  m/s.

**EXAMPLE 5** A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is C = f(x) dollars.

(a) What is the meaning of the derivative f'(x)? What are its units?

(b) In practical terms, what does it mean to say that f'(1000) = 9?

(c) Which do you think is greater, f'(50) or f'(500)? What about f'(5000)?

#### SOLUTION

(a) The derivative f'(x) is the instantaneous rate of change of *C* with respect to *x*; that is, f'(x) means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost*. This idea is discussed in more detail in Sections 3.3 and 4.7.)

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for f'(x) are the same as the units for the difference quotient  $\Delta C/\Delta x$ . Since  $\Delta C$  is measured in dollars and  $\Delta x$  in yards, it follows that the units for f'(x) are dollars per yard.

(b) The statement that f'(1000) = 9 means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When x = 1000, C is increasing 9 times as fast as x.)

Since  $\Delta x = 1$  is small compared with x = 1000, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when x = 500 than when x = 50 (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus, it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

The following example shows how to estimate the derivative of a tabular function, that is, a function defined not by a formula but by a table of values.

**EXAMPLE 6** Let P(t) be the population of the United States at time *t*. The table at the left gives approximate values of this function by providing mid-year population estimates from 1984 to 1992. Interpret and estimate the value of P'(1988).

**SOLUTION** The derivative P'(1988) means the rate of change of P with respect to t when t = 1988, that is, the rate of increase of the population in 1988. According to Equation 3,

$$P'(1988) = \lim_{t \to 1988} \frac{P(t) - P(1988)}{t - 1988}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as follows:

Ť	P(t) = P(1988)
I	t = 1988
1984	2,171,750
1986	2,188,500
1990	2,459,000
1992	2,490,750

Another method is to plot the population function and estimate the slope of the tangent line when t = 1988. (See Example 4 in Section 2.6.) From this table we see that P'(1988) lies somewhere between 2,188,500 and 2,459,000. We estimate that the rate of increase of the population of the United States in 1988 was the average of these two numbers, namely

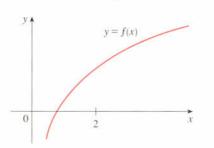
$$P'(1988) \approx 2.3$$
 million people/year

T	P(t)
1984	236,370,000
1986	240,680,000
1988	245,057,000
1990	249.975.000
1992	255,020,000

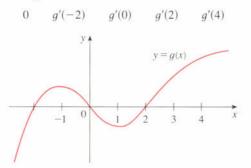
Exercises •



1. On the given graph of f, mark lengths that represent f(2), f(2 + h), f(2 + h) - f(2), and h. (Choose h > 0.) What line has slope  $\frac{f(2 + h) - f(2)}{h}$ ?



- **2.** For the function *f* whose graph is shown in Exercise 1, arrange the following numbers in increasing order and explain your reasoning:
  - 0 f'(2) f(3) f(2)  $\frac{1}{2}[f(4) f(2)]$
- **3.** For the function *g* whose graph is given, arrange the following numbers in increasing order and explain your reasoning:



- **4.** If the tangent line to y = f(x) at (4, 3) passes through the point (0, 2), find f(4) and f'(4).
- 5. Sketch the graph of a function *f* for which f(0) = 0, f'(0) = 3, f'(1) = 0, and f'(2) = -1.
- 6. Sketch the graph of a function g for which g(0) = 0, g'(0) = 3, g'(1) = 0, and g'(2) = 1.
- 7. If  $f(x) = 3x^2 5x$ , find f'(2) and use it to find the equation of the tangent line to the parabola  $y = 3x^2 5x$  at the point (2, 2).
- 8. If  $g(x) = 1 x^3$ , find g'(0) and use it to find an equation of the tangent line to the curve  $y = 1 x^3$  at the point (0, 1).
- 9. (a) If  $F(x) = x^3 5x + 1$ , find F'(1) and use it to find an equation of the tangent line to the curve  $y = x^3 - 5x + 1$  at the point (1, -3).

- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
  - 10. (a) If G(x) = x/(1 + 2x), find G'(a) and use it to find an equation of the tangent line to the curve y = x/(1 + 2x) at the point  $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ .
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
  - II. Let f(x) = 3<sup>x</sup>. Estimate the value of f'(1) in two ways:
    (a) By using Definition 2 and taking successively smaller values of h.
- (b) By zooming in on the graph of y = 3<sup>x</sup> and estimating the slope.
  - 12. Let  $g(x) = \tan x$ . Estimate the value of  $g'(\pi/4)$  in two ways:
    - (a) By using Definition 2 and taking successively smaller values of *h*.
- (b) By zooming in on the graph of y = tan x and estimating the slope.

**13–16** Find f'(a).

1

1

3. 
$$f(x) = 1 + x - 2x^{2}$$
  
4.  $f(x) = x^{3} + 3x$   
5.  $f(x) = \frac{x}{x^{2} - 1}$   
6.  $f(x) = \sqrt{x - 1}$ 

**17–22** Each limit represents the derivative of some function f at some number a. State f and a in each case.

17.  $\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h}$ 18.  $\lim_{h \to 0} \frac{(2+h)^3-8}{h}$ 19.  $\lim_{x \to 1} \frac{x^9-1}{x-1}$ 20.  $\lim_{x \to 3\pi} \frac{\cos x+1}{x-3\pi}$ 21.  $\lim_{t \to 0} \frac{\sin\left(\frac{\pi}{2}+t\right)-1}{t}$ 22.  $\lim_{x \to 0} \frac{3^x-1}{x}$ 

**23–24** A particle moves along a straight line with equation of motion s = f(t), where s is measured in meters and t in seconds. Find the velocity when t = 2.

**23.** 
$$f(t) = t^2 - 6t - 5$$

24. 
$$f(t) = 2t^3 - t + 1$$

a a a

- **25.** The cost of producing x ounces of gold from a new gold mine is C = f(x) dollars.
  - (a) What is the meaning of the derivative f'(x)? What are its units?
  - (b) What does the statement f'(800) = 17 mean?
  - (c) Do you think the values of f'(x) will increase or decrease in the short term? What about the long term? Explain.
- **26.** The number of bacteria after t hours in a controlled laboratory experiment is n = f(t).
  - (a) What is the meaning of the derivative f'(5)? What are its units?
  - (b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, f'(5) or f'(10)? If the supply of nutrients is limited, would that affect your conclusion? Explain.
- 27. The fuel consumption (measured in gallons per hour) of a car traveling at a speed of v miles per hour is c = f(v).
  - (a) What is the meaning of the derivative f'(v)? What are its units?
  - (b) Write a sentence (in layman's terms) that explains the meaning of the equation f'(20) = -0.05.
- **28.** The quantity (in yards) of a certain fabric that is sold by a manufacturer at a price of p dollars per yard is Q = f(p).
  - (a) What is the meaning of the derivative f'(16)? What are its units?
  - (b) Is f'(16) positive or negative? Explain.

**29.** Let C(t) be the price of coffee beans on the commodity market at time *t*. The table gives values of C(t) in dollars per kilogram. (Prices have been adjusted for inflation and are quoted in constant 1990 dollars.) Interpret and estimate the values of C'(1983) and C'(1990).

1	1981	1982	1983	1984	1985	1986
C(t)	2.88	2.44	3.05	3.52	3,29	2.56

T	1987	1988	1989	1990	1991	1992
C(t)	2.24	1.66	1.31	1.27	1.18	1.03

**30.** Life expectancy has improved dramatically in this century. The table gives values of E(t), the life expectancy at birth (in years) of a male born in the year t in the United States. Interpret and estimate the values of E'(1910) and E'(1950).

1	1900	1910	1920	1930	1940
E(t)	48.3	51.1	55.2	57.4	62.5

1	1950	1960	1970	1980	199()
E(t)	65.6	66.6	67.1	70.0	71.8

**31–32** Determine whether or not f'(0) exists.

**31.** 
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0\\ 0 & \text{if } x = 0 \end{cases}$$
  
**32.** 
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0\\ 0 & \text{if } x = 0 \end{cases}$$

# Writing Project

# Early Methods for Finding Tangents

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen farther than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's teacher at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve  $y = x^3 + 2x$  at the point (1, 3) and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

- I. Carl Boyer and Uta Merzbach, A History of Mathematics (New York: John Wiley, 1989), pp. 389, 432.
- C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
- 3. Howard Eves, An Introduction to the History of Mathematics, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
- 4. Morris Kline, Mathematical Thought from Ancient to Modern Times (New York: Oxford University Press, 1972), pp. 344, 346.



# The Derivative as a Function

In the preceding section we considered the derivative of a function *f* at a fixed number *a*:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

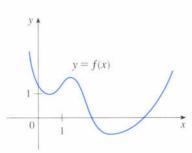
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Given any number x for which this limit exists, we assign to x the number f'(x). So we can regard f' as a new function, called the **derivative of** f and defined by Equation 2. We know that the value of f' at x, f'(x), can be interpreted geometrically as the slope of the tangent line to the graph of f at the point (x, f(x)).

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set  $\{x \mid f'(x) \text{ exists}\}$  and may be smaller than the domain of f.

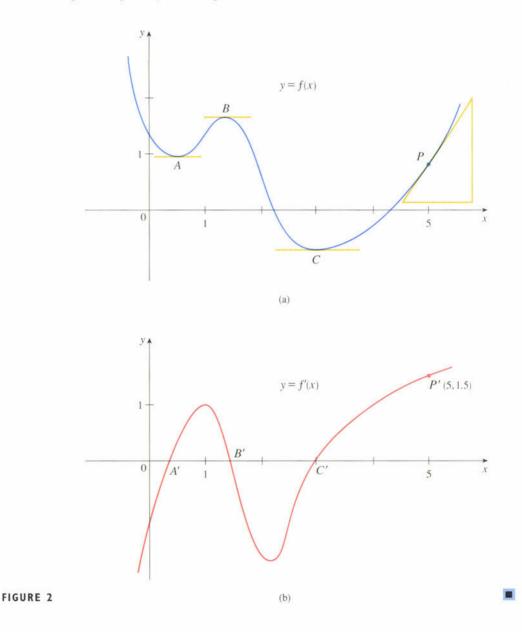
**EXAMPLE 1** The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

**SOLUTION** We can estimate the value of the derivative at any value of x by drawing the tangent at the point (x, f(x)) and estimating its slope. For instance, for





x = 5 we draw the tangent at *P* in Figure 2(a) and estimate its slope to be about  $\frac{3}{2}$ , so  $f'(5) \approx 1.5$ . This allows us to plot the point P'(5, 1.5) on the graph of f' directly beneath *P*. Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at *A*, *B*, and *C* are horizontal, so the derivative is 0 there and the graph of f' crosses the *x*-axis at the points *A'*, *B'*, and *C'*, directly beneath *A*, *B*, and *C*. Between *A* and *B* the tangents have positive slope, so f'(x) is positive there. But between *B* and *C* the tangents have negative slope, so f'(x) is negative there.



If a function is defined by a table of values, we can construct a table of approximate values of its derivative, as in the next example.

7	I(t)
1983	8.62
1984	9.57
1985	7.49
1986	5.97
1987	5.83
1988	6.67
1989	8.11
1990	7.51
1991	5,41
1992	3.46

**EXAMPLE 2** The interest rate on U.S. Treasury bills is a function of time. The table at the left gives midyear values of this function I(t) over a nine-year period (as a percent per year.) Construct a table of values for the derivative of this function.

SOLUTION We assume that there were no wild fluctuations in the interest rate between the stated values. Let's start by approximating I'(1990), the rate of change of the interest rate in 1990. Since

$$I'(1990) = \lim_{h \to 0} \frac{I(1990 + h) - I(1990)}{h}$$
$$I'(1990) \approx \frac{I(1990 + h) - I(1990)}{h}$$

we have

for small values of h.

For h = 1, we get

$$I'(1990) \approx \frac{I(1991) - I(1990)}{1} = 5.41 - 7.51 = -2.10$$

(This is the average rate of change between 1990 and 1991.) For h = -1, we have

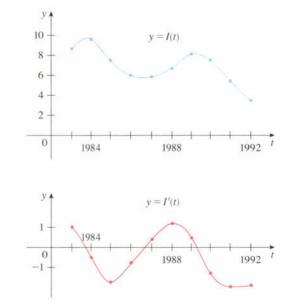
$$I'(1990) \approx \frac{I(1989) - I(1990)}{-1} = -(8.11 - 7.51) = -0.60$$

which is the average rate of change between 1989 and 1990. We get a more accurate approximation if we take the average of these rates of change:

$$I'(1990) \approx \frac{1}{2}(-2.10 - 0.60) = -1.35$$

This means that in 1990 the interest rate on U.S. Treasury bills was decreasing at a rate of about 1.35% per year.

Making similar calculations for the other values (except at the endpoints), we get the table of approximate values for the derivative.



I'(t)
0.95
-0.565
-1.80
-0.83
0.35
1.14
0.42
-1.35
-2.025
-1.95

Figure 3 illustrates Example 2 by showing graphs of the treasury-bill rate function I(t) and its derivative I'(t).

#### EXAMPLE 3

(a) If  $f(x) = x^3 - x$ , find a formula for f'(x).

(b) Illustrate by comparing the graphs of f and f'.

#### SOLUTION

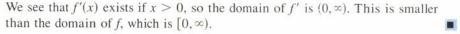
(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[(x+h)^3 - (x+h)\right] - \left[x^3 - x\right]}{h}$$
$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

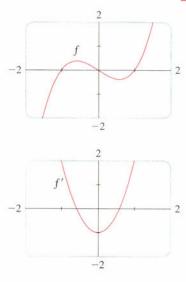
(b) We use a graphing device to graph f and f' in Figure 4. Notice that f'(x) = 0 when f has horizontal tangents and f'(x) is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a).

# **EXAMPLE 4** If $f(x) = \sqrt{x}$ , find the derivative of *f*. State the domain of *f'*. SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$
$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

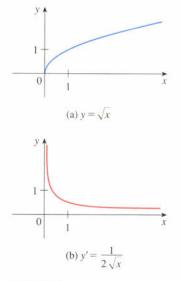


Let's check to see that the result of Example 4 is reasonable. When x is close to 0,  $\sqrt{x}$  is also close to 0, so  $f'(x) = 1/(2\sqrt{x})$  is very large and this corresponds to the steep tangent lines near (0,0) in Figure 5(a). When x is large, f'(x) is very small and this corresponds to the flatter tangent lines at the far right of the graph. Figure 5(b) shows the graph of the derivative y' = f'(x). Notice the relationship between the graphs of f and f'.











Other Notations

If we use the traditional notation y = f(x) to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for f'(x). Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 4 in Section 2.7, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number *a*, we use the notation

$$\frac{dy}{dx}\Big|_{x=a}$$
 or  $\frac{dy}{dx}\Big|_{x=a}$ 

which is a synonym for f'(a).

**3** Definition A function f is differentiable at a if f'(a) exists. It is differentiable on an open interval (a, b) [or  $(a, \infty)$  or  $(-\infty, a)$  or  $(-\infty, \infty)$ ] if it is differentiable at every number in the interval.

**EXAMPLE 5** Where is the function f(x) = |x| differentiable?

**SOLUTION** If x > 0, then |x| = x and we can choose h small enough that x + h > 0 and hence |x + h| = x + h. Therefore, for x > 0 we have

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$
$$= \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so *f* is differentiable for any x > 0.

Similarly, for x < 0 we have |x| = -x and h can be chosen small enough that x + h < 0 and so |x + h| = -(x + h). Therefore, for x < 0,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$
$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so *f* is differentiable for any x < 0.

Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelor's degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for finding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus first. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus first but, because of his fear of controversy, did not publish it immediately. So Leibniz's 1684 account of calculus was the first to be published. For x = 0 we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$
(if it exists)

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

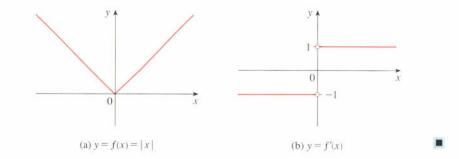
and 
$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, f'(0) does not exist. Thus, f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0\\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 6(b). The fact that f'(0) does not exist is reflected geometrically in the fact that the curve y = |x| does not have a tangent line at (0,0). [See Figure 6(a).]





Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

**4** Theorem If f is differentiable at a, then f is continuous at a.

**Proof** To prove that f is continuous at a, we have to show that  $\lim_{x\to a} f(x) = f(a)$ . We do this by showing that the difference f(x) - f(a) approaches 0.

The given information is that f is differentiable at a, that is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. (See Equation 3 in Section 2.7.) To connect the given and the unknown, we divide and multiply f(x) - f(a) by x - a (which we can do when  $x \neq a$ ):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} (x - a)$$

Thus, using the Product Law and Equation 3, we can write

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} (x - a)$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a)$$
$$= f'(a) \cdot 0 = 0$$

To use what we have just proved, we start with f(x) and add and subtract f(a):

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[ f(a) + (f(x) - f(a)) \right]$$
$$= \lim_{x \to a} f(a) + \lim_{x \to a} \left[ f(x) - f(a) \right]$$
$$= f(a) + 0 = f(a)$$

Therefore, f is continuous at a.

**NOTE** • The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function f(x) = |x| is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

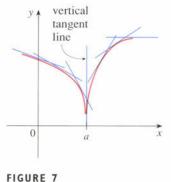
(See Example 7 in Section 2.3.) But in Example 5 we showed that f is not differentiable at 0.

## How Can a Function Fail to Be Differentiable?

In Example 5 we saw that the function y = |x| is not differentiable at 0. In general, if the graph of a function *f* has "corners" or "kinks" in it, then the graph of *f* has no tangent at those points and *f* is not differentiable there. [In trying to compute f'(a), we find that the left and right limits are different.]

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

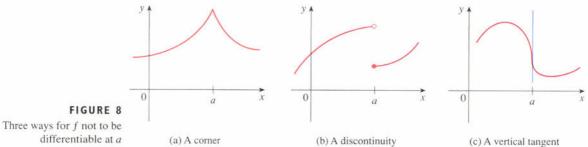
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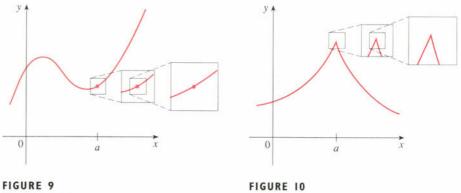
A third possibility is that the curve has a vertical tangent line when x = a, that is, f is continuous at a and

$$\lim_{x \to a} \left| f'(x) \right| = \infty$$

This means that the tangent lines become steeper and steeper as  $x \rightarrow a$ . Figure 7 shows one way that this can happen; Figure 8(c) shows another. Figure 8 illustrates the three possibilities that we have discussed.



A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a, then when we zoom in toward the point (a, f(a))the graph straightens out and appears more and more like a line. (See Figure 9. We saw a specific example of this in Figure 3 in Section 2.7.) But no matter how much we zoom in toward a point like the ones in Figures 7 and 8(a), we cannot eliminate the sharp point or corner. (See Figure 10.)



f is differentiable at a

f is not differentiable at a

# The Second Derivative

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by (f')' = f''. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of y = f(x) as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

**EXAMPLE 6** If  $f(x) = x^3 - x$ , find and interpret f''(x).

**SOLUTION** In Example 3 we found that the first derivative is  $f'(x) = 3x^2 - 1$ . So the second derivative is

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$
$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$
$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$
$$= \lim_{h \to 0} (6x + 3h) = 6x$$

The graphs of f, f', and f'' are shown in Figure 11.

We can interpret f''(x) as the slope of the curve y = f'(x) at the point (x, f'(x)). In other words, it is the rate of change of the slope of the original curve y = f(x).

Notice from Figure 11 that f''(x) is negative when y = f'(x) has negative slope and positive when y = f'(x) has positive slope. So the graphs serve as a check on our calculations.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most famous example of this is *acceleration*, which we define as follows.

If s = s(t) is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity v(t) of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** a(t) of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

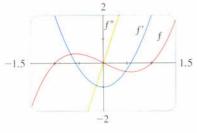
$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

**EXAMPLE 7** • A car starts from rest and the graph of its position function is shown in Figure 12, where *s* is measured in feet and *t* in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at t = 2 seconds?

**SOLUTION** By measuring the slope of the graph of s = f(t) at t = 0, 1, 2, 3, 4, and 5, and using the method of Example 1, we plot the graph of the velocity





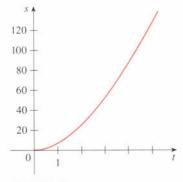


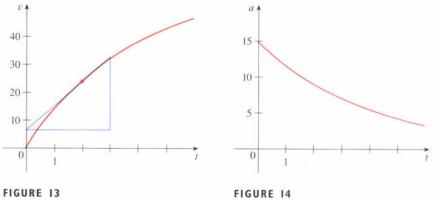
FIGURE 12 Position function of a car

function v = f'(t) in Figure 13. The acceleration when t = 2 s is a = f''(2), the slope of the tangent line to the graph of f' when t = 2. We estimate the slope of this tangent line to be

$$a(2) = f''(2) = v'(2) \approx \frac{27}{3} = 9 \text{ ft/s}^2$$

The units for acceleration are feet per second per second, written as  $ft/s^2$ .

Similar measurements enable us to graph the acceleration function in Figure 14.





Acceleration function

The **third derivative** f''' is the derivative of the second derivative: f''' = (f'')'. So f'''(x) can be interpreted as the slope of the curve y = f''(x) or as the rate of change of f''(x). If y = f(x), then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx} \left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative  $f^{mn}$  is usually denoted by  $f^{(4)}$ . In general, the *n*th derivative of *f* is denoted by  $f^{(n)}$  and is obtained from *f* by differentiating *n* times. If y = f(x), we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

**EXAMPLE 8** If  $f(x) = x^3 - x$ , find f'''(x) and  $f^{(4)}(x)$ .

**SOLUTION** In Example 6 we found that f''(x) = 6x. The graph of the second derivative has equation y = 6x and so it is a straight line with slope 6. Since the derivative f'''(x) is the slope of f''(x), we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

$$f^{(4)}(x) = 0$$

We can interpret the third derivative physically in the case where the function is the position function s = s(t) of an object that moves along a straight line. Since s''' = (s'')' = a', the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Thus, the jerk j is the rate of change of acceleration. It is apply named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 2.10, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Section 8.9 we will see how second and higher derivatives enable us to obtain more accurate approximations of functions than linear approximations and also to represent functions as sums of infinite series.

 $\mathbf{x}_{1} = \mathbf{x}_{1} + \mathbf{x}_{2} + \mathbf{x}_{3}$ 

.

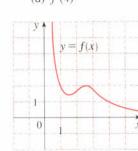
2.8

I-2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

 I. (a) f'(1) (b) f'(2) 

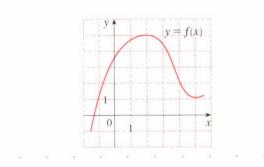
 (c) f'(3) (d) f'(4) 

Exercises

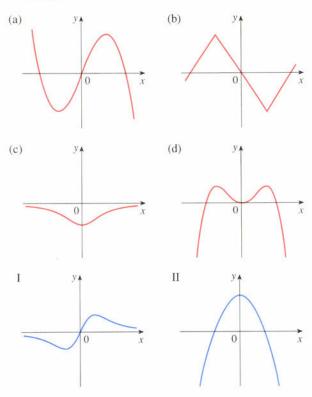


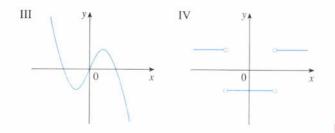
**2.** (a) f'(0)(c) f'(2)(e) f'(4)



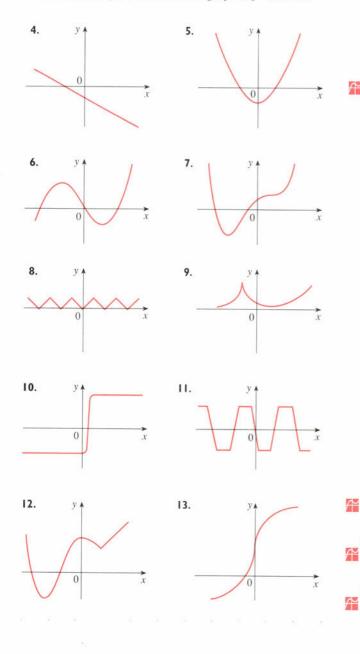


 Match the graph of each function in (a)-(d) with the graph of its derivative in I-IV. Give reasons for your choices.





4-13 = Trace or copy the graph of the given function *f*. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of *f'* below it.



**14–16** Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–13. Can you guess a formula for f'(x) from its graph?

**14.**  $f(x) = \sin x$  **15.**  $f(x) = e^x$ 

**16.**  $f(x) = \ln x$ 

**17.** Let  $f(x) = x^2$ .

- (a) Estimate the values of f'(0), f'(<sup>1</sup>/<sub>2</sub>), f'(1), and f'(2) by using a graphing device to zoom in on the graph of f.
- (b) Use symmetry to deduce the values of f'(−<sup>1</sup>/<sub>2</sub>), f'(−1), and f'(−2).
- (c) Use the results from parts (a) and (b) to guess a formula for f'(x).
- (d) Use the definition of a derivative to prove that your guess in part (c) is correct.

**18.** Let  $f(x) = x^3$ .

- (a) Estimate the values of f'(0), f'(<sup>1</sup>/<sub>2</sub>), f'(1), f'(2), and f'(3) by using a graphing device to zoom in on the graph of f.
- (b) Use symmetry to deduce the values of f'(-<sup>1</sup>/<sub>2</sub>), f'(-1), f'(-2), and f'(-3).
- (c) Use the values from parts (a) and (b) to graph f'.
- (d) Guess a formula for f'(x).
- (e) Use the definition of a derivative to prove that your guess in part (d) is correct.

**19–23** Find the derivative of the given function using the definition of derivative. State the domain of the function and the domain of its derivative.

**19.** 
$$f(x) = 5x + 3$$
  
**20.**  $f(x) = x^3 - x^2 + 2x$   
**21.**  $g(x) = \sqrt{1 + 2x}$   
**22.**  $g(x) = \frac{1}{x^2}$   
**23.**  $f(x) = \frac{x + 1}{x - 1}$ 

- 24. (a) Sketch the graph of  $f(x) = \sqrt{6 x}$  by starting with the graph of  $y = \sqrt{x}$  and using the transformations of Section 1.2.
  - (b) Use the graph from part (a) to sketch the graph of f'.
  - (c) Use the definition of a derivative to find f'(x). What are the domains of f and f'?
  - (d) Use a graphing device to graph f' and compare with your sketch in part (b).

**25.** (a) If f(x) = x - (2/x), find f'(x).

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

**26.** (a) If  $f(t) = 6/(1 + t^2)$ , find f'(t).

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'. 27. The unemployment rate U(t) varies with time. The table gives the percentage of unemployed in the U.S. labor force from 1983 to 1992.

1	1983	1984	1985	1986	1987
U(t)	9.5	7.4	7.1	6.9	6.1

1 I	1988	1989	1990	1991	1992
U(t)	5.4	5.2	5.4	6.6	7.3

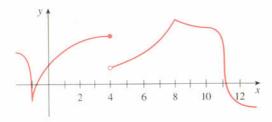
(a) What is the meaning of U'(t)? What are its units?

- (b) Construct a table of values for U'(t).
- **28.** Let the smoking rate among high-school seniors at time t be S(t). The table gives the percentage of seniors who reported that they had smoked in the past 30 days.

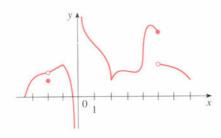
1	1977	1979	1981	1983	1985
<i>S</i> ( <i>t</i> )	38	34	29	31	30

1	1987	1989	1991	1993
S(t)	29	28	27	30

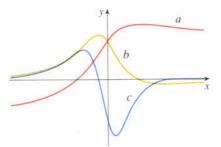
- (a) What is the meaning of S'(t)? What are its units?
- (b) Construct a table of values for S'(t).
- (c) Graph S and S'.
- (d) How would it be possible to get more accurate values for S'(t)?
- **29.** The graph of *f* is given. State, with reasons, the numbers at which *f* is not differentiable.



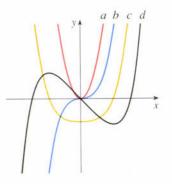
- **30.** The graph of g is given.
  - (a) At what numbers is g discontinuous? Why?
  - (b) At what numbers is g not differentiable? Why?



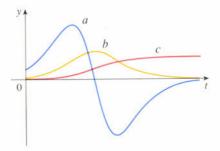
- **31.** Graph the function  $f(x) = x + \sqrt{|x|}$ . Zoom in repeatedly, first toward the point (-1, 0) and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?
- Zoom in toward the points (1, 0), (0, 1), and (-1, 0) on the graph of the function g(x) = (x<sup>2</sup> 1)<sup>2/3</sup>. What do you notice? Account for what you see in terms of the differentiability of g.
  - **33.** The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



**34.** The figure shows graphs of *f*, *f*', *f*", and *f*"". Identify each curve, and explain your choices.

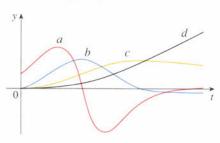


**35.** The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



**36.** The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the

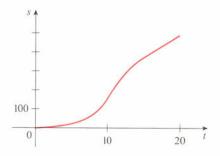
car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



**37–38** Use the definition of a derivative to find f'(x) and f''(x). Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

**37.**  $f(x) = 1 + 4x - x^2$  **38.** f(x) = 1/x

- 6 6 8 8 8 8 8 8 96 9 9
- **39.** If  $f(x) = 2x^2 x^3$ , find f'(x), f''(x), f'''(x), and  $f^{(4)}(x)$ . Graph *f*, *f'*, *f''*, and *f'''* on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?
- 40. (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at t = 10 seconds?



- (b) Use the acceleration curve from part (a) to estimate the jerk at t = 10 s. What are the units for jerk?
- **41.** Let  $f(x) = \sqrt[3]{x}$ .
  - (a) If  $a \neq 0$ , use Equation 3 in Section 2.7 to find f'(a).
  - (b) Show that f'(0) does not exist.
  - (c) Show that  $y = \sqrt[3]{x}$  has a vertical tangent line at (0, 0). (Recall the shape of the graph of *f*. See Figure 5 in Section 1.2.)
- 42. (a) If g(x) = x<sup>2/3</sup>, show that g'(0) does not exist.
  (b) If a ≠ 0, find g'(a).
  - (c) Show that  $y = x^{2/3}$  has a vertical tangent line at (0, 0).
  - (d) Illustrate part (c) by graphing  $y = x^{2/3}$ .
- **43.** Show that the function f(x) = |x 6| is not differentiable at 6. Find a formula for f' and sketch its graph.
- (a) Sketch the graph of the function f(x) = x | x |.
  (b) For what values of x is f differentiable?
  - (c) Find a formula for f'.
- **45.** Recall that a function *f* is called *even* if f(-x) = f(x) for all *x* in its domain and *odd* if f(-x) = -f(x) for all such *x*. Prove each of the following.
  - (a) The derivative of an even function is an odd function.
  - (b) The derivative of an odd function is an even function.
- **46.** When you turn on a hot-water faucet, the temperature *T* of the water depends on how long the water has been running.
  - (a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
  - (b) Describe how the rate of change of T with respect to t varies as t increases.
  - (c) Sketch a graph of the derivative of T.

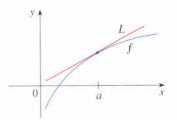


# Linear Approximations

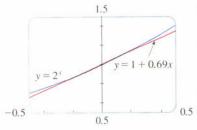
We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2 in Section 2.6 and Figure 3 in Section 2.7.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value f(a) of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the

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easily computed values of the linear function L whose graph is the tangent line of f at (a, f(a)). (See Figure 1.) The following example illustrates the method.

**EXAMPLE 1** Use a linear approximation to estimate the values of  $2^{0.1}$  and  $2^{0.4}$ .

**SOLUTION** The desired values are values of the function  $f(x) = 2^x$  near a = 0. From Example 3 in Section 2.7 we know that the slope of the tangent line to the curve  $y = 2^x$  at the point (0, 1) is  $f'(0) \approx 0.69$ . So the equation of the tangent line is approximately

$$y - 1 = 0.69(x - 0)$$
 or  $y = 1 + 0.69x$ 

Because the tangent line lies close to the curve when x = 0.1 (see Figure 2), the value of the function is almost the same as the height of the tangent line when x = 0.1. Thus

$$2^{0.1} = f(0.1) \approx 1 + 0.69(0.1) = 1.069$$
  
 $2^{0.4} = f(0.4) \approx 1 + 0.69(0.4) = 1.276$ 

It appears from Figure 2 that our estimate for  $2^{0.1}$  is better than our estimate for  $2^{0.4}$  and that both estimates are less than the true values because the tangent line lies below the curve. In fact, this is correct because the true values of these numbers are

$$2^{0.1} = 1.07177...$$
  $2^{0.4} = 1.31950...$ 

In general, we use the tangent line at (a, f(a)) as an approximation to the curve y = f(x) when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

Similarly,

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

**EXAMPLE 2** Find the linear approximation for the function  $f(x) = \sqrt{x}$  at a = 1. Then use it to approximate the numbers  $\sqrt{0.99}$ ,  $\sqrt{1.01}$ , and  $\sqrt{1.05}$ . Are these approximations overestimates or underestimates?

**SOLUTION** We first have to find f'(1), the slope of the tangent line to  $y = \sqrt{x}$  when x = 1. We could estimate f'(1) using numerical or graphical methods as in Section 2.7, or we could find the value exactly using the definition of a derivative. In fact, in Example 4 in Section 2.8, we found that

$$f'(x) = \frac{1}{2\sqrt{x}}$$

We will see in Sections 3.8 and 8.9 that linear approximations are very useful in physics for the purpose of simplifying a calculation or even an entire theory.

#### SECTION 2.9 LINEAR APPROXIMATIONS **I** 173

and so  $f'(1) = \frac{1}{2}$ . Therefore, an equation of the tangent line at (1, 1) is

$$y - 1 = \frac{1}{2}(x - 1)$$
 or  $y = \frac{1}{2}x + \frac{1}{2}$ 

and the linear approximation is

$$\sqrt{x} \approx L(x) = \frac{1}{2}x + \frac{1}{2}$$

In particular, we have

$$\sqrt{0.99} \approx L(0.99) = \frac{1}{2}(0.99) + \frac{1}{2} = 0.995$$
$$\sqrt{1.01} \approx L(1.01) = \frac{1}{2}(1.01) + \frac{1}{2} = 1.005$$
$$\sqrt{1.05} \approx L(1.05) = \frac{1}{2}(1.05) + \frac{1}{2} = 1.025$$

In Figure 3 we graph the root function  $y = \sqrt{x}$  and its linear approximation  $L(x) = \frac{1}{2}x + \frac{1}{2}$ . We see that our approximations are overestimates because the tangent line lies above the curve.

In the following table we compare the estimates from the linear approximation with the true values. Notice from this table, and also from Figure 3, that  $_5$  the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

	From $L(x)$	Actual value
$\sqrt{0.99}$	0.995	0.99498743
x1.001	1.0005	1.00049987
$\sqrt{1.01}$	1.005	1.00498756
$\sqrt{1.05}$	1.025	1.02469507
$\sqrt{1.1}$	1.05	1.04880884
$\sqrt{1.5}$	1.25	1.22474487
× 2	1.5	1.41421356

Of course, a calculator can give us better approximations than the linear approximations we found in Examples 1 and 2. But a linear approximation gives an approximation over an entire *interval* and that is the reason that scientists often use such approximations. (See Sections 3.8 and 8.9.)

The following example is typical of situations in which we use a linear approximation to predict the future behavior of a function given by empirical data.

**EXAMPLE 3** Suppose that after you stuff a turkey its temperature is 50 °F and you then put it in a 325 °F oven. After an hour the meat thermometer indicates that the temperature of the turkey is 93 °F and after two hours it indicates 129 °F. Predict the temperature of the turkey after three hours.

**SOLUTION** If T(t) represents the temperature of the turkey after t hours, we are given that T(0) = 50, T(1) = 93, and T(2) = 129. In order to make a linear approximation with a = 2, we need an estimate for the derivative T'(2). Because

$$T'(2) = \lim_{t \to 2} \frac{T(t) - T(2)}{t - 2}$$

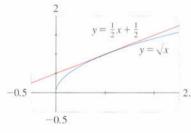


FIGURE 3

we could estimate T'(2) by the difference quotient with t = 1:

$$T'(2) \approx \frac{T(1) - T(2)}{1 - 2} = \frac{93 - 129}{-1} = 36$$

This amounts to approximating the instantaneous rate of temperature change by the average rate of change between t = 1 and t = 2, which is 36 °F/h. With this estimate, the linear approximation for the temperature after 3 h is

$$T(3) \approx T(2) + T'(2)(3 - 2)$$
  
$$\approx 129 + 36 \cdot 1 = 165$$

So the predicted temperature after three hours is 165 °F.

We obtain a more accurate estimate for T'(2) by plotting the given data, as in Figure 4, and estimating the slope of the tangent line at t = 2 to be

$$T'(2) \approx 33$$

Then our linear approximation becomes

$$T(3) \approx T(2) + T'(2) \cdot 1 \approx 129 + 33 = 162$$

and our improved estimate for the temperature is 162 °F.

Because the temperature curve lies below the tangent line, it appears that the actual temperature after three hours will be somewhat less than 162 °F, perhaps closer to 160 °F.

and a restance of a second second





- **1.** (a) If  $f(x) = 3^x$ , estimate the value of f'(0) either numerically or graphically.
  - (b) Use the tangent line to the curve  $y = 3^x$  at (0, 1) to find approximate values for  $3^{0.05}$  and  $3^{0.1}$ .
  - (c) Graph the curve and its tangent line. Are the approximations in part (b) less than or greater than the true values? Why?
- **2.** (a) If  $f(x) = \ln x$ , estimate the value of f'(1) graphically.
  - (b) Use the tangent line to the curve y = ln x at (1,0) to estimate the values of ln 0.9 and ln 1.3.
  - (c) Graph the curve and its tangent line. Are the estimates in part (b) less than or greater than the true values? Why?
- **3.** (a) If  $f(x) = \sqrt[3]{x}$ , estimate the value of f'(1).
  - (b) Find the linear approximation for f at a = 1.
  - (c) Use part (b) to estimate the cube roots of the numbers 0.5, 0.9, 0.99, 1.01, 1.1, 1.5, and 2. Compare these estimates with the values of the cube roots from your calculator. Did you obtain underestimates or overestimates? Which of your estimates are the most accurate?

- (d) Graph the curve  $y = \sqrt[3]{x}$  and its tangent line at (1, 1). Use these graphs to explain your results from part (c).
- **4.** (a) If  $f(x) = \cos x$ , estimate the value of  $f'(\pi/3)$ .
  - (b) Find the linear approximation for f at  $a = \pi/3$ .
  - (c) Use the linear approximation to estimate the values of cos 1, cos 1.1, cos 1.5, and cos 2. Are these underestimates or overestimates? Which of your estimates are the most accurate?
  - (d) Graph the curve y = cos x and its tangent line at (π/3, ½). Use these graphs to explain your results from part (c).
  - 5-6 =
  - (a) Use the definition of a derivative to compute f'(1).
  - (b) Use the linear approximation for f at a = 1 to estimate f(x) for x = 0.9, 0.95, 0.99, 1.01, 1.05, and 1.1. How do these estimates compare with the actual values?
- (c) Graph f and its tangent line at (1, 1). Do the graphs support your comments in part (b)?

**5.** 
$$f(x) = x^2$$
 **6.**  $f(x) = x^3$ 

7. The turkey in Example 3 is removed from the oven when its temperature reaches 185 °F and is placed on a

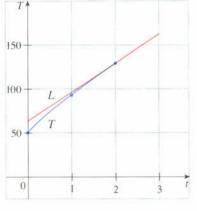
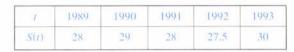
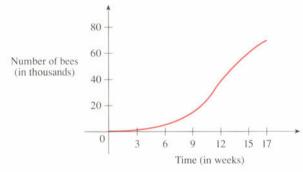


table in a room where the temperature is 75 °F. After 10 minutes the temperature of the turkey is 172 °F and after 20 minutes it is 160 °F. Use a linear approximation to predict the temperature of the turkey after half an hour. Do you think your prediction is an overestimate or an underestimate? Why?

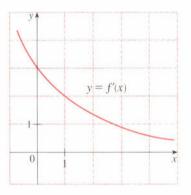
- 8. Atmospheric pressure P decreases as altitude h increases. At a temperature of 15 °C, the pressure is 101.3 kilopascals (kPa) at sea level, 87.1 kPa at h = 1 km, and 74.9 kPa at h = 2 km. Use a linear approximation to estimate the atmospheric pressure at an altitude of 3 km.
- **9.** The table gives the percentage of high-school seniors who said they were smokers (in the sense that they had smoked in the past 30 days). Use a linear approximation to estimate the smoking rate among high-school seniors in 1994 and 1995.



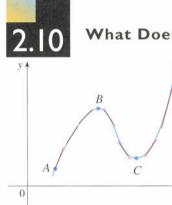
**10.** The figure shows the graph of a population of Cyprian honeybees raised in an apiary.



- (a) Use a linear approximation to predict the bee population after 18 weeks and after 20 weeks.
- (b) Are your predictions underestimates or overestimates? Why?
- (c) Which of your predictions do you think is the more accurate? Why?
- **II.** Suppose that the only information we have about a function f is that f(1) = 5 and the graph of its *derivative* is as shown.
  - (a) Use a linear approximation to estimate f(0.9) and f(1.1).
  - (b) Are your estimates in part (a) too large or too small? Explain.



- 12. Suppose that we don't have a formula for g(x) but we know that g(2) = −4 and g'(x) = √x² + 5 for all x.
  (a) Use a linear approximation to estimate g(1.95) and g(2.05).
  - (b) Are your estimates in part (a) too large or too small? Explain.



What Does f' Say about f? ••••••••••••••••••••••••••••••••••

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivative f'. Because f'(x) represents the slope of the curve y = f(x) at the point (x, f(x)), it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about f'(x) will provide us with information about f(x).

In particular, to see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1. (Increasing functions and decreasing functions were defined in Section 1.1.) Between A and B and between C and D, the tangent lines have positive slope and so f'(x) > 0. Between B and C, the tangent lines have negative slope and so f'(x) < 0. Thus, it appears that f increases when f'(x) is positive and decreases when f'(x) is negative.



It turns out, as we will see in Chapter 4, that what we observed for the function graphed in Figure 1 is always true. We state the general result as follows.

If f'(x) > 0 on an interval, then *f* is increasing on that interval.

If f'(x) < 0 on an interval, then f is decreasing on that interval.

## EXAMPLE I

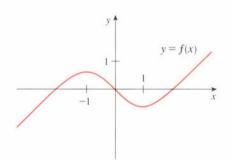
(a) If it is known that the graph of the derivative f' of a function is as shown in Figure 2, what can we say about f?

(b) If it is known that f(0) = 0, sketch a possible graph of f.

### SOLUTION

(a) We observe from Figure 2 that f'(x) is negative when -1 < x < 1, so the original function f must be decreasing on the interval (-1, 1). Similarly, f'(x) is positive for x < -1 and for x > 1, so f is increasing on the intervals  $(-\infty, -1)$  and  $(1, \infty)$ . Also note that, since f'(-1) = 0 and f'(1) = 0, the graph of f has horizontal tangents when  $x = \pm 1$ .

(b) We use the information from part (a), and the fact that the graph passes through the origin, to sketch a possible graph of f in Figure 3. Notice that f'(0) = -1, so we have drawn the curve y = f(x) passing through the origin with a slope of -1. Notice also that  $f'(x) \rightarrow 1$  as  $x \rightarrow \pm \infty$  (from Figure 2). So the slope of the curve y = f(x) approaches 1 as x becomes large (positive or negative). That is why we have drawn the graph of f in Figure 3 progressively straighter as  $x \rightarrow \pm \infty$ .

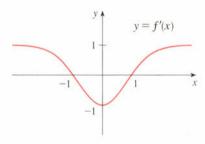


#### FIGURE 3

We say that the function f in Example 1 has a **local maximum** at -1 because near x = -1 the values of f(x) are at least as big as the neighboring values. Note that f'(x) is positive to the left of -1 and negative just to the right of -1. Similarly, f has a **local minimum** at 1, where the derivative changes from negative to positive. In Chapter 4 we will develop these observations into a general method for finding optimal values of functions.

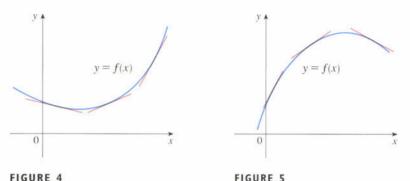
## What Does f" Say about f?

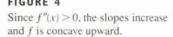
Let's see how the sign of f''(x) affects the appearance of the graph of f. Since f'' = (f')', we know that if f''(x) is positive, then f' is an increasing function. This says that the slopes of the tangent lines of the curve y = f(x) increase from left to





right. Figure 4 shows the graph of such a function. The slope of this curve becomes progressively larger as x increases and we observe that, as a consequence, the curve bends upward. Such a curve is called **concave upward**. In Figure 5, however, f''(x) is negative, which means that f' is decreasing. Thus, the slopes of fdecrease from left to right and the curve bends downward. This curve is called **concave downward**. We summarize our discussion as follows. (Concavity is discussed in greater detail in Section 4.3.)

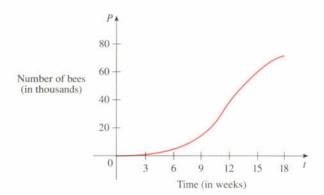




**FIGURE 5** Since f''(x) < 0, the slopes decrease and *f* is concave downward.

If f''(x) > 0 on an interval, then *f* is concave upward on that interval. If f''(x) < 0 on an interval, then *f* is concave downward on that interval.

**EXAMPLE 2** Figure 6 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is *P* concave upward or concave downward?



# FIGURE 6

**SOLUTION** By looking at the slope of the curve as *t* increases, we see that the rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about t = 12 weeks, and decreases as the population begins to level off. As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of increase, P'(t), approaches 0. The curve is concave upward on (0, 12) and concave downward on (12, 18).

In Example 2, the population curve changed from concave upward to concave downward at approximately the point (12, 38,000). This point is called an *inflection point* of the curve. The significance of this point is that the rate of population increase has its maximum value there. In general, a point where a curve changes its direction of concavity is called an **inflection point**.

**EXAMPLE 3** Sketch a possible graph of a function f that satisfies the following conditions:

- (i) f'(x) > 0 on  $(-\infty, 1)$ , f'(x) < 0 on  $(1, \infty)$
- (ii) f''(x) > 0 on  $(-\infty, -2)$  and  $(2, \infty)$ , f''(x) < 0 on (-2, 2)
- (iii)  $\lim_{x \to 0} f(x) = -2$ ,  $\lim_{x \to 0} f(x) = 0$

**SOLUTION** Condition (i) tells us that f is increasing on  $(-\infty, 1)$  and decreasing on  $(1, \infty)$ . Condition (ii) says that f is concave upward on  $(-\infty, -2)$  and  $(2, \infty)$ , and concave downward on (-2, 2). From condition (iii) we know that the graph of f has two horizontal asymptotes: y = -2 and y = 0.

We first draw the horizontal asymptote y = -2 as a dashed line. (See Figure 7.) We then draw the graph of *f* approaching this asymptote at the far left, increasing to its maximum point at x = 1 and decreasing toward the *x*-axis as  $x \to \infty$ . We also make sure that the graph has inflection points when x = -2 and 2. Notice that the curve bends upward for x < -2 and x > 2, and bends downward when *x* is between -2 and 2.

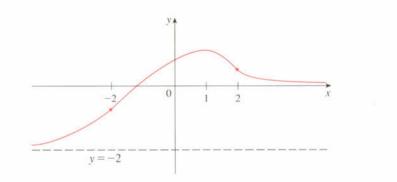


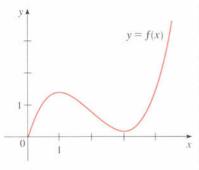
FIGURE 7

# **Antiderivatives**

In many problems in mathematics and its applications, we are given a function f and we are required to find a function F whose derivative is f. If such a function F exists, we call it an *antiderivative* of f. In other words, an **antiderivative** of f is a function F such that F' = f. (In Example 1 we sketched an antiderivative f of the function f'.)

**EXAMPLE 4** Let F be an antiderivative of the function f whose graph is shown in Figure 8.

- (a) Where is F increasing or decreasing?
- (b) Where is F concave upward or concave downward?
- (c) At what values of x does F have an inflection point?





(d) If F(0) = 1, sketch the graph of F.
(e) How many antiderivatives does f have?

## SOLUTION

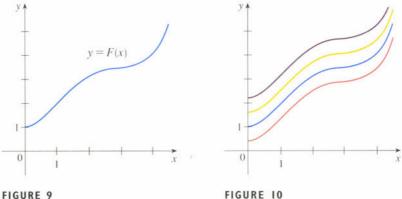
(a) We see from Figure 8 that f(x) > 0 for all x > 0. Since F is an antiderivative of f, we have F'(x) = f(x) and so F'(x) is positive when x > 0. This means that F is increasing on  $(0, \infty)$ .

(b) *F* is concave upward when F''(x) > 0. But F''(x) = f'(x), so *F* is concave upward when f'(x) > 0, that is, when *f* is increasing. From Figure 8 we see that *f* is increasing when 0 < x < 1 and when x > 3. So *F* is concave upward on (0, 1) and  $(3, \infty)$ . *F* is concave downward when F''(x) = f'(x) < 0, that is, when *f* is decreasing. So *F* is concave downward on (1, 3).

(c) *F* has an inflection point when the direction of concavity changes. From part (b) we know that *F* changes from concave upward to concave downward at x = 1, so *F* has an inflection point there. *F* changes from concave downward to concave upward when x = 3, so *F* has another inflection point at x = 3.

(d) In sketching the graph of F, we use the information from parts (a), (b), and (c). But, for finer detail, we also bear in mind the meaning of an antiderivative: Because F'(x) = f(x), the slope of y = F(x) at any value of x is equal to the height of y = f(x). (Of course, this is the exact opposite of the procedure we used in Example 1 in Section 2.8 to sketch a derivative.)

Therefore, since f(0) = 0, we start drawing the graph of *F* at the given point (0, 1) with slope 0, always increasing, with upward concavity to x = 1, downward concavity to x = 3, and upward concavity when x > 3. (See Figure 9.) Notice that  $f(3) \approx 0.2$ , so y = F(x) has a gentle slope at the second inflection point. But we see that the slope becomes steeper when x > 3.



An antiderivative of f

Members of the family of antiderivatives of f

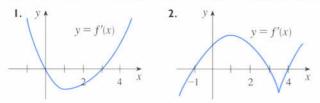
(e) The antiderivative of f that we sketched in Figure 9 satisfies F(0) = 1, so its graph starts at the point (0, 1). But there are many other antiderivatives, whose graphs start at other points on the y-axis. In fact, f has infinitely many anti-derivatives; their graphs are obtained from the graph of F by shifting upward or downward as in Figure 10.

Exercises · · ·

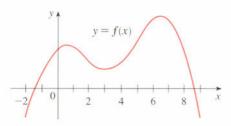


1-2 The graph of the *derivative* f' of a function f is shown.

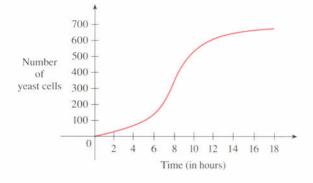
- (a) On what intervals is *f* increasing or decreasing?
- (b) At what values of x does f have a local maximum or minimum?
- (c) If it is known that f(0) = 0, sketch a possible graph of f.



**3.** Use the given graph of *f* to estimate the intervals on which the derivative *f'* is increasing or decreasing.



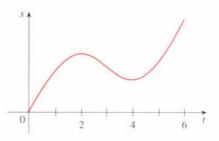
- **4.** (a) Sketch a curve whose slope is always positive and increasing.
  - (b) Sketch a curve whose slope is always positive and decreasing.
  - (c) Give equations for curves with these properties.
- The president announces that the national deficit is increasing, but at a decreasing rate. Interpret this statement in terms of a function and its derivatives.
- **6.** A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
  - (a) Describe how the rate of population increase varies.
  - (b) When is this rate highest?
  - (c) On what intervals is the population function concave upward or downward?



- (d) Estimate the coordinates of the inflection point.
- **7.** The table gives population densities for ring-necked pheasants (in number of pheasants per acre) on Pelee Island, Ontario.
  - (a) Describe how the rate of change of population varies.
  - (b) Estimate the inflection points of the graph. What is the significance of these points?

1	1927	1930	1932	1934	1936	1938	1940
P(t)	0. I	0.6	2.5	4.6	4.8	3.5	3.0

- **8.** A particle is moving along a horizontal straight line. The graph of its position function (the distance to the right of a fixed point as a function of time) is shown.
  - (a) When is the particle moving toward the right and when is it moving toward the left?
  - (b) When does the particle have positive acceleration and when does it have negative acceleration?

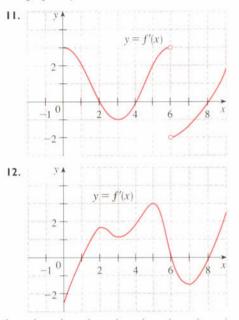


- **9.** Let K(t) be a measure of the knowledge you gain by studying for a test for *t* hours. Which do you think is larger, K(8) K(7) or K(3) K(2)? Is the graph of *K* concave upward or concave downward? Why?
- 10. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?



**II-12** The graph of the derivative f' of a function f is shown.

- (a) On what intervals is f increasing or decreasing?
- (b) At what values of x does f have a local maximum or minimum?
- (c) On what intervals is f concave upward or downward?
- (d) State the x-coordinates of the points of inflection.
- (e) Assuming that f is continuous and f(0) = 0, sketch a graph of f.



- **13.** Sketch the graph of a function whose first and second derivatives are always negative.
- Sketch the graph of a function whose first derivative is always negative and whose second derivative is always positive.

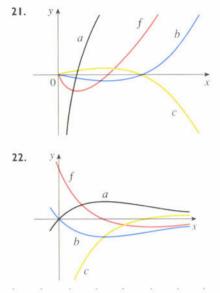
**15–18** Sketch the graph of a function that satisfies the given conditions.

- **15.** f'(-1) = f'(1) = 0, f'(x) < 0 if |x| < 1, f'(x) > 0 if |x| > 1, f(-1) = 4, f(1) = 0, f''(x) < 0 if x < 0, f''(x) > 0 if x > 0
- **16.** f'(-1) = 0, f'(1) does not exist, f'(x) < 0 if |x| < 1, f'(x) > 0 if |x| > 1, f(-1) = 4, f(1) = 0, f''(x) < 0 if  $x \neq 1$
- **17.** f'(2) = 0, f(2) = -1, f(0) = 0, f'(x) < 0 if 0 < x < 2, f'(x) > 0 if x > 2, f''(x) < 0 if  $0 \le x < 1$  or if x > 4, f''(x) > 0 if 1 < x < 4,  $\lim_{x \to \infty} f(x) = 1$ , f(-x) = f(x) for all x
- **18.**  $\lim_{x\to 3} f(x) = -\infty$ , f''(x) < 0 if  $x \neq 3$ , f'(0) = 0, f'(x) > 0 if x < 0 or x > 3, f'(x) < 0 if 0 < x < 3

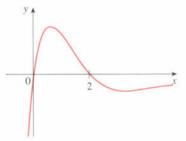
- **19.** Suppose  $f'(x) = xe^{-x^2}$ .
  - (a) On what interval is *f* increasing? On what interval is *f* decreasing?
  - (b) Does f have a maximum or minimum value?

**20.** If  $f'(x) = e^{-x^2}$ , what can you say about f?

**21–22** The graph of a function f is shown. Which graph is an antiderivative of f and why?



**23.** The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that F(0) = 0.



**24.** The graph of the velocity function of a car is shown in the figure. Sketch the graph of the position function.



25-26 Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

**25.**  $f(x) = \sin(x^2)$ ,  $0 \le x \le 4$  **26.**  $f(x) = 1/(x^4 + 1)$ 



# **Chapter 2 Review**

- CONCEPT CHECK •
- 1. Define each of the following in your own words. (Check by referring to the definition in the text.)
  - (a)  $\lim f(x) = L$ (b)  $\lim f(x) = L$  $x \rightarrow a$  $x \rightarrow a^{*}$
  - (c)  $\lim f(x) = L$ (d)  $\lim f(x) = \infty$  $x \rightarrow a$
  - (f) Vertical asymptote (e)  $\lim f(x) = L$  $\chi \rightarrow \alpha$

 $x \rightarrow a$ 

- (g) Horizontal asymptote
- 2. State the basic Limit Laws.
- **3.** (a) Define the derivative of a function *f* at a number *a*. (Give two expressions for the limit that defines this derivative.) Check by referring to Section 2.7.
  - (b) Give two interpretations of the derivative.
- 4. Define the second derivative of a function.
- 5. Let s = f(t) be the position function of a particle that moves along a straight line. Define
  - (a) the velocity of the particle at t = a,

- (b) the speed of the particle at t = a,
- (c) the acceleration of the particle at t = a.
- **6.** (a) What does it mean for f to be continuous at a? (b) What does it mean for f to be differentiable at a?
  - (c) What is the relation between the differentiability and continuity of a function?
- 7. (a) State the Squeeze Theorem. (b) State the Intermediate Value Theorem.
- 8. Which of the following curves have vertical asymptotes? Which have horizontal asymptotes?
  - (a)  $y = x^4$ (b)  $y = \sin x$ (c)  $y = \tan x$ (d)  $y = \tan^{-1}x$
  - (e)  $y = e^x$ (f)  $y = \ln x$
  - (g) y = 1/x(h)  $v = \sqrt{x}$
- **9.** (a) What does the sign of f'(x) tell us about f? (b) What does the sign of f''(x) tell us about f?
- **10.** (a) Define the linear approximation to f at a. (b) Define an antiderivative of f.

# 🔺 TRUE-FALSE QUIZ 🔺

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. 
$$\lim_{x \to 4} \left( \frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \to 4} \frac{2x}{x-4} - \lim_{x \to 4} \frac{8}{x-4}$$
  
2. 
$$\lim_{x \to 4} \frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x \to 1} (x^2 + 6x - 7)}{\lim_{x \to 1} (x^2 + 5x - 6)}$$

$$x \to 1$$
  $x \to -5$   $x \to -5$   $\lim_{x \to 1} (x \to -5)$   
 $\lim_{x \to 1} (x - 3)$ 

3. 
$$\lim_{x \to 1} \frac{x-3}{x^2+2x-4} = \frac{\lim_{x \to 1} (x^2+2x-4)}{\lim_{x \to 1} (x^2+2x-4)}$$

- **4.** If  $\lim_{x\to 5} f(x) = 2$  and  $\lim_{x\to 5} g(x) = 0$ , then  $\lim_{x\to 5} [f(x)/g(x)]$  does not exist.
- **5.** If  $\lim_{x\to 5} f(x) = 0$  and  $\lim_{x\to 5} g(x) = 0$ , then  $\lim_{x\to 5} [f(x)/g(x)]$  does not exist.
- **6.** If  $\lim_{x\to 6} f(x)g(x)$  exists, then the limit must be f(6)g(6).
- 7. If p is a polynomial, then  $\lim_{x\to b} p(x) = p(b)$ .

- 8. If  $\lim_{x\to 0} f(x) = \infty$  and  $\lim_{x\to 0} g(x) = \infty$ , then  $\lim_{x \to 0} [f(x) - g(x)] = 0.$
- **9.** If the line x = 1 is a vertical asymptote of y = f(x), then f is not defined at 1.
- 10. If f(1) > 0 and f(3) < 0, then there exists a number c between 1 and 3 such that f(c) = 0.
- **11.** If f is continuous at 5 and f(5) = 2 and f(4) = 3, then  $\lim_{x \to 2} f(4x^2 - 11) = 2.$
- 12. If f is continuous on [-1, 1] and f(-1) = 4 and f(1) = 3, then there exists a number r such that |r| < 1and  $f(r) = \pi$ .
- **13.** If f is continuous at a, then f is differentiable at a.

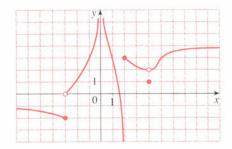
$$14. \ \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

- **15.** An equation of the tangent line to the parabola  $y = x^2$ at (-2, 4) is y - 4 = 2x(x + 2).
- 16. If f'(r) exists, then  $\lim_{x\to r} f(x) = f(r)$ .

- I. The graph of f is given.
  - (a) Find each limit, or explain why it does not exist. (i)  $\lim_{x \to \infty} f(x)$ (ii)  $\lim_{x \to \infty} f(x)$

(i) 
$$\lim_{x \to 2^+} f(x)$$
 (ii) 
$$\lim_{x \to -3^+} f(x)$$
  
(iii) 
$$\lim_{x \to -3} f(x)$$
 (iv) 
$$\lim_{x \to 4} f(x)$$
  
(v) 
$$\lim_{x \to 0} f(x)$$
 (vi) 
$$\lim_{x \to 2^-} f(x)$$
  
(vii) 
$$\lim_{x \to 0} f(x)$$
 (viii) 
$$\lim_{x \to -\infty} f(x)$$

- (b) State the equations of the horizontal asymptotes.
- (c) State the equations of the vertical asymptotes.
- (d) At what numbers is f discontinuous?



2. Sketch the graph of an example of a function f that satisfies all of the following conditions:  $\lim f(x) = -2, \quad \lim f(x) = 1, \quad f(0) = -1,$  $x \rightarrow 0^{-1}$  $x \rightarrow 0^+$  $\lim f(x) = \infty$ ,  $\lim f(x) = -\infty$ ,  $x \rightarrow 2^+$  $x \rightarrow 2^{-}$  $\lim f(x) = 3$ ,  $\lim f(x) = 4$  $\chi \rightarrow \infty$ 

**3–16** ■ Find the limit.

3. 
$$\lim_{x \to 0} \tan(x^2)$$
4. 
$$\lim_{t \to -1} \frac{t+1}{t^3 - t}$$
5. 
$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$
6. 
$$\lim_{h \to 0} \frac{(1+h)^{-2} - 1}{h}$$
7. 
$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 + 3x - 2}$$
8. 
$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 + 3x + 2}$$
9. 
$$\lim_{t \to 6} \frac{17}{(t-6)^2}$$
10. 
$$\lim_{x \to -6^+} \frac{x}{x+6}$$
11. 
$$\lim_{x \to 8^-} \frac{|x-8|}{x-8}$$
12. 
$$\lim_{x \to 10^-} \ln(100 - x^2)$$

**13.** 
$$\lim_{x \to 0} \frac{1 - \sqrt{1 - x^2}}{x}$$

**15.** lim  $e^{-x^2}$  $\chi \rightarrow \infty$ 

17-18 Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

17. 
$$y = \frac{\cos^2 x}{x^2}$$
  
18.  $y = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$ 

- **19.** If  $2x 1 \le f(x) \le x^2$  for 0 < x < 3, find  $\lim_{x \to 1} f(x)$ .
- **20.** Prove that  $\lim_{x\to 0} x^2 \cos(1/x^2) = 0$ .

21. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0\\ 3 - x & \text{if } 0 \le x < 3\\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

- (a) Evaluate each limit, if it exists.
  - (i)  $\lim f(x)$ (ii)  $\lim f(x)$ (iii)  $\lim_{x \to \infty} f(x)$  $x \rightarrow 0^{-1}$  $x \rightarrow 0$  $x \rightarrow 0^+$ (vi)  $\lim f(x)$ (iv)  $\lim f(x)$ (v)  $\lim f(x)$  $x \rightarrow 3^{\circ}$  $x \rightarrow 3^{+}$  $x \rightarrow 3$
- (b) Where is *f* discontinuous?
- (c) Sketch the graph of f.
- 22. Show that each function is continuous on its domain. State the domain.

(a) 
$$g(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 2}$$
 (b)  $h(x) = xe^{\sin x}$ 

23-24 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

**23.** 
$$2x^3 + x^2 + 2 = 0$$
,  $(-2, -1)$   
**24.**  $e^{-x^2} = x$ ,  $(0, 1)$ 

2

 $x^2$ 

14.  $\lim_{x \to \infty} \frac{1 + 2x - x^2}{1 - x + 2x^2}$ 

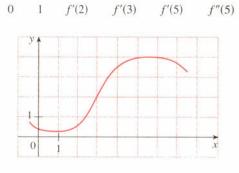
16. lim  $\arctan(-x^2)$ 

 $X \rightarrow \infty$ 

- 25. The displacement (in meters) of an object moving in a straight line is given by  $s = 1 + 2t + t^2/4$ , where t is measured in seconds.
  - (a) Find the average velocity over the following time periods.
    - (i) [1,3] (ii) [1,2]
    - (iv) [1, 1.1] (iii) [1, 1.5]
  - (b) Find the instantaneous velocity when t = 1.
- 26. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, PV = 800, where P is measured in pounds per square inch and V is measured in cubic inches.
  - (a) Find the average rate of change of P as V increases from 200 in3 to 250 in3.

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- (b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.
- **27.** For the function *f* whose graph is shown, arrange the following numbers in increasing order:



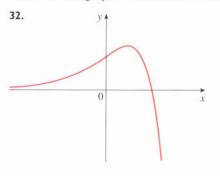
- **28.** (a) Use the definition of a derivative to find f'(2), where  $f(x) = x^3 2x$ .
  - (b) Find an equation of the tangent line to the curve  $y = x^3 2x$  at the point (2, 4).
  - (c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.
- **29.** (a) If  $f(x) = e^{-x^2}$ , estimate the value of f'(1) graphically and numerically.
  - (b) Find an approximate equation of the tangent line to the curve  $y = e^{-x^2}$  at the point where x = 1.
  - (c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.
  - **30.** Find a function f and a number a such that

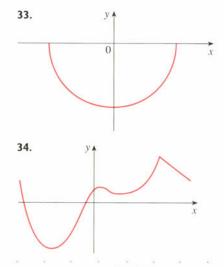
 $\mathcal{M}$ 

$$\lim_{h \to 0} \frac{(2+h)^6 - 64}{h} = f'(a)$$

- **31.** The total cost of paying off a student loan at an interest rate of r% per year is C = f(r).
  - (a) What is the meaning of the derivative f'(r)? What are its units?
  - (b) What does the statement f'(10) = 1200 mean?
  - (c) Is f'(r) always positive or does it change sign?

**32–34** Trace or copy the graph of the given function. Then sketch a graph of its derivative directly beneath.

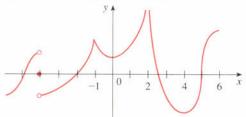




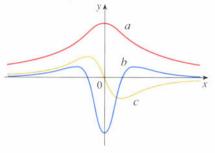
- **35.** (a) If  $f(x) = \sqrt{3 5x}$ , use the definition of a derivative to find f'(x).
  - (b) Find the domains of f and f'.

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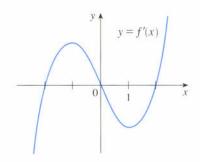
- (c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.
- **36.** (a) Find the asymptotes of the graph of f(x) = (4 x)/(3 + x) and use them to sketch the graph.
  - (b) Use your graph from part (a) to sketch the graph of *f*'.
  - (c) Use the definition of a derivative to find f'(x).
  - (d) Use a graphing device to graph f' and compare with your sketch in part (b).
- **37.** The graph of *f* is shown. State, with reasons, the numbers at which *f* is not differentiable.



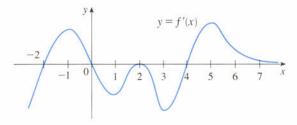
**38.** The figure shows the graphs of *f*, *f'*, and *f''*. Identify each curve, and explain your choices.



- **39.** (a) If  $f(x) = e^x$ , what is the value of f'(0)?
  - (b) Find the linear approximation for f at a = 0.
  - (c) Use the linear approximation to estimate the values of  $e^{-0.2}$ ,  $e^{-0.1}$ ,  $e^{-0.01}$ ,  $e^{0.01}$ ,  $e^{0.1}$ , and  $e^{0.2}$ .
  - (d) Are your approximations overestimates or underestimates? Which of your estimates are the most accurate?
- **40.** The cost of living continues to rise, but at a slower rate. In terms of a function and its derivatives, what does this statement mean?
- **41.** The graph of the derivative f' of a function f is given.
  - (a) On what intervals is *f* increasing or decreasing?
  - (b) At what values of x does f have a local maximum or minimum?
  - (c) Where is f concave upward or downward?
  - (d) If f(0) = 0, sketch a possible graph of f.



- **42.** The figure shows the graph of the derivative f' of a function *f*.
  - (a) Sketch the graph of f''.
  - (b) Sketch a possible graph of f.

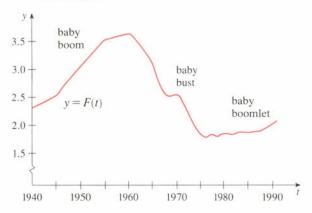


**43.** Sketch the graph of a function that satisfies the given conditions:

 $\begin{aligned} f(0) &= 0, \quad f'(-2) = f'(1) = f'(9) = 0, \\ \lim_{x \to \infty} f(x) &= 0, \quad \lim_{x \to 6} f(x) = -\infty, \\ f'(x) &< 0 \text{ on } (-\infty, -2), (1, 6), \text{ and } (9, \infty), \\ f'(x) &> 0 \text{ on } (-2, 1) \text{ and } (6, 9), \end{aligned}$ 

$$f''(x) > 0$$
 on  $(-\infty, 0)$  and  $(12, \infty)$ ,  
 $f''(x) < 0$  on  $(0, 6)$  and  $(6, 12)$ 

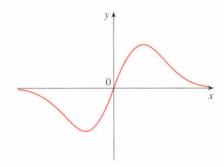
- **44.** The *total fertility rate* at time *t*, denoted by F(t), is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 1990.
  - (a) Estimate the values of F'(1950), F'(1965), and F'(1987).
  - (b) What are the meanings of these derivatives?
  - (c) Can you suggest reasons for the values of these derivatives?



**45.** A car starts from rest and its distance traveled is recorded in the table in 2-second intervals.

t (se	conds)	0	2	4	6	8	10	12	14
8.0	feet)	0	8	40	95	180	260	319	373

- (a) Estimate the speed after 6 seconds.
- (b) Estimate the coordinates of the inflection point of the graph of the position function.
- (c) What is the significance of the inflection point?
- **46.** The graph of a function is shown. Sketch the graph of an antiderivative *F*, given that F(0) = 0.





In our discussion of the principles of problem solving we considered the problem solving strategy called *Introduce Something Extra* (see page 88). In the following example we show how this principle is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 5.5, we will make more extensive use of this general idea.

**Example 1** Evaluate 
$$\lim_{x \to 0} \frac{\sqrt[3]{1 + cx} - 1}{x}$$
, where c is a constant.

Solution As it stands, this limit looks challenging. In Section 2.3 we evaluated several limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable *t* by the equation

1

$$t = \sqrt[3]{1 + cx}$$

We also need to express x in terms of t, so we solve this equation:

$$t^3 = 1 + cx$$
  $x = \frac{t^3 - 1}{c}$ 

Notice that  $x \to 0$  is equivalent to  $t \to 1$ . This allows us to convert the given limit into one involving the variable *t*:

$$\lim_{x \to 0} \frac{\sqrt[3]{1 + cx - 1}}{x} = \lim_{t \to 1} \frac{t - 1}{(t^3 - 1)/c}$$
$$= \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1}$$

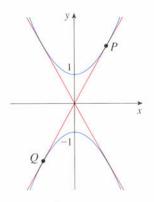
The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\lim_{t \to 1} \frac{c(t-1)}{t^3 - 1} = \lim_{t \to 1} \frac{c(t-1)}{(t-1)(t^2 + t + 1)}$$
$$= \lim_{t \to 1} \frac{c}{t^2 + t + 1} = \frac{c}{3}$$

Before you look at Example 2, cover up the solution and try it yourself first.

**Example 2** How many lines are tangent to both of the parabolas  $y = -1 - x^2$  and  $y = 1 + x^2$ ? Find the coordinates of the points at which these tangents touch the parabolas.

**Solution** To gain insight into this problem it is essential to draw a diagram. So we sketch the parabolas  $y = 1 + x^2$  (which is the standard parabola  $y = x^2$  shifted 1 unit upward) and  $y = -1 - x^2$  (which is obtained by reflecting the first parabola about the *x*-axis). If we try to draw a line tangent to both





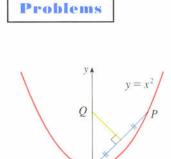


Figure for Problem 4

parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

Let *P* be a point at which one of these tangents touches the upper parabola and let *a* be its *x*-coordinate. (The choice of notation for the unknown is important. Of course we could have used *b* or *c* or  $x_0$  or  $x_1$  instead of *a*. However, it's not advisable to use *x* in place of *a* because that *x* could be confused with the variable *x* in the equation of the parabola.) Then, since *P* lies on the parabola  $y = 1 + x^2$ , its *y*-coordinate must be  $1 + a^2$ . Because of the symmetry shown in Figure 1, the coordinates of the point *Q* where the tangent touches the lower parabola must be  $(-a, -(1 + a^2))$ .

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P. We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If  $f(x) = 1 + x^2$ , then the slope of the tangent line at *P* is f'(a). Using the definition of the derivative as in Section 2.7, we find that f'(a) = 2a. Thus, the condition that we need to use is that

$$\frac{1+a^2}{a} = 2a$$

Solving this equation, we get  $1 + a^2 = 2a^2$ , so  $a^2 = 1$  and  $a = \pm 1$ . Therefore, the points are (1, 2) and (-1, -2). By symmetry, the two remaining points are (-1, 2) and (1, -2).

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving on page 87.

1. Evaluate 
$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$$
.  
2. Find numbers *a* and *b* such that  $\lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1$ 

- **3.** Evaluate  $\lim_{x \to 0} \frac{|2x 1| |2x + 1|}{x}$ .
- 4. The figure shows a point P on the parabola  $y = x^2$  and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.
- **5.** If [x] denotes the greatest integer function, find  $\lim_{x \to \infty} \frac{x}{[x]}$ .

- 6. Sketch the region in the plane defined by each of the following equations.
  (a) [[x]]<sup>2</sup> + [[y]]<sup>2</sup> = 1
  (b) [[x]]<sup>2</sup> [[y]]<sup>2</sup> = 3
  - (c)  $[x + y]^2 = 1$  (d) [x] + [y] = 1
- **7.** Find all values of *a* such that *f* is continuous on  $\mathbb{R}$ :

$$f(x) = \begin{cases} x+1 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases}$$

- 8. A fixed point of a function f is a number c in its domain such that f(c) = c. (The function doesn't move c; it stays fixed.)
  - (a) Sketch the graph of a continuous function with domain [0, 1] whose range also lies in [0, 1]. Locate a fixed point of f.
  - (b) Try to draw the graph of a continuous function with domain [0, 1] and range in [0, 1] that does *not* have a fixed point. What is the obstacle?
  - (c) Use the Intermediate Value Theorem to prove that any continuous function with domain [0, 1] and range in [0, 1] must have a fixed point.
- **9.** (a) If we start from  $0^{\circ}$  latitude and proceed in a westerly direction, we can let T(x) denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.
  - (b) Does the result in part (a) hold for points lying on any circle on Earth's surface?
  - (c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?
- 10. (a) The figure shows an isosceles triangle ABC with ∠B = ∠C. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude |AM| of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.
  - (b) Try to sketch the path traced out by *P* during this process. Then find the equation of this curve and use this equation to sketch the curve.
- **II.** Find points P and Q on the parabola  $y = 1 x^2$  so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle.
- 12. Water is flowing at a constant rate into a spherical tank. Let V(t) be the volume of water in the tank and H(t) be the height of the water in the tank at time t.
  - (a) What are the meanings of V'(t) and H'(t)? Are these derivatives positive, negative, or zero?
  - (b) Is V''(t) positive, negative, or zero? Explain.
  - (c) Let t<sub>1</sub>, t<sub>2</sub>, and t<sub>3</sub> be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values H"(t<sub>1</sub>), H"(t<sub>2</sub>), and H"(t<sub>3</sub>) positive, negative, or zero? Why?
- **13.** Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

(b) Find f'(0).

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(a) Find f(0).

(c) Find f'(x).

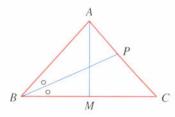


Figure for Problem 10

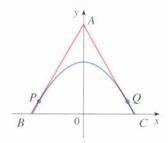


Figure for Problem 11

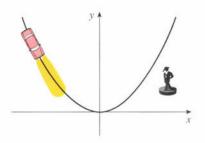


Figure for Problem 14

14. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

**5.** If 
$$\lim_{x \to a} [f(x) + g(x)] = 2$$
 and  $\lim_{x \to a} [f(x) - g(x)] = 1$ , find  $\lim_{x \to a} f(x)g(x)$ .

16. If f is a differentiable function and g(x) = x f(x), use the definition of a derivative to show that

$$g'(x) = xf'(x) + f(x)$$

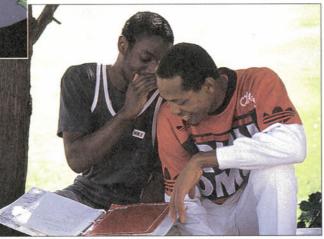
- 17. Suppose f is a function with the property that  $|f(x)| \le x^2$  for all x. Show that f(0) = 0. Then show that f'(0) = 0.
- 18. (a) Suppose  $\lim_{x\to\infty} f(x) = 2$  and f'(x) > 0 for all x. Does  $\lim_{x\to\infty} f'(x)$  exist? If so, what is its value? Give an intuitive explanation, supported by a sketch.
  - (b) Sketch the graph of a function f for which lim<sub>x→∞</sub> f(x) = 2 but lim<sub>x→∞</sub> f'(x) does not exist.





Differentiation Rules

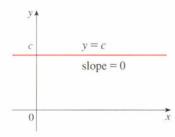
• • • These photographs represent derivatives in various contexts. The racing car driver wants to know his speed at a given time. Because blood flows more slowly near the wall of a blood vessel, we might want to know the rate at which blood velocity increases with respect to distance from the wall. The rate at which a rumor spreads depends on the number of people involved and the way they react to information. The rate of expansion of the internet is exponential. (The depiction of the global structure of the multicast backbone shows the major arteries of the internet and is used to help plan its future expansion.) These rates of change are all special cases of a single mathematical idea, the derivative.



◆ We have seen how to interpret derivatives as slopes and rates of change. We have seen how to estimate derivatives of functions given by tables of values. We have learned how to graph derivatives of functions that are defined graphically. We have used the definition of a derivative to calculate the derivatives of functions defined by formulas. But it would be tedious if we always had to use the definition, so in this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, trigonometric and inverse trigonometric functions. We then use these rules to solve problems involving rates of change, tangents to parametric curves, and the approximation of functions.



# **Derivatives of Polynomials and Exponential Functions**



In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Let's start with the simplest of all functions, the constant function f(x) = c. The graph of this function is the horizontal line y = c, which has slope 0, so we must have f'(x) = 0 (see Figure 1). A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h}$$
$$= \lim_{h \to 0} 0 = 0$$

FIGURE I The graph of f(x) = c is the line y = c, so f'(x) = 0.

In Leibniz notation, we write this rule as follows.

## **Derivative of a Constant Function**

$$\frac{d}{dx}\left(c\right) = 0$$

y A v = xslope = 1×

#### FIGURE 2

The graph of f(x) = x is the line y = x, so f'(x) = 1.

# **Power Functions**

We next look at the functions  $f(x) = x^n$ , where n is a positive integer. If n = 1, the graph of f(x) = x is the line y = x, which has slope 1 (see Figure 2). So

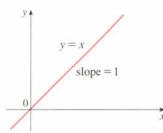
$$\frac{d}{dx}(x) = 1$$

(You can also verify Equation 1 from the definition of a derivative.) We have already investigated the cases n = 2 and n = 3. In fact, in Section 2.8 (Exercises 17) and 18) we found that

2 
$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

For n = 4 we find the derivative of  $f(x) = x^4$  as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$
$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$
$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$
$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$$



Thus

3

$$\frac{d}{dx}(x^4) = 4x^3$$

Comparing the equations in (1), (2), and (3), we see a pattern emerging. It seems to be a reasonable guess that, when *n* is a positive integer,  $(d/dx)(x^n) = nx^{n-1}$ . This turns out to be true.

**The Power Rule** If *n* is a positive integer, then

$$\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$$

**Proof** If  $f(x) = x^n$ , then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

The Binomial Theorem is given on the front endpapers.

In finding the derivative of  $x^4$  we had to expand  $(x + h)^4$ . Here we need to expand  $(x + h)^n$  and we use the Binomial Theorem to do so:

$$f'(x) = \lim_{h \to 0} \frac{\left[ x^n + nx^{n-1}h + \frac{n(n-1)}{2} x^{n-2}h^2 + \dots + nxh^{n-1} + h^n \right] - x^n}{h}$$
$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2} x^{n-2}h^2 + \dots + nxh^{n-1} + h^n}{h}$$
$$= \lim_{h \to 0} \left[ nx^{n-1} + \frac{n(n-1)}{2} x^{n-2}h + \dots + nxh^{n-2} + h^{n-1} \right]$$
$$= nx^{n-1}$$

because every term except the first has h as a factor and therefore approaches 0.

We illustrate the Power Rule using various notations in Example 1.

# EXAMPLE I

(a) If 
$$f(x) = x^6$$
, then  $f'(x) = 6x^5$ . (b) If  $y = x^{1000}$ , then  $y' = 1000x^{999}$ .  
(c) If  $y = t^4$ , then  $\frac{dy}{dt} = 4t^3$ . (d)  $\frac{d}{dr}(r^3) = 3r^2$ 

What about power functions with negative integer exponents? In Exercise 51 we ask you to verify from the definition of a derivative that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

We can rewrite this equation as

$$\frac{d}{dx}(x^{-1}) = (-1)x^{-2}$$

and so the Power Rule is true when n = -1. In fact, we will show in the next section (Exercise 41) that it holds for all negative integers.

What if the exponent is a fraction? In Example 4 in Section 2.8 we found that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

which can be written as

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

This shows that the Power Rule is true even when  $n = \frac{1}{2}$ . In fact, we will show in Section 3.7 that it is true for all real numbers *n*.

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$$

**EXAMPLE 2** Differentiate:

(b)

(a) 
$$f(x) = \frac{1}{x^2}$$
 (b)  $y = \sqrt[3]{x^2}$ 

**SOLUTION** In each case we rewrite the function as a power of x. (a) Since  $f(x) = x^{-2}$ , we use the Power Rule with n = -2:

$$f'(x) = \frac{d}{dx} (x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt[3]{x^2} = \frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$$

**EXAMPLE 3** Find an equation of the tangent line to the curve 
$$y = x\sqrt{x}$$
 at the point (1, 1). Illustrate by graphing the curve and its tangent line.

**SOLUTION** The derivative of  $f(x) = x\sqrt{x} = xx^{1/2} = x^{3/2}$  is

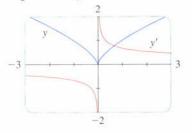
$$f'(x) = \frac{3}{2}x^{(3/2)-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$$

So the slope of the tangent line at (1, 1) is  $f'(1) = \frac{3}{2}$ . Therefore, an equation of the tangent line is

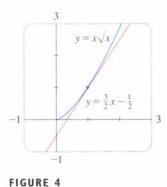
$$y - 1 = \frac{3}{2}(x - 1)$$
 or  $y = \frac{3}{2}x - \frac{1}{2}$ 

We graph the curve and its tangent line in Figure 4.

Figure 3 shows the function y in Example 2(b) and its derivative y'. Notice that y is not differentiable at 0 (y' is not defined there). Observe that y' is positive when y increases and is negative when y decreases.







# New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions. In particular, the following formula says that the derivative of a constant times a function is the constant times the derivative of the function.

**The Constant Multiple Rule** If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}\left[cf(x)\right] = c \frac{d}{dx}f(x)$$

**Proof** Let q(x) = c f(x). Then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$
$$= \lim_{h \to 0} c \left[ \frac{f(x+h) - f(x)}{h} \right]$$
$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(by Law 3 of limits)}$$
$$= cf'(x)$$

# EXAMPLE 4

(a) 
$$\frac{d}{dx}(3x^4) = 3 \frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$$
  
(b)  $\frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x] = (-1) \frac{d}{dx}(x) = -1(1) = -1$ 

The next rule tells us that the derivative of a sum of functions is the sum of the derivatives.

Using the prime notation, we can write the Sum Rule as

(f+g)'=f'+g'

d d

**The Sum Rule** If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

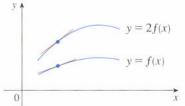
d

**Proof** Let F(x) = f(x) + g(x). Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$
  
= 
$$\lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$
  
= 
$$\lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$
  
= 
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
 (by Law 1)  
= 
$$f'(x) + g'(x)$$

of the Constant Multiple Rule y A

**Geometric Interpretation** 



Multiplying by c = 2 stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are doubled, too.

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

By writing f - g as f + (-1)g and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

**The Difference Rule** If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

These three rules can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

### EXAMPLE 5

$$\frac{d}{dx} (x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx} (x^8) + 12 \frac{d}{dx} (x^5) - 4 \frac{d}{dx} (x^4) + 10 \frac{d}{dx} (x^3) - 6 \frac{d}{dx} (x) + \frac{d}{dx} (5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$

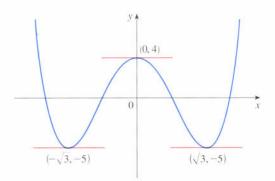
$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

**EXAMPLE 6** Find the points on the curve  $y = x^4 - 6x^2 + 4$  where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx} (x^4) - 6 \frac{d}{dx} (x^2) + \frac{d}{dx} (4)$$
$$= 4x^3 - 12x + 0 = 4x(x^2 - 3)$$

Thus, dy/dx = 0 if x = 0 or  $x^2 - 3 = 0$ , that is,  $x = \pm\sqrt{3}$ . So the given curve has horizontal tangents when x = 0,  $\sqrt{3}$ , and  $-\sqrt{3}$ . The corresponding points are (0, 4),  $(\sqrt{3}, -5)$ , and  $(-\sqrt{3}, -5)$ . (See Figure 5.)



**FIGURE 5** The curve  $y = x^4 - 6x^2 + 4$  and its horizontal tangents

**EXAMPLE 7** The equation of motion of a particle is  $s = 2t^3 - 5t^2 + 3t + 4$ , where *s* is measured in centimeters and *t* in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

SOLUTION The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$
$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is  $a(2) = 14 \text{ cm/s}^2$ .

# **Exponential Functions**

4

Let us try to compute the derivative of the exponential function  $f(x) = a^x$  using the definition of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x (a^h - 1)}{h}$$

The factor  $a^x$  doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

Therefore, we have shown that if the exponential function  $f(x) = a^x$  is differentiable at 0, then it is differentiable everywhere and

$$f'(x) = f'(0)a^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself.* (The slope is proportional to the height.)

Numerical evidence for the existence of f'(0) is given in the table at the left for the cases a = 2 and a = 3. (Values are stated correct to four decimal places. For the case a = 2, see also Example 3 in Section 2.7.) It appears that the limits exist and

for 
$$a = 2$$
,  $f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$ 

for 
$$a = 3$$
,  $f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$ 

12	$2^{h} = 1$	3'' = 1	
h	h	h	
0.1	0.7177	1.1612	
0.01	0.6956	1.1047	
0.001	0.6934	1.0992	
0.0001	0.6932	1.0987	

In fact, it can be proved that the limits exist and, correct to six decimal places, the values are

$$\frac{d}{dx}(2^x)\Big|_{x=0} \approx 0.693147$$
  $\frac{d}{dx}(3^x)\Big|_{x=0} \approx 1.098612$ 

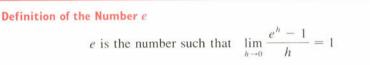
Thus, from Equation 4 we have

5 
$$\frac{d}{dx}(2^x) \approx (0.69)2^x$$
  $\frac{d}{dx}(3^x) \approx (1.10)3^x$ 

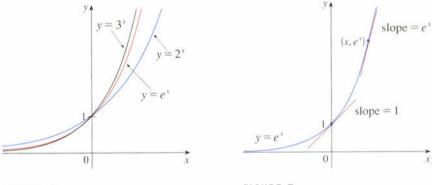
Of all possible choices for the base a in Equation 4, the simplest differentiation formula occurs when f'(0) = 1. In view of the estimates of f'(0) for a = 2 and a = 3, it seems reasonable that there is a number a between 2 and 3 for which f'(0) = 1. It is traditional to denote this value by the letter e. (In fact, that is how we introduced e in Section 1.5.) Thus, we have the following definition.

In Exercise I we will see that e lies between 2.7 and 2.8. Later we will be able to show that, correct to five decimal places,

 $e \approx 2.71828$ 



Geometrically, this means that of all the possible exponential functions  $y = a^x$ , the function  $f(x) = e^x$  is the one whose tangent line at (0, 1) has a slope f'(0) that is exactly 1. (See Figures 6 and 7.)



## FIGURE 6

FIGURE 7

If we put a = e and, therefore, f'(0) = 1 in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function  $\frac{d}{dx} (e^x) = e^x$ 

Thus, the exponential function  $f(x) = e^x$  has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve  $y = e^x$  is equal to the y-coordinate of the point (see Figure 7).

**EXAMPLE 8** If 
$$f(x) = e^x - x$$
, find f' and f''.

**SOLUTION** Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx} (e^x - x) = \frac{d}{dx} (e^x) - \frac{d}{dx} (x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx} (e^x - 1) = \frac{d}{dx} (e^x) - \frac{d}{dx} (1) = e^x$$

We know that  $e^x$  is positive for all x, so f''(x) > 0 for all x. Thus, the graph of f is concave upward on  $(-\infty, \infty)$ . This is confirmed in Figure 8.

**EXAMPLE 9** At what point on the curve  $y = e^x$  is the tangent line parallel to the line y = 2x?

**SOLUTION** Since  $y = e^x$ , we have  $y' = e^x$ . Let the *x*-coordinate of the point in question be *a*. Then the slope of the tangent line at that point is  $e^a$ . This tangent line will be parallel to the line y = 2x if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \qquad a = \ln 2$$

Therefore, the required point is  $(a, e^a) = (\ln 2, 2)$ . (See Figure 9.)



**I.** (a) How is the number *e* defined?

Exercises · ·

(b) Use a calculator to estimate the values of the limits

2

$$\lim_{h \to 0} \frac{2.7^n - 1}{h} \text{ and } \lim_{h \to 0} \frac{2.8^n - 1}{h}$$

correct to two decimal places. What can you conclude about the value of *e*?

- (a) Sketch, by hand, the graph of the function
   f(x) = e<sup>x</sup>, with particular attention to how the
   graph crosses the y-axis. What fact allows you to
   do this?
  - (b) What types of functions are f(x) = e<sup>x</sup> and g(x) = x<sup>e</sup>? Compare the differentiation formulas for f and g.
  - (c) Which of the two functions in part (b) grows more rapidly when *x* is large?
- **3–20** Differentiate the function.
- **3.**  $y = x^8$  **4.**  $y = \sqrt[3]{x}$
- **5.**  $y = x^{-2/5}$  **6.**  $y = 5e^x + 3$
- **7.**  $f(x) = x^2 10x + 100$  **8.**  $q(x) = x^{100} + 50x + 1$
- **9.**  $V(r) = \frac{4}{3}\pi r^3$  **10.**  $s(t) = t^8 + 6t^7 18t^2 + 2t$

 11.  $Y(t) = 6t^{-9}$  12.  $R(x) = \frac{\sqrt{10}}{x^7}$  

 13.  $F(x) = (16x)^3$  14.  $H(t) = \sqrt[3]{t} (t+2)$  

 15.  $g(x) = x^2 + \frac{1}{x^2}$  16.  $f(t) = \sqrt{t} - \frac{1}{\sqrt{t}}$  

 17.  $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$  18.  $y = \frac{x^2 - 2\sqrt{x}}{x}$  

 19.  $y = 3x + 2e^x$  20.  $y = e^{x^{+1}} + 1$ 

**21–26** Find f'(x). Compare the graphs of f and f' and use them to explain why your answer is reasonable.

**21.**  $f(x) = 2x^2 - x^4$  **22.**  $f(x) = 3x^5 - 20x^3 + 50x$  **23.**  $f(x) = 3x^{15} - 5x^3 + 3$  **24.**  $f(x) = x + \frac{1}{x}$  **25.**  $f(x) = x - 3x^{1/3}$ **26.**  $f(x) = x^2 + 2e^x$ 

**27.** (a) By zooming in on the graph of  $f(x) = x^{2/5}$ , estimate the value of f'(2).

(b) Use the Power Rule to find the exact value of f'(2) and compare with your estimate in part (a).



 $v = e^x - x$ 

3

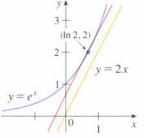


FIGURE 9

- **28.** (a) By zooming in on the graph of  $f(x) = x^2 2e^x$ , estimate the value of f'(1).
  - (b) Find the exact value of f'(1) and compare with your estimate in part (a).

29-32 Find an equation of the tangent line to the given curve at the specified point. Illustrate by graphing the curve and the tangent line on the same screen.

**29.** 
$$y = x + \frac{4}{x}$$
, (2,4)  
**30.**  $y = x^{5/2}$ , (4,32)  
**31.**  $y = x + \sqrt{x}$ , (1,2)  
**32.**  $y = x^2 + 2e^x$ , (0,2)

**33–34** Find the first and second derivatives of the function.

- **33.**  $f(x) = x^4 3x^3 + 16x$
- **34.**  $G(r) = \sqrt{r} + \sqrt[3]{r}$

**35–36** Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of *f*, *f'*, and *f''*.

- **35.**  $f(x) = 2x 5x^{3/4}$  **36.**  $f(x) = e^x x^3$
- 37. The equation of motion of a particle is s = t<sup>3</sup> 3t, where s is in meters and t is in seconds. Find
  (a) the velocity and acceleration as functions of t,
  - (b) the acceleration after 2 s,
  - (c) the acceleration when the velocity is 0.
- 38. The equation of motion of a particle is

 $s = 2t^3 - 7t^2 + 4t + 1$ , where s is in meters and t is in seconds.

- (a) Find the velocity and acceleration as functions of t.
- (b) Find the acceleration after 1 s.
- (c) Graph the position, velocity, and acceleration functions on the same screen.
- **39.** The table gives the population of the world in the 20th century.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3020
1910	1750	1970	3700
1920	1860	1980	4450
1930	2070	1990	5300
1940	2300	1996	5770
1950	2520		

- (a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
- (b) In Example 2 in Section 1.7 we found a cubic function that models these data remarkably well:

$$P = at^3 + bt^2 + ct + d$$

 $a = 2325.67 \qquad b = -1.306488 \times 10^{7}$  $c = 2.44631 \times 10^{10} \qquad d = -1.52658 \times 10^{13}$ 

Use this expression to find a model for the rate of population growth in the 20th century.

- (c) Use part (b) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).(d) Estimate the rate of growth in 1985.
- 40. The interest rate on treasury bills is a function of time. The following table gives midyear values of this function *I(t)* over a nine-year period (as a percent per year).

I.	1983	1984	1985	1986	1987
I(t)	8.62	9.57	7.49	5.97	5.83

Î.	1988	1989	1990	1991	1992
I(t)	6.67	8.11	7.51	5.41	3.46

- (a) Use the methods of Section 1.7 to model these data by a fourth-degree polynomial.
- (b) Use part (a) to find a model for I'(t).
- (c) Estimate the rate of change of interest rates in 1988 and 1991.
- (d) Graph the data points and the models for I and I'.
- **41.** On what interval is the function  $f(x) = 1 + 2e^x 3x$  increasing?
- **42.** On what interval is the function  $f(x) = x^3 4x^2 + 5x$  concave upward?
- **43.** Find the points on the curve  $y = x^3 x^2 x + 1$  where the tangent is horizontal.
- **44.** For what values of x does the graph of  $f(x) = 2x^3 3x^2 6x + 87$  have a horizontal tangent?
- **45.** Show that the curve  $y = 6x^3 + 5x 3$  has no tangent line with slope 4.
- **46.** At what point on the curve  $y = 1 + 2e^x 3x$  is the tangent line parallel to the line 3x y = 5? Illustrate by graphing the curve and both lines.
  - **47.** Draw a diagram to show that there are two tangent lines to the parabola  $y = x^2$  that pass through the point (0, -4). Find the coordinates of the points where these tangent lines intersect the parabola.
  - **48.** Find the equations of both lines through the point (2, -3) that are tangent to the parabola  $y = x^2 + x$ .
  - **49.** The **normal line** to a curve *C* at a point *P* is, by definition, the line that passes through *P* and is perpendicular to the tangent line to *C* at *P*. Find an equation of the normal line to the parabola  $y = 1 x^2$

at the point (2, -3). Sketch the parabola and its normal line.

- **50.** Where does the normal line to the parabola  $y = x x^2$  at the point (1,0) intersect the parabola a second time? Illustrate with a sketch.
- **51.** Use the definition of a derivative to show that if f(x) = 1/x, then  $f'(x) = -1/x^2$ . (This proves the Power Rule for the case n = -1.)
- **52.** Find the *n*th derivative of the function by calculating the first few derivatives and observing the pattern that occurs.

(a)  $f(x) = x^n$  (b) f(x) = 1/x

- 53. Find a second-degree polynomial P such that P(2) = 5, P'(2) = 3, and P''(2) = 2.
- 54. The equation  $y'' + y' 2y = x^2$  is called a **differential** equation because it involves an unknown function y and its derivatives y' and y". Find constants A, B, and C such that the function  $y = Ax^2 + Bx + C$  satisfies this equation. (Differential equations will be studied in detail in Chapter 7.)
- **55.** (a) In Section 2.10 we defined an antiderivative of f to be a function F such that F' = f. Try to guess a formula for an antiderivative of  $f(x) = x^2$ . Then

check your answer by differentiating it. How many antiderivatives does *f* have?

- (b) Find antiderivatives for  $f(x) = x^3$  and  $f(x) = x^4$ .
- (c) Find an antiderivative for  $f(x) = x^n$ , where  $n \neq -1$ . Check by differentiation.
- 56. Use the result of Exercise 55(c) to find an antiderivative of each function.
  (a) f(x) = √x
  (b) f(x) = e<sup>x</sup> + 8x<sup>3</sup>
- 57. Find a parabola with equation  $y = ax^2 + bx$  whose tangent line at (1, 1) has equation y = 3x 2.
- **58.** A tangent line is drawn to the hyperbola xy = c at a point *P*.
  - (a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is *P*.
  - (b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

**59.** Evaluate 
$$\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$$
.

**60.** Draw a diagram showing two perpendicular lines that intersect on the *y*-axis and are both tangent to the parabola  $y = x^2$ . Where do these lines intersect?



## The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions formed from old functions by multiplication or division.

## The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let f(x) = x and  $g(x) = x^2$ . Then the Power Rule gives f'(x) = 1and g'(x) = 2x. But  $(fg)(x) = x^3$ , so  $(fg)'(x) = 3x^2$ . Thus,  $(fg)' \neq f'g'$ . The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

Before stating the Product Rule, let's see how we might discover it. In the case where u = f(x) and v = g(x) are both positive functions, we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount  $\Delta x$ , then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product,  $(u + \Delta u)(v + \Delta v)$ , can be interpreted as the area of the large rectangle in Figure 1 (provided that  $\Delta u$  and  $\Delta v$  happen to be positive).

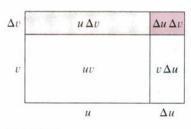


FIGURE I The geometry of the Product Rule

The change in the area of the rectangle is

$$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v$$

= the sum of the three shaded areas

If we divide by  $\Delta x$ , we get

$$\frac{\Delta(uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

Recall that in Leibniz notation the defini- If we now let  $\Delta x \rightarrow 0$ , we get the derivative of uv: tion of a derivative can be written as

2

$$\frac{y}{x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

 $\frac{d}{d}$ 

$$\frac{d}{dx}(uv) = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0} \left( u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right)$$
$$= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left( \lim_{\Delta x \to 0} \Delta u \right) \left( \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right)$$
$$= u \frac{dv}{dx} + v \frac{du}{dx} + 0 \cdot \frac{dv}{dx}$$
$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

(Notice that  $\Delta u \rightarrow 0$  as  $\Delta x \rightarrow 0$  since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u, v,  $\Delta u$ , and  $\Delta v$  are positive or negative.) So we have proved Equation 2, known as the Product Rule, for all differentiable functions u and v.

**The Product Rule** If 
$$f$$
 and  $g$  are both differentiable, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

In words, the Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

**EXAMPLE 1** A telephone company wants to estimate the number of new residential phone lines that it will need to install during the upcoming month. At the beginning of January, 1997, the company had 100,000 subscribers, each of whom had 1.2 phone lines, on average. The company estimated that its subscribership was increasing at the rate of 1000 monthly. By polling its existing subscribers, the company found that each intended to install an average of 0.01 new phone lines by the end of January. Estimate the number of new lines the company will have to install in January, 1997, by calculating the rate of increase of lines at the beginning of the month.

**SOLUTION** Let s(t) be the number of subscribers and let n(t) be the number of phone lines per subscriber at time t, where t is measured in years and t = 0 corresponds to the beginning of 1997. Then the total number of lines is given by

$$L(t) = s(t)n(t)$$

and we want to find L'(0). According to the Product Rule, we have

$$L'(t) = \frac{d}{dt} \left[ s(t)n(t) \right] = s(t) \frac{d}{dt} n(t) + n(t) \frac{d}{dt} s(t)$$

We are given that s(0) = 100,000 and n(0) = 1.2. The company's estimates concerning rates of increase are that  $s'(0) \approx 1000$  and  $n'(0) \approx 0.01$ . Therefore,

$$\mathcal{L}'(0) = s(0)n'(0) + n(0)s'(0)$$
  
\$\approx 100,000 \cdot 0.01 + 1.2 \cdot 1000 = 2200

The company will need to install approximately 2200 new phone lines during January of 1997.

Notice that the two terms arising from the Product Rule come from different sources—old subscribers and new subscribers. One contribution to L' is the number of existing subscribers (100,000) times the rate at which they order new lines (about 0.01 per subscriber monthly). A second contribution is the average number of lines per subscriber (1.2 at the beginning of the month) times the rate of increase of subscribers (1000 monthly).

## EXAMPLE 2

- (a) If  $f(x) = xe^x$ , find f'(x).
- (b) Find the *n*th derivative,  $f^{(n)}(x)$ .

#### SOLUTION

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx} (xe^x) = x \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x)$$
$$= xe^x + e^x \cdot 1 = (x+1)e^x$$

(b) Using the Product Rule a second time, we get

$$f''(x) = \frac{d}{dx} \left[ (x+1)e^x \right] = (x+1) \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x+1)$$
$$= (x+1)e^x + e^x \cdot 1 = (x+2)e^x$$

Further applications of the Product Rule give

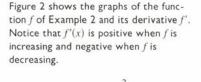
$$f'''(x) = (x + 3)e^x$$
  $f^{(4)}(x) = (x + 4)e^x$ 

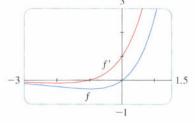
In fact, each successive differentiation adds another term  $e^x$ , so

$$f^{(n)}(x) = (x + n)e^x$$

**EXAMPLE 3** Differentiate the function  $f(t) = \sqrt{t} (1 - t)$ . SOLUTION 1 Using the Product Rule, we have

$$f'(t) = \sqrt{t} \frac{d}{dt} (1-t) + (1-t) \frac{d}{dt} \sqrt{t}$$
$$= \sqrt{t} (-1) + (1-t) \cdot \frac{1}{2} t^{-1/2}$$
$$= -\sqrt{t} + \frac{1-t}{2\sqrt{t}} = \frac{1-3t}{2\sqrt{t}}$$







**SOLUTION 2** If we first use the laws of exponents to rewrite f(t), then we can proceed directly without using the Product Rule.

$$f(t) = \sqrt{t} - t\sqrt{t} = t^{1/2} - t^{3/2}$$
$$f'(t) = \frac{1}{2}t^{-1/2} - \frac{3}{2}t^{1/2}$$

which is equivalent to the answer given in Solution 1.

Example 3 shows that it is sometimes easier to simplify a product of functions than to use the Product Rule. In Example 2, however, the Product Rule is the only possible method.

**EXAMPLE 4** If  $f(x) = \sqrt{x} q(x)$ , where q(4) = 2 and q'(4) = 3, find f'(4).

**SOLUTION** Applying the Product Rule, we get

$$f'(x) = \frac{d}{dx} \left[ \sqrt{x} g(x) \right] = \sqrt{x} \frac{d}{dx} \left[ g(x) \right] + g(x) \frac{d}{dx} \left[ \sqrt{x} \right]$$
$$= \sqrt{x} g'(x) + g(x) \cdot \frac{1}{2} x^{-1/2}$$
$$= \sqrt{x} g'(x) + \frac{g(x)}{2\sqrt{x}}$$

Therefore

$$f'(4) = \sqrt{4} g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$



#### The Quotient Rule

Suppose that f and g are differentiable functions. If we make the prior assumption that the quotient function F = f/g is differentiable, then it is not difficult to find a formula for F' in terms of f' and g'.

Since F(x) = f(x)/g(x), we can write f(x) = F(x)g(x) and apply the Product Rule:

$$f'(x) = F(x)g'(x) + g(x)F'(x)$$

Solving this equation for F'(x), we get

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)} = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$
$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$
$$\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Although we derived this formula under the assumption that F is differentiable, it can be proved without that assumption (see Exercise 42).

The Quotient Rule If f and g are both differentiable, then  

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} \left[ f(x) \right] - f(x) \frac{d}{dx} \left[ g(x) \right]}{\left[ g(x) \right]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denomi*nator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

EXAMPLE 5 Let 
$$y = \frac{x^2 + x - 2}{x^3 + 6}$$
.  
Then  $y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$   
 $= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$   
 $= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$   
 $= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$ 

**EXAMPLE 6** Find an equation of the tangent line to the curve  $y = e^x/(1 + x^2)$  at the point (1, e/2).

**SOLUTION** According to the Quotient Rule, we have

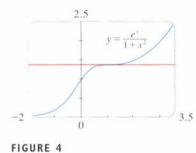
$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$
$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$$

So the slope of the tangent line at (1, e/2) is

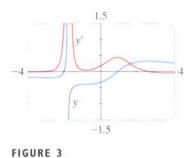
$$\left. \frac{ly}{lx} \right|_{x=1} = 0$$

This means that the tangent line at (1, e/2) is horizontal and its equation is y = e/2. [See Figure 4. Notice that the function is increasing and crosses its tangent line at (1, e/2).]

**NOTE** • Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the pur-



We can use a graphing device to check  
that the answer to Example 5 is plausible.  
Figure 3 shows the graphs of the func-  
tion of Example 5 and its derivative.  
Notice that when y grows rapidly (near  
$$-2$$
), y' is large. And when y grows  
slowly, y' is near 0.



pose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.



 $\mathcal{A}$ 

- **I.** Find the derivative of  $y = (x^2 + 1)(x^3 + 1)$  in two ways: by using the Product Rule and by performing the multiplication first. Show that your answers are equivalent.
- 2. Find the derivative of the function

Exercises · ·

$$F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$

in two ways: by using the Quotient Rule and by simplifying first. Do your answers agree?

**3–18** ■ Differentiate the function.

- 3.  $f(x) = x^2 e^x$ 4.  $g(x) = \sqrt{x} e^x$ 5.  $y = \frac{e^x}{x^2}$ 6.  $y = \frac{e^x}{1+x}$ 7.  $h(x) = \frac{x+2}{x-1}$ 8.  $f(u) = \frac{1-u^2}{1+u^2}$ 9.  $G(s) = (s^2 + s + 1)(s^2 + 2)$ 10.  $g(x) = (1 + \sqrt{x})(x - x^3)$ 11.  $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$ 12.  $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$ 13.  $y = (r^2 - 2r)e^r$ 14.  $y = \frac{u^2 - u - 2}{u+1}$ 15.  $y = \frac{1}{x^4 + x^2 + 1}$ 16.  $y = \frac{e^x}{x + e^x}$ 17.  $f(x) = \frac{x}{x + \frac{c}{x}}$ 18.  $f(x) = \frac{ax + b}{cx + d}$
- 19. (a) The curve y = 1/(1 + x<sup>2</sup>) is called a witch of Maria Agnesi. Find an equation of the tangent line to this curve at the point (-1, 1/2).
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

- **20.** (a) The curve  $y = x/(1 + x^2)$  is called a serpentine. Find an equation of the tangent line to this curve at the point (3, 0.3).
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

**21.** (a) If  $f(x) = e^{x}/x^{3}$ , find f'(x).

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

**22.** (a) If  $f(x) = x/(x^2 - 1)$ , find f'(x).

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

- **23.** (a) If  $f(x) = (x 1)e^{x}$ , find f'(x) and f''(x).
- (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f".

**24.** (a) If  $f(x) = x/(x^2 + 1)$ , find f'(x) and f''(x).

- (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f".
  - **25.** Suppose that f(5) = 1, f'(5) = 6, g(5) = -3, and g'(5) = 2. Find the values of (a) (fg)'(5), (b) (f/g)'(5), and (c) (g/f)'(5).
  - **26.** If f(3) = 4, g(3) = 2, f'(3) = -6, and g'(3) = 5, find the following numbers.

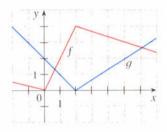
(a) 
$$(f + g)'(3)$$
 (b)  $(fg)'(3)$   
(c)  $(f/g)'(3)$  (d)  $(\frac{f}{f - g})'(3)$ 

**27.** If  $f(x) = e^{x}g(x)$ , where g(0) = 2 and g'(0) = 5, find f'(0).

**28.** If h(2) = 4 and h'(2) = -3, find

$$\left. \frac{d}{dx} \left( \frac{h(x)}{x} \right) \right|_{x=2}$$

(a) Find u'(1). (b) Find v'(5).



**30.** If *f* is a differentiable function, find an expression for the derivative of each of the following functions.

(a) 
$$y = x^2 f(x)$$
 (b)  $y = \frac{f(x)}{x^2}$   
(c)  $y = \frac{x^2}{f(x)}$  (d)  $y = \frac{1 + xf(x)}{\sqrt{x}}$ 

- **31.** In this exercise we estimate the rate at which the total personal income is rising in the Miami–Ft. Lauderdale metropolitan area. In July, 1993, the population of this area was 3,354,000, and the population was increasing at roughly 45,000 people per year. The average annual income was \$21,107 per capita, and this average was increasing at about \$1900 per year (well above the national average of about \$660 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in Miami–Ft. Lauderdale in July, 1993. Explain the meaning of each term in the Product Rule.
- 32. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write q = f(p). Then the total revenue earned with selling price p is R(p) = pf(p).
  (a) What does it mean to say that f(20) = 10,000 and
  - f'(20) = -350? (b) Assuming the values in part (a), find R'(20) and
  - (b) Assuming the values in part (a), find *R* (20) and interpret your answer.
- **33.** On what interval is the function  $f(x) = x^3 e^x$  increasing?
- **34.** On what interval is the function  $f(x) = x^2 e^x$  concave downward?

- **35.** How many tangent lines to the curve y = x/(x + 1) pass through the point (1, 2)? At which points do these tangent lines touch the curve?
- 36. Find the equations of the tangent lines to the curve y = (x 1)/(x + 1) that are parallel to the line x 2y = 2.
- **37.** (a) Use the Product Rule twice to prove that if *f*, *g*, and *h* are differentiable, then

$$(fgh)' = f'gh + fg'h + fgh'$$

(b) Taking f = g = h in part (a), show that

$$\frac{d}{dx} [f(x)]^3 = 3[f(x)]^2 f'(x) .$$

- (c) Use part (b) to differentiate  $y = e^{3x}$ .
- **38.** (a) If F(x) = f(x)g(x), where f and g have derivatives of all orders, show that

$$F'' = f''g + 2f'g' + fg''$$

- (b) Find similar formulas for F''' and  $F^{(4)}$ .
- (c) Guess a formula for  $F^{(n)}$ .
- **39.** Find expressions for the first five derivatives of  $f(x) = x^2 e^x$ . Do you see a pattern in these expressions? Guess a formula for  $f^{(n)}(x)$  and prove it using mathematical induction.
- **40.** (a) Use the definition of a derivative to prove the **Reciprocal Rule**: If *g* is differentiable, then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$$

- (b) Use the Reciprocal Rule to differentiate the function in Exercise 15.
- **41.** Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1}$$

for all positive integers n.

**42.** Use the Product Rule and the Reciprocal Rule to prove the Quotient Rule.



# Rates of Change in the Natural and Social Sciences

Recall from Section 2.7 that if y = f(x), then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.6 the basic idea behind rates of change. If x changes from  $x_1$  to  $x_2$ , then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of** *y* **with respect to** *x* **over the interval**  $[x_1, x_2]$  and can be interpreted as the slope of the secant line *PQ* in Figure 1. Its limit as  $\Delta x \rightarrow 0$  is the derivative  $f'(x_1)$ , which can therefore be interpreted as the **instantaneous rate of change of** *y* **with respect to** *x* **or the slope of the tangent line at**  $P(x_1, f(x_1))$ . Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Whenever the function y = f(x) has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. (As we discussed in Section 2.6, the units for dy/dx are the units for y divided by the units for x.) We now look at some of these interpretations in the natural and social sciences.

## Physics

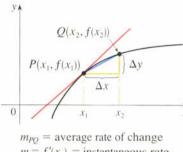
If s = f(t) is the position function of a particle that is moving in a straight line, then  $\Delta s/\Delta t$  represents the average velocity over a time period  $\Delta t$ , and v = ds/dtrepresents the instantaneous velocity (the rate of change of displacement with respect to time). This was discussed in Sections 2.6 and 2.7, but now that we know the differentiation formulas, we are able to solve velocity problems more easily.

**EXAMPLE 1** The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

- (a) Find the velocity at time t.
- (b) What is the velocity after 2 s? After 4 s?
- (c) When is the particle at rest?
- (d) When is the particle moving forward (that is, in the positive direction)?
- (e) Draw a diagram to represent the motion of the particle.
- (f) Find the total distance traveled by the particle during the first five seconds.
- (g) Find the acceleration at time t and after 4 s.
- (h) Graph the position, velocity, and acceleration functions for  $0 \le t \le 5$ .
  - (i) When is the particle speeding up? When is it slowing down?



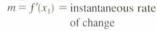


FIGURE I

#### SOLUTION

(a) The velocity function is the derivative of the position function.

$$s = f(t) = t^{3} - 6t^{2} + 9t$$
$$v(t) = \frac{ds}{dt} = 3t^{2} - 12t + 9$$

(b) The velocity after 2 s means the instantaneous velocity when t = 2, that is,

$$v(2) = \frac{ds}{dt}\Big|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

(c) The particle is at rest when v(t) = 0, that is,

$$3t^{2} - 12t + 9 = 3(t^{2} - 4t + 3) = 3(t - 1)(t - 3) = 0$$

and this is true when t = 1 or t = 3. Thus, the particle is at rest after 1 s and after 3 s.

(d) The particle moves in the positive direction when v(t) > 0, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive (t > 3) or when both factors are negative (t < 1). Thus, the particle moves in the positive direction in the time intervals t < 1 and t > 3. It moves backward when 1 < t < 3.

(e) The motion of the particle is illustrated schematically in Figure 2.

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately. The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4$$
 m

From t = 1 to t = 3 the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4$$
 m

From t = 3 to t = 5 the distance traveled is

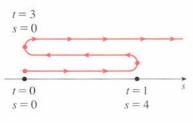
$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is 4 + 4 + 20 = 28 m.

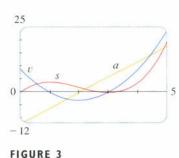
(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$
$$a(4) = 6(4) - 12 = 12 \text{ m/s}^2$$

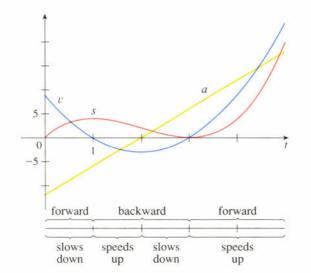
(h) Figure 3 shows the graphs of s, v, and a.







(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.) From Figure 3 we see that this happens when 1 < t < 2 and when t > 3. The particle slows down when v and a have opposite signs, that is, when  $0 \le t < 1$  and when 2 < t < 3. Figure 4 summarizes the motion of the particle.



#### FIGURE 4

**EXAMPLE 2** If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ( $\rho = m/l$ ) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is m = f(x) as shown in Figure 5.



#### FIGURE 5

This part of the rod has mass f(x).

The mass of the part of the rod that lies between  $x = x_1$  and  $x = x_2$  is given by  $\Delta m = f(x_2) - f(x_1)$ , so the average density of that part of the rod is

average density 
$$= \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let  $\Delta x \to 0$  (that is,  $x_2 \to x_1$ ), we are computing the average density over a smaller and smaller interval. The **linear density**  $\rho$  at  $x_1$  is the limit of these average densities as  $\Delta x \to 0$ ; that is, the linear density is the rate of change of mass with respect to length. Symbolically,

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus, the linear density of the rod is the derivative of mass with respect to length.

For instance, if  $m = f(x) = \sqrt{x}$ , where x is measured in meters and m in kilograms, then the average density of the part of the rod given by  $1 \le x \le 1.2$  is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2 - 1}}{0.2} \approx 0.48 \text{ kg/m}$$

while the density right at x = 1 is

$$\rho = \frac{dm}{dx} \bigg|_{x=1} = \frac{1}{2\sqrt{x}} \bigg|_{x=1} = 0.50 \text{ kg/m}$$

**EXAMPLE 3** • A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a shaded plane surface. If  $\Delta Q$  is the net charge that passes through this surface during a time period  $\Delta t$ , then the average current during this time interval is defined as

average current = 
$$\frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** I at a given time  $t_1$ :

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus, the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

## Chemistry

**EXAMPLE 4** A chemical reaction results in the formation of one or more substances (called products) from one or more starting materials (called reactants). For instance, the "equation"

$$2H_2 + O_2 \longrightarrow 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. Let us consider the reaction

$$A + B \longrightarrow C$$

where A and B are the reactants and C is the product. The **concentration** of a reactant A is the number of moles  $(6.022 \times 10^{23} \text{ molecules})$  per liter and is denoted by [A]. The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (*t*). The average rate of reaction of the product C over a

FIGURE 6

time interval  $t_1 \leq t \leq t_2$  is

$$\frac{\Delta[\mathbf{C}]}{\Delta t} = \frac{[\mathbf{C}](t_2) - [\mathbf{C}](t_1)}{t_2 - t_1}$$

But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval  $\Delta t$  approaches 0:

rate of reaction = 
$$\lim_{\Delta t \to 0} \frac{\Delta[C]}{\Delta t} = \frac{d[C]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative d[C]/dt will be positive and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives d[A]/dt and d[B]/dt. Since [A] and [B] each decrease at the same rate that [C] increases, we have

rate of reaction 
$$= \frac{d[C]}{dt} = -\frac{d[A]}{dt} = -\frac{d[B]}{dt}$$

More generally, it turns out that for a reaction of the form

$$aA + bB \longrightarrow cC + dD$$

we have

$$-\frac{1}{a}\frac{d[A]}{dt} - \frac{1}{b}\frac{d[B]}{dt} = \frac{1}{c}\frac{d[C]}{dt} + \frac{1}{d}\frac{d[D]}{dt}$$

The rate of reaction can be determined by graphical methods (see Exercise 16). In some cases we can use the rate of reaction to find explicit formulas for the concentrations as functions of time (see Exercises 7.4).

**EXAMPLE 5** One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP. As P increases, V decreases, so dV/dP < 0. The compressibility is defined by introducing a minus sign and dividing this derivative by the volume V:

isothermal compressibility 
$$= \beta = -\frac{1}{V} \frac{dV}{dP}$$

Thus,  $\beta$  measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25 °C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

### SECTION 3.3 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES 213

The rate of change of V with respect to P when P = 50 kPa is

$$\frac{dV}{dP}\Big|_{P=50} = -\frac{5.3}{P^2}\Big|_{P=50}$$
$$= -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa}$$

The compressibility at that pressure is

$$\beta = -\frac{1}{V} \left. \frac{dV}{dP} \right|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02 \text{ (m}^3/\text{kPa)/m}^3$$

## Biology

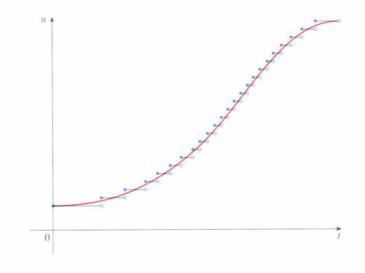
**EXAMPLE 6** Let n = f(t) be the number of individuals in an animal or plant population at time *t*. The change in the population size between the times  $t = t_1$  and  $t = t_2$  is  $\Delta n = f(t_2) - f(t_1)$ , and so the average rate of growth during the time period  $t_1 \le t \le t_2$  is

average rate of growth 
$$=$$
  $\frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$ 

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period  $\Delta t$  approach 0:

growth rate = 
$$\lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function n = f(t) would be a step function that is discontinuous whenever a birth or death occurs and, therefore, not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.





To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is  $n_0$  and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$
  

$$f(2) = 2f(1) = 2^2n_0$$
  

$$f(3) = 2f(2) = 2^3n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is  $n = n_0 2'$ .

In Section 3.1 we discussed derivatives of exponential functions and found that

$$\frac{d}{dx}\left(2^{x}\right)\approx\left(0.69\right)2^{x}$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt} \left( n_0 2^t \right) \approx n_0(0.69) 2^t$$

For example, suppose that we start with an initial population of  $n_0 = 100$  bacteria. Then the rate of growth after 4 hours is

$$\left. \frac{dn}{dt} \right|_{t=4} \approx 100(0.69)2^4 = 1104$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1100 bacteria per hour.

**EXAMPLE 7** When we consider the flow of blood through a blood vessel, such as a vein or artery, we can take the shape of the blood vessel to be a cylindrical tube with radius R and length l as illustrated in Figure 8.

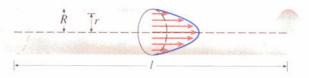


FIGURE 8 Blood flow in an artery

> Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Poiseuille in 1840. This states that

$$v = \frac{P}{4nl} \left( R^2 - r^2 \right)$$

where  $\eta$  is the viscosity of the blood and *P* is the pressure difference between the ends of the tube. If *P* and *l* are constant, then *v* is a function of *r* with domain [0, *R*]. [For more detailed information, see W. Nichols and M. O'Rourke (eds.), *McDonald's Blood Flow in Arteries: Theoretic, Experimental, and Clinical Principles*, 3d ed. (Philadelphia: Lea & Febiger, 1990).]

The average rate of change of the velocity as we move from  $r = r_1$  outward to  $r = r_2$  is

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let  $\Delta r \rightarrow 0$ , we obtain the instantaneous rate of change of velocity with respect to r:

velocity gradient = 
$$\lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l} \left( 0 - 2r \right) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take  $\eta = 0.027$ , R = 0.008 cm, l = 2 cm, and P = 4000 dynes/cm<sup>2</sup>, which gives

$$v = \frac{4000}{4(0.027)2} (0.000064 - r^2)$$
  

$$\approx 1.85 \times 10^4 (6.4 \times 10^{-5} - r^2)$$

At r = 0.002 cm the blood is flowing at a speed of

$$v(0.002) \approx 1.85 \times 10^4 (64 \times 10^{-6} - 4 \times 10^{-6})$$
  
= 1.11 cm/s

and the velocity gradient at that point is

$$\left. \frac{dv}{dr} \right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74 \, (\text{cm/s})/\text{cm}$$

To get a feeling for what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000  $\mu$ m). Then the radius of the artery is 80  $\mu$ m. The velocity at the central axis is 11,850  $\mu$ m/s, which decreases to 11,110  $\mu$ m/s at a distance of  $r = 20 \ \mu$ m. The fact that  $dv/dr = -74 \ (\mu m/s)/\mu$ m means that, when  $r = 20 \ \mu$ m, the velocity is decreasing at a rate of about 74  $\mu$ m/s for each micrometer that we proceed away from the center.

## Economics

**EXAMPLE 8** Suppose C(x) is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from  $x_1$  to  $x_2$ , the additional cost is  $\Delta C = C(x_2) - C(x_1)$ , and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as  $\Delta x \rightarrow 0$ , that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

marginal cost = 
$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x can usually take on only integer values, it may not make literal sense to let  $\Delta x$  approach 0, but we can always replace C(x) by a smooth approximating function as in Example 6.]

Taking  $\Delta x = 1$  and *n* large (so that  $\Delta x$  is small compared to *n*), we have

$$C'(n) \approx C(n+1) - C(n)$$

Thus, the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the (n + 1)st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where *a* represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) =$$
\$15/item

This gives the rate at which costs are increasing with respect to the production level when x = 500 and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C(501) - C(500) = [10,000 + 5(501) + 0.01(501)^{2}]$$
$$- [10,000 + 5(500) + 0.01(500)^{2}]$$
$$= $15.01$$

Notice that  $C'(500) \approx C(501) - C(500)$ .

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 4 after we have developed techniques for finding the maximum and minimum values of functions.

## Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height. (See Exercise 15 in Section 7.5.)

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance P(t) of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as time passes, that is, dP/dt.

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If p(t) denotes the proportion of a population that knows a rumor by time t, then the derivative dp/dt represents the rate of spread of the rumor. (See Exercise 56 in Section 3.5.)

### Summary

Velocity, density, current, power, and temperature gradient in physics, rate of reaction and compressibility in chemistry, rate of growth and blood velocity gradient in biology, marginal cost and marginal profit in economics, rate of heat flow in geology, rate of improvement of performance in psychology, rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them."

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Exercises · · · · · · · · · · · ·

- 1. A particle moves according to a law of motion  $s = f(t) = t^3 - 12t^2 + 36t, t \ge 0$ , where t is measured in seconds and s in meters.
  - (a) Find the velocity at time *t*.
  - (b) What is the velocity after 3 s?
  - (c) When is the particle at rest?
  - (d) When is the particle moving forward?
  - (e) Find the total distance traveled during the first 8 s.
  - (f) Draw a diagram like Figure 2 to illustrate the motion of the particle.
  - (g) Find the acceleration at time t and after 3 s.

- (h) Graph the position, velocity, and acceleration functions for  $0 \le t \le 8$ .
- (i) When is the particle speeding up? When is it slowing down?
- A particle moves along the x-axis, its position at time t given by x(t) = t/(1 + t<sup>2</sup>), t ≥ 0, where t is measured in seconds and x in meters.
  - (a) Find the velocity at time t.
  - (b) When is the particle moving to the right? When is it moving to the left?

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- (c) Find the total distance traveled during the first 4 s.
- (d) Find the acceleration at time *t*. When is it 0?
- (e) Graph the position, velocity, and acceleration functions for  $0 \le t \le 4$ .
- (f) When is the particle speeding up? When is it slowing down?
- 3. The position function of a particle is given by  $s = t^3 4.5t^2 7t$ ,  $t \ge 0$ .
  - (a) When does the particle reach a velocity of 5 m/s?
  - (b) When is the acceleration 0? What is the significance of this value of *t*?
- **4.** If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is  $s = 80t 16t^2$ .
  - (a) What is the maximum height reached by the ball?(b) What is the velocity of the ball when it is 96 ft
  - above the ground on its way up? On its way down?
- 5. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area A(x) of a wafer changes when the side length x changes. Find A'(15) and explain its meaning in this situation.
  - (b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area ΔA if Δx is small?
- 6. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x, calculate dV/dx when x = 3 mm and explain its meaning.
  - (b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 5(b).
- 7. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from (i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1
  - (b) Find the instantaneous rate of change when r = 2.
  - (c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount  $\Delta r$ . How can you approximate the resulting change in area  $\Delta A$  if  $\Delta r$  is small?
- **8.** A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

- **9.** A spherical balloon is being inflated. Find the rate of increase of the surface area  $(S = 4\pi r^2)$  with respect to the radius *r* when *r* is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?
- 10. (a) The volume of a growing spherical cell is  $V = \frac{4}{3}\pi r^3$ , where the radius *r* is measured in micrometers (1  $\mu$ m = 10<sup>-6</sup> m). Find the average rate of change of *V* with respect to *r* when *r* changes from (i) 5 to 8  $\mu$ m (ii) 5 to 6  $\mu$ m (iii) 5 to 5.1  $\mu$ m
  - (b) Find the instantaneous rate of change of V with respect to r when  $r = 5 \,\mu \text{m}$ .
  - (c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 7(c).
- 11. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is 3x<sup>2</sup> kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?
- 12. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 min, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000 \left(1 - \frac{t}{40}\right)^2 \qquad 0 \le t \le 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

- 13. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by  $Q(t) = t^3 - 2t^2 + 6t + 2$ . Find the current when (a) t = 0.5 s and (b) t = 1 s. [See Example 3. The unit of current is an ampere (1 A = 1 C/s).] At what time is the current lowest?
- 14. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

- (a) If the bodies are moving, find *dF/dr* and explain its meaning. What does the minus sign indicate?
- (b) Suppose it is known that Earth attracts an object with a force that decreases at the rate of 2 N/km when r = 20,000 km. How fast does this force change when r = 10,000 km?
- **15.** Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the

pressure and the volume remains constant: PV = C.

- (a) Find the rate of change of volume with respect to pressure.
- (b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.
- (c) Prove that the isothermal compressibility (see Example 5) is given by  $\beta = 1/P$ .
- 16. The data in the table concern the lactonization of hydroxyvaleric acid at 25 °C. They give the concentration C(t) of this acid in moles per liter after t minutes.

T	0	2	4	6	8
C(t)	0.0800	0.0570	0.0408	0.0295	0.0210

(a) Find the average rate of reaction for the following time intervals:

$$2 \le t \le 6$$
 (ii)  $2 \le t \le 4$  (iii)  $0 \le t \le 2$ 

- (b) Plot the points from the table and draw a smooth curve through them as an approximation to the graph of the concentration function. Then draw the tangent at t = 2 and use it to estimate the instantaneous rate of reaction when t = 2.
- (c) Is the reaction speeding up or slowing down?
- 17. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value [A] = [B] = a moles/L, then [C] = a<sup>2</sup>kt/(akt + 1), where k is a constant.
  - (a) Find the rate of reaction at time t.
  - (b) Show that if x = [C], then

(i)

$$\frac{dx}{dt} = k(a - x)^2$$

- (c) What happens to the concentration as  $t \to \infty$ ?
- (d) What happens to the rate of reaction as  $t \to \infty$ ?
- (e) What do the results of parts (c) and (d) mean in practical terms?
- **18.** Suppose that a bacteria population starts with 500 bacteria and triples every hour.
  - (a) What is the population after 3 hours? After 4 hours? After *t* hours?
  - (b) Use the result of (5) in Section 3.1 to estimate the rate of increase of the bacteria population after 6 hours.
- 19. Refer to the law of laminar flow in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm<sup>2</sup>, and viscosity  $\eta = 0.027$ .

(a) Find the velocity of the blood along the centerline

r = 0, at radius r = 0.005 cm, and at the wall r = R = 0.01 cm.

- (b) Find the velocity gradient at r = 0, r = 0.005, and r = 0.01.
- (c) Where is the velocity the greatest? Where is the velocity changing most?
- **20.** The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L} \sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and  $\rho$  is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 2d ed. (Pacific Grove, CA: Brooks/ Cole, 1991).]

- (a) Find the rate of change of the frequency with respect to
  - (i) the length (when T and  $\rho$  are constant),
  - (ii) the tension (when L and  $\rho$  are constant), and
  - (iii) the linear density (when L and T are constant).
- (b) The pitch of a note (how high or low the note sounds) is determined by the frequency *f*. (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
  - (i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
  - (ii) when the tension is increased by turning a tuning peg,
  - (iii) when the linear density is increased by changing to another string.
- **21.** The cost, in dollars, of producing *x* yards of a certain fabric is

 $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$ 

- (a) Find the marginal cost function.
- (b) Find C'(100) and explain its meaning. What does it predict?
- (c) Compare C'(100) with the cost of manufacturing the 101st yard.
- 22. The cost function for a commodity is

$$C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3$$

- (a) Find and interpret C'(100).
- (b) Compare C'(100) with the cost of producing the 101st item.
- (c) Graph the cost function and estimate the inflection point.
- (d) Calculate the value of *x* for which *C* has an inflection point. What is the significance of this value of *x*?
- **23.** If p(x) is the total value of the production when there are x workers in a plant, then the *average productivity*

of the work force at the plant is

$$A(x) = \frac{p(x)}{x}$$

- (a) Find A'(x). Why does the company want to hire more workers if A'(x) > 0?
- (b) Show that A'(x) > 0 if p'(x) is greater than the average productivity.
- **24.** If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.

(a) Find the sensitivity.

- (b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?
- **25.** The gas law for an ideal gas at absolute temperature *T* (in kelvins), pressure *P* (in atmospheres), and volume *V* (in liters) is PV = nRT, where *n* is the number of moles of the gas and R = 0.0821 is the gas constant. Suppose that, at a certain instant, P = 8.0 atm and is increasing at a rate of 0.10 atm/min and V = 10 L and is decreasing at a rate of 0.15 L/min. Find the rate of change of *T* with respect to time at that instant if n = 10 mol.

**26.** In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0 \left( 1 - \frac{P(t)}{P_c} \right) P(t) - \beta P(t)$$

where  $r_0$  is the birth rate of the fish,  $P_c$  is the maximum population that the pond can sustain (called the *carry-ing capacity*), and  $\beta$  is the percentage of the population that is harvested.

- (a) What value of *dP/dt* corresponds to a stable population?
- (b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
- (c) What happens if  $\beta$  is raised to 5%?
- 27. In the study of ecosystems, *predator-prey* models are often used to study the interaction between species. Consider a population of tundra wolves, given by W(t), and caribou, given by C(t), in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = -cW + dCW$$

- (a) What values of *dC/dt* and *dW/dt* correspond to stable populations?
- (b) How would the statement "The caribou go extinct" be represented mathematically?
- (c) Suppose that a = 0.05, b = 0.001, c = 0.05, and d = 0.0001. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the species to live in harmony or will one or both species become extinct?

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# Derivatives of Trigonometric Functions • • •

A review of the trigonometric functions is given in Appendix C.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that  $\sin x$  means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot. Recall from Section 2.4 that all of the trigonometric functions are continuous at every number in their domains.

If we sketch the graph of the function  $f(x) = \sin x$  and use the interpretation of f'(x) as the slope of the tangent to the sine curve in order to sketch the graph of f'

(see Exercise 14 in Section 2.8), then it looks as if the graph of f' may be the same as the cosine curve (see Figure 1).

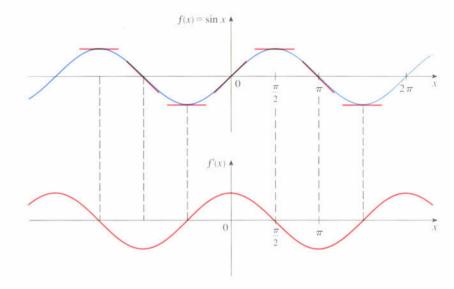


FIGURE I

П

2

Let's try to confirm our guess that if  $f(x) = \sin x$ , then  $f'(x) = \cos x$ . From the definition of a derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h \to 0} \left[ \sin x \left( \frac{\cos h - 1}{h} \right) + \cos x \left( \frac{\sin h}{h} \right) \right]$$
$$= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

We have used the addition formula for sine. See Appendix C.

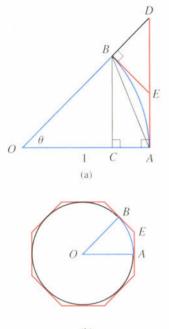
Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as  $h \rightarrow 0$ , we have

$$\lim_{h \to 0} \sin x = \sin x \quad \text{and} \quad \lim_{h \to 0} \cos x = \cos x$$

The limit of  $(\sin h)/h$  is not so obvious. In Example 3 in Section 2.2 we made the guess, on the basis of numerical and graphical evidence, that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

We now use a geometric argument to prove Equation 2. Assume first that  $\theta$  lies



(b)

FIGURE 2

SO

between 0 and  $\pi/2$ . Figure 2(a) shows a sector of a circle with center *O*, central angle  $\theta$ , and radius 1. *BC* is drawn perpendicular to *OA*. By the definition of radian measure, we have arc  $AB = \theta$ . Also,  $|BC| = |OB| \sin \theta = \sin \theta$ . From the diagram we see that

$$|BC| < |AB| < \operatorname{arc} AB$$
  
 $\sin \theta < \theta$  so  $\frac{\sin \theta}{\theta} < 1$ 

θ

Let the tangents at *A* and *B* intersect at *E*. You can see from Figure 2(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, so  $\operatorname{arc} AB < |AE| + |EB|$ . Thus

$$= \operatorname{arc} AB < |AE| + |EB|$$
$$< |AE| + |ED|$$
$$= |AD| = |OA| \tan \theta$$
$$= \tan \theta$$

Therefore, we have

Therefore

 $\cos\theta < \frac{\sin\theta}{\theta} < 1$ 

 $\theta < \frac{\sin\theta}{\cos\theta}$ 

We know that  $\lim_{\theta\to 0} 1 = 1$  and  $\lim_{\theta\to 0} \cos\theta = 1$ , so by the Squeeze Theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

But the function  $(\sin \theta)/\theta$  is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

so we have proved Equation 2.

We can deduce the value of the remaining limit in (1) as follows:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \left[ \frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right] = \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)}$$
$$= \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} = -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1}$$
$$= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1}$$
$$= -1 \cdot \left( \frac{\theta}{1 + 1} \right) = 0 \qquad \text{(by Equation 2)}$$

We multiply numerator and denominator by  $\cos \theta + 1$  in order to put the function in a form in which we can use the limits we know.

SECTION 3.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS 223

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

If we now put the limits (2) and (3) in (1), we get

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$
$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

So we have proved the formula for the derivative of the sine function:

$$\frac{d}{dx}(\sin x) = \cos x$$

**EXAMPLE I** Differentiate  $y = x^2 \sin x$ .

SOLUTION Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (x^2)$$
$$= x^2 \cos x + 2x \sin x$$

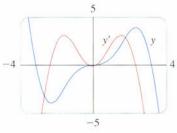
Using the same methods as in the proof of Formula 4, one can prove (see Exercise 16) that

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx} (\tan x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right)$$
$$= \frac{\cos x \frac{d}{dx} (\sin x) - \sin x \frac{d}{dx} (\cos x)}{\cos^2 x}$$
$$= \frac{\cos x \cdot \cos x - \sin x (-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Figure 3 shows the graphs of the function of Example 1 and its derivative. Notice that y' = 0 whenever y has a horizontal tangent.





6

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule (see Exercises 13–15). We collect all the differentiation formulas for trigonometric functions in the following table.

**Derivatives of Trigonometric Functions** 

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$
$\frac{d}{dx}\left(\cos x\right) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx} (\tan x) = \sec^2 x$	$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$

**EXAMPLE 2** Differentiate  $f(x) = \frac{\sec x}{1 + \tan x}$ . For what values of x does the graph of f have a horizontal tangent?

**SOLUTION** The Quotient Rule gives

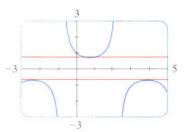
$$f'(x) = \frac{(1 + \tan x) \frac{d}{dx} (\sec x) - \sec x \frac{d}{dx} (1 + \tan x)}{(1 + \tan x)^2}$$
$$= \frac{(1 + \tan x) \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$
$$= \frac{\sec x [\tan x + \tan^2 x - \sec^2 x]}{(1 + \tan x)^2}$$
$$= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$$

In simplifying the answer we have used the identity  $\tan^2 x + 1 = \sec^2 x$ .

Since sec x is never 0, we see that f'(x) = 0 when  $\tan x = 1$ , and this occurs when  $x = n\pi + \pi/4$ , where n is an integer. (See Figure 4.)

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

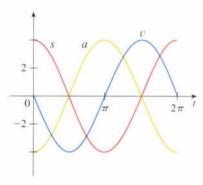
When you memorize this table it is helpful to notice that the minus signs go with the derivatives of the "cofunctions," that is, cosine, cosecant, and cotangent.



**FIGURE 4** The horizontal tangents in Example 2



FIGURE 5





Look for a pattern

t

**EXAMPLE 3** An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time t = 0. (See Figure 5 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4\cos t$$

Find the velocity and acceleration at time *t* and use them to analyze the motion of the object.

**SOLUTION** The velocity and acceleration are

$$v = \frac{ds}{dt} = \frac{d}{dt} (4\cos t) = 4\frac{d}{dt} (\cos t) = -4\sin t$$
$$a = \frac{dv}{dt} = \frac{d}{dt} (-4\sin t) = -4\frac{d}{dt} (\sin t) = -4\cos t$$

The object oscillates from the lowest point (s = 4 cm) to the highest point (s = -4 cm). The period of the oscillation is  $2\pi$ , the period of cos t.

The speed is  $|v| = 4|\sin t|$ , which is greatest when  $|\sin t| = 1$ , that is, when  $\cos t = 0$ . So the object moves fastest as it passes through its equilibrium position (s = 0). Its speed is 0 when  $\sin t = 0$ , that is, at the high and low points.

The acceleration  $a = -4 \cos t = 0$  when s = 0. It has greatest magnitude at the high and low points. See the graphs in Figure 6.

**EXAMPLE 4** Find the 27th derivative of  $\cos x$ .

**SOLUTION** The first few derivatives of  $f(x) = \cos x$  are as follows:

 $f'(x) = -\sin x$   $f''(x) = -\cos x$   $f'''(x) = \sin x$   $f^{(4)}(x) = \cos x$  $f^{(5)}(x) = -\sin x$ 

We see that the successive derivatives occur in a cycle of length 4 and, in particular,  $f^{(n)}(x) = \cos x$  whenever *n* is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$



Exercises ·

 1-12 = Find  $\frac{dy}{dx}$ .
 5.  $y = 2 \cot x - \sqrt{x} \sec x$  

 1.  $y = \sin x + \cos x$  2.  $y = \cos x - 2 \tan x$  

 3.  $y = x^2 \cos x$  4.  $y = e^x \sin x$ 

8. 
$$y = \frac{\sin x}{1 + \cos x}$$
  
9. 
$$y = \frac{x}{\sin x + \cos x}$$
  
10. 
$$y = \frac{\tan x - 1}{\sec x}$$
  
11. 
$$y = e^{x}(\tan x - x)$$
  
12. 
$$y = x \sin x \cos x$$
  
13. Prove that  $\frac{d}{dx}(\csc x) = -\csc x \cot x$ .  
14. Prove that  $\frac{d}{dx}(\sec x) = \sec x \tan x$ .

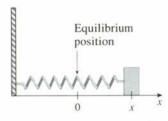
**15.** Prove that 
$$\frac{d}{dx}(\cot x) = -\csc^2 x.$$

16. Prove, using the definition of derivative, that if  $f(x) = \cos x$ , then  $f'(x) = -\sin x$ .

**17–18** Find the equation of the tangent line to the given curve at the specified point.

- **17.**  $y = \tan x$ ,  $(\pi/4, 1)$  **18.**  $y = 2\sin x$ ,  $(\pi/6, 1)$
- 19. (a) Find an equation of the tangent line to the curve  $y = x \cos x$  at the point  $(\pi, -\pi)$ .
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
  - **20.** (a) Find an equation of the tangent line to the curve  $y = \sec x 2\cos x$  at the point  $(\pi/3, 1)$ .
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
  - **21.** (a) If  $f(x) = 2x + \cot x$ , find f'(x).
- (b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for  $0 < x < \pi$ .
  - **22.** (a) If  $f(x) = e^x \cos x$ , find f'(x) and f''(x).
- (b) Check to see that your answers to part (a) are reasonable by graphing f, f', and f".
  - **23.** If  $g(s) = s^2 \cos s$ , find g' and g''.
  - **24.** If  $f(x) = \sec x$ , find  $f''(\pi/4)$ .
  - **25.** For what values of x does the graph of  $f(x) = x + 2 \sin x$  have a horizontal tangent?
  - **26.** Find the points on the curve  $y = (\cos x)/(2 + \sin x)$  at which the tangent is horizontal.
  - **27.** Let  $f(x) = x 2\sin x$ ,  $0 \le x \le 2\pi$ . On what interval is *f* increasing?
  - **28.** Let  $f(x) = x \sin x$ ,  $0 \le x \le 2\pi$ . On what interval is f concave upward?

- 29. A mass on a spring vibrates horizontally on a smooth level surface in simple harmonic motion. (See the figure.) Its equation of motion is x(t) = 8 sin t, where t is in seconds and x in centimeters.
  - (a) Find the velocity and acceleration at time *t*.
  - (b) Find the position, velocity, and acceleration of the mass at time  $t = 2\pi/3$ . In what direction is it moving at that time? Is it speeding up or slowing down?



- **30.** An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically in simple harmonic motion. The equation of motion is  $s = 2 \cos t + 3 \sin t$ ,  $t \ge 0$ , where s is measured in centimeters and t in seconds. (We take the positive direction to be downward.)
  - (a) Find the velocity and acceleration at time t.
  - (b) Graph the velocity and acceleration functions.
  - (c) When does the mass pass through the equilibrium position for the first time?
  - (d) How far from its equilibrium position does the mass travel?
  - (e) When is the speed the greatest? When is the mass speeding up?
  - **31.** A ladder 10 ft long rests against a vertical wall. Let  $\theta$  be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to  $\theta$  when  $\theta = \pi/3$ ?
  - **32.** An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle  $\theta$  with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where  $\mu$  is a constant called the *coefficient of friction*.

- (a) Find the rate of change of F with respect to  $\theta$ .
- (b) When is this rate of change equal to 0?

A

(c) If W = 50 lb and  $\mu = 0.6$ , draw the graph of F as a function of  $\theta$  and use it to locate the value of  $\theta$  for which  $dF/d\theta = 0$ . Is the value consistent with your answer to part (b)?

**33–34** Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

**33.** 
$$\frac{d^{99}}{dx^{99}}(\sin x)$$
 **34.**  $\frac{d^{35}}{dx^{35}}(x\sin x)$ 

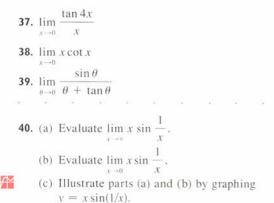
**35.** Find constants A and B such that the function  $y = A \sin x + B \cos x$  satisfies the differential equation  $y'' + y' - 2y = \sin x$ .

**36.** (a) Use the substitution  $\theta = 5x$  to evaluate

$$\lim_{x \to 0} \frac{\sin 5x}{x}$$

(b) Use part (a) and the definition of a derivative to find  $\frac{d}{dx}$  (sin 5x).

**37–39** Use Formula 2 and trigonometric identities to evaluate the limit.



The Chain Rule



Suppose you were asked to differentiate the function

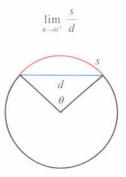
$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate F'(x).

Observe that *F* is a composite function. If we let  $y = f(u) = \sqrt{u}$  and let  $u = g(x) = x^2 + 1$ , then we can write y = F(x) = f(g(x)), that is,  $F = f \circ g$ . We know how to differentiate both *f* and *g*, so it would be useful to have a rule that tells us how to find the derivative of  $F = f \circ g$  in terms of the derivatives of *f* and *g*.

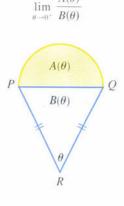
It turns out that the derivative of the composite function  $f \circ g$  is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives

**41.** The figure shows a circular arc of length *s* and a chord of length *d*, both subtended by a central angle  $\theta$ . Find



**42.** A semicircle with diameter *PQ* sits on an isosceles triangle *PQR* to form a region shaped like an ice cream cone, as shown in the figure. If  $A(\theta)$  is the area of the semicircle and  $B(\theta)$  is the area of the triangle, find

 $A(\theta)$ 



See Section 1.2 for a review of composite functions.

as rates of change. Regard du/dx as the rate of change of u with respect to x, dy/du as the rate of change of y with respect to u, and dy/dx as the rate of change of y with respect to x. If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

**The Chain Rule** If f and g are both differentiable and  $F = f \circ g$  is the composite function defined by F(x) = f(g(x)), then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if y = f(u) and u = g(x) are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

**Comments on the Proof of the Chain Rule** Let  $\Delta u$  be the change in *u* corresponding to a change of  $\Delta x$  in *x*, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad \text{(Note that } \Delta u \to 0 \text{ as } \Delta x \to 0 \text{ since } g \text{ is continuous.)}$$

$$= \frac{dy}{du} \frac{du}{dx}$$

The only flaw in this reasoning is that in (1) it might happen that  $\Delta u = 0$  (even when  $\Delta x \neq 0$ ) and, of course, we can't divide by 0. Nonetheless, this reasoning does at least *suggest* that the Chain Rule is true. A more subtle argument can be given to fix the flaw, but we will not present it here.

The Chain Rule can be written either in the prime notation

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

or, if y = f(u) and u = g(x), in Leibniz notation:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du. Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

**EXAMPLE I** Find F'(x) if  $F(x) = \sqrt{x^2 + 1}$ .

**SOLUTION 1** (using Equation 2): At the beginning of this section we expressed *F* as  $F(x) = (f \circ g)(x) = f(g(x))$  where  $f(u) = \sqrt{u}$  and  $g(x) = x^2 + 1$ . Since

$$f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$$
 and  $g'(x) = 2x$   
 $F'(x) = f'(g(x))g'(x)$ 

we have

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

**SOLUTION 2** (using Equation 3): If we let  $u = x^2 + 1$  and  $y = \sqrt{u}$ , then

$$F'(x) = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{u}} (2x)$$
$$= \frac{1}{2\sqrt{x^2 + 1}} (2x) = \frac{x}{\sqrt{x^2 + 1}}$$

When using Formula 3 we should bear in mind that dy/dx refers to the derivative of y when y is considered as a function of x (called the *derivative of y with respect to x*), whereas dy/du refers to the derivative of y when considered as a function of u (the derivative of y with respect to u). For instance, in Example 1, y can be considered as a function of x ( $y = \sqrt{x^2 + 1}$ ) and also as a function of u ( $y = \sqrt{u}$ ). Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2 + 1}}$$
 whereas  $\frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$ 

**NOTE** • In using the Chain Rule we work from the outside to the inside. Formula 2 says that we differentiate the outer function f [at the inner function g(x)] and then we multiply by the derivative of the inner function.



**EXAMPLE 2** Differentiate (a)  $y = \sin(x^2)$  and (b)  $y = \sin^2 x$ .

#### SOLUTION

(a) If  $y = \sin(x^2)$ , then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$\frac{dy}{dx} = \frac{d}{dx}$	sin	$(x^2)$	-	cos	$(x^2)$	•	2 <i>x</i>
ux ux	outer function	evaluated at inner function		derivative of outer function	evaluated at inner function		derivative of inner function
= 2x	$\cos(x^2)$						

(b) Note that  $\sin^2 x = (\sin x)^2$ . Here the outer function is the squaring function and the inner function is the sine function. So

$\frac{dy}{dx} =$	$\frac{d}{dx}$	$(\sin x)^2$	=	$(\sin x)$	cos x
		inner function	derivative of outer	evaluated at inner	derivative of inner
		Tunction	function	function	function

The answer can be left as  $2 \sin x \cos x$  or written as  $\sin 2x$  (by a trigonometric identity known as the double-angle formula).

In Example 2(a) we combined the Chain Rule with the rule for differentiating the sine function. In general, if  $y = \sin u$ , where u is a differentiable function of x, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \cos u \frac{du}{dx}$$
$$\frac{d}{dx} (\sin u) = \cos u \frac{du}{dx}$$

Thus

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

Let's make explicit the special case of the Chain Rule where the outer function f is a power function. If  $y = [g(x)]^n$ , then we can write  $y = f(u) = u^n$  where u = g(x). By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = nu^{n-1} \frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

**4** The Power Rule Combined with the Chain Rule If n is any real number and u = g(x) is differentiable, then

$$\frac{d}{dx}\left(u^{n}\right) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx} [g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Notice that the derivative in Example 1 could be calculated by taking  $n = \frac{1}{2}$  in Rule (4).

## **EXAMPLE 3** Differentiate $y = (x^3 - 1)^{100}$ .

**SOLUTION** Taking  $u = g(x) = x^3 - 1$  and n = 100 in (4), we have

$$\frac{dy}{dx} = \frac{d}{dx} (x^3 - 1)^{100} = 100(x^3 - 1)^{99} \frac{d}{dx} (x^3 - 1)$$
$$= 100(x^3 - 1)^{99} \cdot 3x^2$$
$$= 300x^2(x^3 - 1)^{99}$$

**EXAMPLE 4** Find f'(x) if  $f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$ .

SOLUTION First rewrite f:  $f(x) = (x^2 + x + 1)^{-1/3}$ . Thus

$$f'(x) = -\frac{1}{3}(x^2 + x + 1)^{-4/3} \frac{d}{dx}(x^2 + x + 1)$$
$$= -\frac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)$$

**EXAMPLE 5** Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt} \left(\frac{t-2}{2t+1}\right)$$
$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2}$$
$$= \frac{45(t-2)^8}{(2t+1)^{10}}$$

**EXAMPLE 6** Differentiate  $y = (2x + 1)^5(x^3 - x + 1)^4$ .

**SOLUTION** In this example we must use the Product Rule before using the Chain Rule:

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx} (x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx} (2x+1)^5$$
$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx} (x^3 - x + 1)$$
$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx} (2x+1)$$
$$= 4(2x+1)^5 (x^3 - x + 1)^3 (3x^2 - 1) + 5(x^3 - x + 1)^4 (2x+1)^4 \cdot 2$$

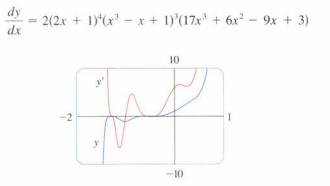
The graphs of the functions y and y' in Example 6 are shown in Figure I. Notice

that y' is large when y increases rapidly and y' = 0 when y has a horizontal

tangent. So our answer appears to be

reasonable.

By using common factors, we could write the answer as



#### FIGURE I

**EXAMPLE 7** Differentiate  $y = e^{\sin x}$ .

**SOLUTION** Here the inner function is  $g(x) = \sin x$  and the outer function is the exponential function  $f(x) = e^x$ . So, by the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx} (e^{\sin x}) = e^{\sin x} \frac{d}{dx} (\sin x) = e^{\sin x} \cos x$$

We can use the Chain Rule to differentiate an exponential function with any base a > 0. Recall from Section 1.6 that  $a = e^{\ln a}$ . So

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

and the Chain Rule gives

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{(\ln a)x}) = e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$
$$= e^{(\ln a)x} \cdot \ln a = a^x \ln a$$

because  $\ln a$  is a constant. So we have the formula

Don't confuse Formula 5 (where *x* is the *exponent*) with the Power Rule (where *x* is the *base*):

 $\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$ 

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 $\frac{d}{dx}\left(a^{x}\right) = a^{x}\ln a$ 

In particular, if a = 2, we get

$$\frac{d}{dx}\left(2^{x}\right) = 2^{x}\ln 2$$

In Section 3.1 we gave the estimate

$$\frac{d}{dx}\left(2^{x}\right)\approx\left(0.69\right)2^{x}$$

This is consistent with the exact formula (6) because  $\ln 2 \approx 0.693147$ .

In Example 6 in Section 3.3 we considered a population of bacteria cells that doubles every hour and saw that the population after *t* hours is  $n = n_0 2^t$ , where  $n_0$  is the initial population. Formula 6 enables us to find the rate of growth of the bacteria population:

$$\frac{dn}{dt} = n_0 2^t \ln 2$$

The reason for the name "Chain Rule" becomes clear when we make a longer chain by adding another link. Suppose that y = f(u), u = g(x), and x = h(t), where f, g, and h are differentiable functions. Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

**EXAMPLE 8** If  $f(x) = \sin(\cos(\tan x))$ , then

$$f'(x) = \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x)$$
$$= \cos(\cos(\tan x)) [-\sin(\tan x)] \frac{d}{dx} (\tan x)$$
$$= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x$$

Notice that the Chain Rule has been used twice.

## **Tangents to Parametric Curves**

In Section 1.4 we discussed curves defined by parametric equations

$$x = f(t) \qquad y = g(t)$$

The Chain Rule helps us find tangent lines to such curves. Suppose f and g are differentiable functions and we want to find the tangent line at a point on the curve where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If  $dx/dt \neq 0$ , we can solve for dy/dx:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

Equation 7 (which you can remember by thinking of canceling the dt's) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t. If we think of the curve as being traced out by a moving particle, then dy/dt and dx/dt are the vertical and horizontal velocities of the par-

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ticle and Formula 7 says that the slope of the tangent is the ratio of these velocities. We see from (7) that the curve has a horizontal tangent when dy/dt = 0 (provided that  $dx/dt \neq 0$ ) and it has a vertical tangent when dx/dt = 0 (provided that  $dy/dt \neq 0$ ).

**EXAMPLE 9** Find an equation of the tangent line to the parametric curve

$$x = 2\sin 2t$$
  $y = 2\sin t$ 

at the point  $(\sqrt{3}, 1)$ . Where does this curve have horizontal or vertical tangents? SOLUTION At the point with parameter value *t*, the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2\sin t)}{\frac{d}{dt}(2\sin 2t)} = \frac{2\cos t}{2(\cos 2t)(2)} = \frac{\cos t}{2\cos 2t}$$

The point  $(\sqrt{3}, 1)$  corresponds to the parameter value  $t = \pi/6$ , so the slope of the tangent at that point is

$$\frac{dy}{dx}\Big|_{t=\pi/6} = \frac{\cos(\pi/6)}{2\cos(\pi/3)} = \frac{\sqrt{3}/2}{2(\frac{1}{2})} = \frac{\sqrt{3}}{2}$$

An equation of the tangent line is therefore

$$y - 1 = \frac{\sqrt{3}}{2}(x - \sqrt{3})$$
 or  $y = \frac{\sqrt{3}}{2}x - \frac{1}{2}$ 

Figure 2 shows the curve and its tangent line.

The tangent line is horizontal when dy/dx = 0, which occurs when  $\cos t = 0$  (and  $\cos 2t \neq 0$ ), that is, when  $t = \pi/2$  or  $3\pi/2$ . Thus, the curve has horizontal tangents at the points (0, 2) and (0, -2), which we could have guessed from Figure 2.

The tangent is vertical when  $dx/dt = 4 \cos 2t = 0$  (and  $\cos t \neq 0$ ), that is, when  $t = \pi/4$ ,  $3\pi/4$ ,  $5\pi/4$ , or  $7\pi/4$ . The corresponding four points on the curve are  $(\pm 2, \pm \sqrt{2})$ . If we look again at Figure 2, we see that our answer appears to be reasonable.



**1–6** Write the composite function in the form f(g(x)). [Identify the inner function u = g(x) and the outer function y = f(u).] Then find the derivative dy/dx.

1. 
$$y = (x^2 + 4x + 6)^5$$
 2.  $y = \tan 3x$ 

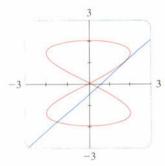
 3.  $y = \cos(\tan x)$ 
 4.  $y = \sqrt[3]{1 + x^3}$ 

 5.  $y = e^{\sqrt{x}}$ 
 6.  $y = \sin(e^x)$ 

**7–30** ■ Find the derivative of the function.

**7.** 
$$g(x) = \sqrt{x^2 - 7x}$$
 **8.**  $f(t) = \frac{1}{(t^2 - 2t - 5)^4}$ 

9. 
$$y = \cos(x^3)$$
  
10.  $y = \cos^3 x$   
11.  $y = 5^{-1/x}$   
12.  $y = 4 \sec 5x$   
13.  $y = xe^{-x^2}$   
14.  $y = e^{-5x}\cos 3x$   
15.  $G(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12}$   
16.  $g(t) = (6t^2 + 5)^3(t^3 - 7)^4$   
17.  $y = e^{x\cos x}$   
18.  $F(s) = \sqrt{s^3 + 1}(s^2 + 1)^4$   
19.  $F(y) = \left(\frac{y - 6}{y + 7}\right)^3$   
20.  $s(t) = \sqrt[4]{\frac{t^3 + 1}{t^3 - 1}}$ 





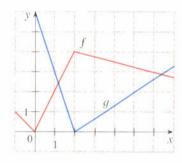
21. 
$$f(z) = \frac{1}{\sqrt[5]{2z - 1}}$$
  
22.  $f(x) = \frac{x}{\sqrt{7 - 3x}}$   
23.  $y = x \sin \frac{1}{x}$   
24.  $y = \tan(x^2) + \tan^2 x$   
25.  $y = \tan^2(x^3)$   
26.  $y = \sin(\sin(\sin x))$   
27.  $y = \sqrt{x + \sqrt{x}}$   
28.  $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$   
29.  $y = \sin(\tan\sqrt{\sin x})$   
30.  $y = 2^{3^{1/2}}$ 

**31–32** Find an equation of the tangent line to the curve at the given point.

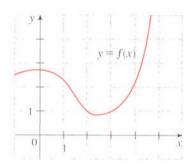
**31.** 
$$y = \frac{8}{\sqrt{4+3x}}$$
, (4,2) **32.**  $y = x^2 e^{-x}$ , (1,1/e)

- **33.** (a) Find an equation of the tangent line to the curve  $y = 2/(1 + e^{-y})$  at the point (0, 1).
  - (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
- **34.** (a) The curve  $y = |x|/\sqrt{2 x^2}$  is called a **bullet-nose** curve. Find an equation of the tangent line to this curve at the point (1, 1).
  - (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
- **35.** (a) If  $f(x) = \sqrt{1 x^2}/x$ , find f'(x).
  - (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.
- **36.** (a) If  $f(x) = 2\cos x + \sin^2 x$ , find f'(x) and f''(x).
  - (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f".
- **37.** Suppose that F(x) = f(g(x)) and g(3) = 6, g'(3) = 4, f'(3) = 2, and f'(6) = 7. Find F'(3).
- 38. Suppose that w = u ∘ v and u(0) = 1, v(0) = 2, u'(0) = 3, u'(2) = 4, v'(0) = 5, and v'(2) = 6. Find w'(0).
- **39.** If f and g are the functions whose graphs are shown, let u(x) = f(g(x)), v(x) = g(f(x)), and w(x) = g(g(x)). Find each derivative, if it exists. If it does not exist, explain why.

(a) u'(1) (b) v'(1) (c) w'(1)



40. If f is the function whose graph is shown, let h(x) = f(f(x)) and g(x) = f(x<sup>2</sup>). Use the graph of f to estimate the value of each derivative.
(a) h'(2)
(b) g'(2)



**41.** Use the table to estimate the value of h'(0.5), where h(x) = f(g(x)).

$\langle N \rangle$	0	0,1	0,2	0.3	0.4	0.5	0.6
f(x)	12.6	14.8	18.4	23.0	25,9	27.5	29,1
g(x)	0.58	0.40	0.37	0.26	0.17	0.10	0.05

**42.** If g(x) = f(f(x)), use the table to estimate the value of g'(1).

Â,	0.0	0.5	1.0	1.5	2.0	2.5
f(x)	1.7	1.8	2.0	2.4	3.1	4,4

- **43.** Let *h* be differentiable on  $[0, \infty)$  and define *G* by  $G(x) = h(\sqrt{x})$ .
  - (a) Where is G differentiable?
  - (b) Find an expression for G'(x).
- 44. Suppose f is differentiable on R and α is a real number. Let F(x) = f(x<sup>a</sup>) and G(x) = [f(x)]<sup>a</sup>. Find expressions for (a) F'(x) and (b) G'(x).
- **45.** Suppose f is differentiable on  $\mathbb{R}$ . Let  $F(x) = f(e^x)$  and  $G(x) = e^{f(x)}$ . Find expressions for (a) F'(x) and (b) G'(x).
- **46.** If g is a twice differentiable function and  $f(x) = xg(x^2)$ , find f" in terms of g, g', and g".
- **47.** Find the *x*-coordinates of all points on the curve  $y = \sin 2x 2 \sin x$  at which the tangent line is horizontal.
- **48.** On what interval is the curve  $y = e^{-x^2}$  concave downward?
- **49.** Show that the function  $y = Ae^{-x} + Bxe^{-x}$  satisfies the differential equation y'' + 2y' + y = 0.
- **50.** For what values of *r* does the function  $y = e^{rx}$  satisfy the equation y'' + 5y' 6y = 0?

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- **51.** Find the 50th derivative of  $y = \cos 2x$ .
- **52.** Find the 1000th derivative of  $f(x) = xe^{-x}$ .
- **53.** The displacement of a particle on a vibrating string is given by the equation  $s(t) = 10 + \frac{1}{4}\sin(10\pi t)$ , where *s* is measured in centimeters and *t* in seconds. Find the velocity and acceleration of the particle after *t* seconds.
- 54. If the equation of motion of a particle is given by  $s = A \cos(\omega t + \delta)$ , the particle is said to undergo simple harmonic motion.
  - (a) Find the velocity of the particle at time t.
  - (b) When is the velocity 0? When is the acceleration 0?
- **55.** A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by  $\pm 0.35$ . In view of these data, the brightness of Delta Cephei at time *t*, where *t* is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin(2\pi t/5.4)$$

- (a) Find the rate of change of the brightness after t days.
- (b) Find, correct to two decimal places, the rate of increase after one day.
- **56.** Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where p(t) is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 7.6 we will see that this is a reasonable equation for p(t).]

(a) Find  $\lim_{t \to \infty} p(t)$ .

- (b) Find the rate of spread of the rumor.
- (c) Graph p for the case a = 10, k = 0.5 with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.
- **57.** A particle moves along a straight line with displacement s(t), velocity v(t), and acceleration a(t). Show that

$$a(t) = v(t)\frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds.

- **58.** Air is being pumped into a spherical weather balloon. At any time *t*, the volume of the balloon is V(t) and its radius is r(t).
  - (a) What do the derivatives dV/dr and dV/dt represent?
    (b) Express dV/dt in terms of dr/dt.

59. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

1	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

- (a) Use the methods of Section 1.7 to find an exponential model for the charge.
- (b) The derivative Q'(t) represents the electric current (measured in microamperes,  $\mu A$ ) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when t = 0.04 s. Compare with the result of Example 2 in Section 2.1.

<b>6</b> 0.	The	table	gives	the	U.S.	population	from	1790	to	1860.
-------------	-----	-------	-------	-----	------	------------	------	------	----	-------

Year	Population
1790	3,929,000
1800	5,308,000
1810	7,240,000
1820	9,639,000
1830	12,861,000
1840	17,063,000
1850	23,192,000
1860	31,443,000

- (a) Use the methods of Section 1.7 to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
- (b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
- (c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
- (d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?
- **61.** Find an equation of the tangent line to the curve with parametric equations  $x = t \sin t$ ,  $y = t \cos t$  at the point  $(0, -\pi)$ .
- **62.** Show that the curve with parametric equations  $x = \sin t$ ,  $y = \sin(t + \sin t)$  has two tangent lines at the origin and find their equations. Illustrate by graphing the curve and its tangents.

### LABORATORY PROJECT BÉZIER CURVES . 237

- **63.** A curve C is defined by the parametric equations  $x = t^2$ ,  $y = t^3 3t$ .
  - (a) Show that C has two tangents at the point (3,0) and find their equations.
  - (b) Find the points on *C* where the tangent is horizontal or vertical.
  - (c) Illustrate parts (a) and (b) by graphing *C* and the tangent lines.
- **64.** The cycloid  $x = r(\theta \sin \theta)$ ,  $y = r(1 \cos \theta)$  was discussed in Example 6 in Section 1.4.
  - (a) Find an equation of the tangent to the cycloid at the point where  $\theta = \pi/3$ .
  - (b) At what points is the tangent horizontal? Where is it vertical?
  - (c) Graph the cycloid and its tangent lines for the case r = 1.
- **65.** Computer algebra systems have commands that differentiate functions, but the form of the answer may not be convenient and so further commands may be necessary to simplify the answer.
  - (a) Use a CAS to find the derivative in Example 5 and compare with the answer in that example. Then use the simplify command and compare again.
  - (b) Use a CAS to differentiate the function in Example 6. What happens if you use the simplify command? What happens if you use the factor command? Which form of the answer would be best for locating horizontal tangents?
- **CAS** 66. (a) Use a CAS to differentiate the function

$$f(x) = \sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}$$

and to simplify the result.

- (b) Where does the graph of *f* have horizontal tangents?(c) Graph *f* and *f'* on the same screen. Are the graphs
- consistent with your answer to part (b)?
- **67.** (a) If *n* is a positive integer, prove that

$$\frac{d}{dx}\left(\sin^n x \cos nx\right) = n\,\sin^{n-1} x \cos(n+1)x$$

- (b) Find a formula for the derivative of  $y = \cos^n x \cos nx$  that is similar to the one in part (a).
- **68.** Suppose y = f(x) is a curve that always lies above the *x*-axis and never has a horizontal tangent, where *f* is differentiable everywhere. For what value of *y* is the rate of change of  $y^5$  with respect to *x* eighty times the rate of change of *y* with respect to *x*?
- **69.** Use the Chain Rule to show that if  $\theta$  is measured in degrees, then

$$\frac{d}{d\theta} (\sin \theta) = \frac{\pi}{180} \cos \theta$$

(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: the differentiation formulas would not be as simple if we used degree measure.)

**70.** (a) Write  $|x| = \sqrt{x^2}$  and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

- (b) If f(x) = |sin x|, find f'(x) and sketch the graphs of f and f'. Where is f not differentiable?
- (c) If g(x) = sin |x|, find g'(x) and sketch the graphs of g and g'. Where is g not differentiable?
- **71.** If y = f(u) and u = g(x), where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} \left(\frac{du}{dx}\right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2}$$

**72.** Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

Laboratory		
Project	Bézier Curves	

The **Bézier curves** are used in computer-aided design and are named after a mathematician working in the automotive industry. A cubic Bézier curve is determined by four *control points*,  $P_0(x_0, y_0)$ ,  $P_1(x_1, y_1)$ ,  $P_2(x_2, y_2)$ , and  $P_3(x_3, y_3)$ , and is defined by the parametric equations

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$
  
$$y = y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3$$

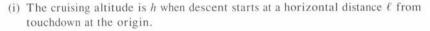
where  $0 \le t \le 1$ . Notice that when t = 0 we have  $(x, y) = (x_0, y_0)$  and when t = 1 we have  $(x, y) = (x_3, y_3)$ , so the curve starts at  $P_0$  and ends at  $P_3$ .

- 1. Graph the Bézier curve with control points  $P_0(4, 1)$ ,  $P_1(28, 48)$ ,  $P_2(50, 42)$ , and  $P_3(40, 5)$ . Then, on the same screen, graph the line segments  $P_0P_1$ ,  $P_1P_2$ , and  $P_2P_3$ . (Exercise 19 in Section 1.4 shows how to do this.) Notice that the middle control points  $P_1$  and  $P_2$  don't lie on the curve; the curve starts at  $P_0$ , heads toward  $P_1$  and  $P_2$  without reaching them, and ends at  $P_3$ .
- 2. From the graph in Problem 1 it appears that the tangent at  $P_0$  passes through  $P_1$  and the tangent at  $P_3$  passes through  $P_2$ . Prove it.
- **3.** Try to produce a Bézier curve with a loop by changing the second control point in Problem 1.
- 4. Some laser printers use Bézier curves to represent letters and other symbols. Experiment with control points until you find a Bézier curve that gives a reasonable representation of the letter C.
- 5. More complicated shapes can be represented by piecing together two or more Bézier curves. Suppose the first Bézier curve has control points  $P_0$ ,  $P_1$ ,  $P_2$ ,  $P_3$  and the second one has control points  $P_3$ ,  $P_4$ ,  $P_5$ ,  $P_6$ . If we want these two pieces to join together smoothly, then the tangents at  $P_3$  should match and so the points  $P_2$ ,  $P_3$ , and  $P_4$  all have to lie on this common tangent line. Using this principle, find control points for a pair of Bézier curves that represent the letter S.



## Where Should a Pilot Start Descent?

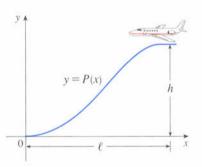
An approach path for an aircraft landing is shown in the figure and satisfies the following conditions:



- (ii) The pilot must maintain a constant horizontal speed v throughout descent.
- (iii) The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).
- 1. Find a cubic polynomial  $P(x) = ax^3 + bx^2 + cx + d$  that satisfies condition (i) by imposing suitable conditions on P(x) and P'(x) at the start of descent and at touchdown.
- 2. Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{\ell^2} \le k$$

- **3.** Suppose that an airline decides not to allow vertical acceleration of a plane to exceed  $k = 860 \text{ mi/h}^2$ . If the cruising altitude of a plane is 35,000 ft and the speed is 300 mi/h, how far away from the airport should the pilot start descent?
- **4.** Graph the approach path if the conditions stated in Problem 3 are satisfied.





#### Implicit Differentiation

Most of the functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

 $y = \sqrt{x^3 + 1}$  or  $y = x \sin x$ 

or, in general, y = f(x). Some functions, however, are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 25$$

or

$$x^3 + y^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve Equation 1 for y, we get  $y = \pm \sqrt{25 - x^2}$ , so two functions determined by the implicit Equation 1 are  $f(x) = \sqrt{25 - x^2}$  and  $g(x) = -\sqrt{25 - x^2}$ . The graphs of f and g are the upper and lower semicircles of the circle  $x^2 + y^2 = 25$  (see Figure 1).

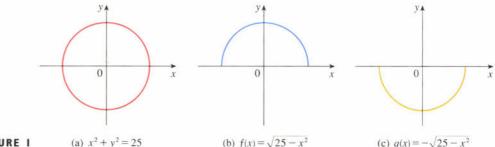
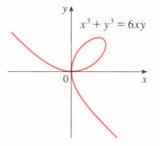


FIGURE I

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.) Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines y as several functions of x. The graphs of three such functions are shown in Figure 3. When we say that f is a



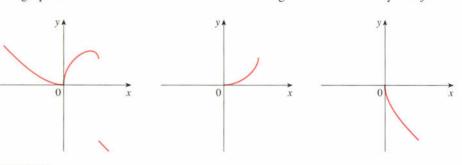


FIGURE 3 Graphs of three functions defined by the folium of Descartes

FIGURE 2 The folium of Descartes function defined implicitly by Equation 2, we mean that the equation

$$x^{3} + [f(x)]^{3} = 6xf(x)$$

is true for all values of x in the domain of f.

Fortunately, it is not necessary to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentia-tion**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

#### EXAMPLE I

- (a) If  $x^2 + y^2 = 25$ , find  $\frac{dy}{dx}$ .
- (b) Find the equation of the tangent to the circle  $x^2 + y^2 = 25$  at the point (3, 4).

#### SOLUTION I

(a) Differentiate both sides of the equation  $x^2 + y^2 = 25$ :

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}\left(y^2\right) = 2y\,\frac{dy}{dx}$$

Thus 
$$2x + 2y \frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) At the point (3, 4) we have x = 3 and y = 4, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\frac{3}{4}(x - 3)$$
 or  $3x + 4y = 25$ 

#### **SOLUTION 2**

(b) Solving the equation  $x^2 + y^2 = 25$ , we get  $y = \pm \sqrt{25 - x^2}$ . The point (3, 4) lies on the upper semicircle  $y = \sqrt{25 - x^2}$  and so we consider the function

#### SECTION 3.6 IMPLICIT DIFFERENTIATION 241

 $f(x) = \sqrt{25 - x^2}$ . Differentiating f using the Chain Rule, we have

$$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2)$$
$$= \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$
$$= -\frac{x}{\sqrt{25 - x^2}}$$
$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, the equation of the tangent is 3x + 4y = 25.

NOTE I • Example 1 illustrates that even when it is possible to solve an equation explicitly for y in terms of x, it may be easier to use implicit differentiation.

**NOTE 2** • The expression dy/dx = -x/y gives the derivative in terms of both x and y. It is correct no matter which function y is determined by the given equation. For instance, for  $y = f(x) = \sqrt{25 - x^2}$  we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

whereas for  $y = q(x) = -\sqrt{25 - x^2}$  we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

#### EXAMPLE 2

(a) Find y' if  $x^3 + y^3 = 6xy$ .

- (b) Find the tangent to the folium of Descartes  $x^3 + y^3 = 6xy$  at the point (3, 3).
- (c) At what points on the curve is the tangent line horizontal or vertical?

#### SOLUTION

(a) Differentiating both sides of  $x^3 + y^3 = 6xy$  with respect to x, regarding y as a function of x, and using the Chain Rule on the  $y^3$  term and the Product Rule on the 6xy term, we get

> $3x^2 + 3y^2y' = 6y + 6xy'$  $x^2 + y^2 y' = 2y + 2xy'$

or

We now solve for y':

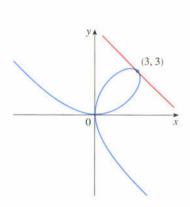
$$(y^{2} - 2x)y' = 2y - x^{2}$$
  
 $y' = \frac{2y - x^{2}}{y^{2} - 2x}$ 

(b) When x = y = 3,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

FIGURE 4

and a glance at Figure 4 confirms that this is a reasonable value for the slope



at (3, 3). So an equation of the tangent to the folium at (3, 3) is

$$y - 3 = -1(x - 3)$$
 or  $x + y = 6$ 

(c) The tangent line is horizontal if y' = 0. Using the expression for y' from part (a), we see that y' = 0 when  $2y - x^2 = 0$ . Substituting  $y = \frac{1}{2}x^2$  in the equation of the curve, we get

$$x^{3} + \left(\frac{1}{2}x^{2}\right)^{3} = 6x\left(\frac{1}{2}x^{2}\right)$$

which simplifies to  $x^6 = 16x^3$ . So either x = 0 or  $x^3 = 16$ . If  $x = 16^{1/3} = 2^{4/3}$ , then  $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$ . Thus, the tangent is horizontal at (0, 0) and at  $(2^{4/3}, 2^{5/3})$ , which is approximately (2.5198, 3.1748). Looking at Figure 5, we see that our answer is reasonable.

The tangent line is vertical when the denominator in the expression for dy/dx is 0. Another method is to observe that the equation of the curve is unchanged when x and y are interchanged, so the curve is symmetric about the line y = x. This means that the horizontal tangents at (0, 0) and  $(2^{4/3}, 2^{5/3})$  correspond to vertical tangents at (0, 0) and  $(2^{5/3}, 2^{4/3})$ . (See Figure 5.)

**NOTE** • There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If we use this formula (or a computer algebra system) to solve the equation  $x^3 + y^3 = 6xy$  for y in terms of x, we get three functions determined by the equation:

$$y = f(x) = \sqrt[3]{-\frac{1}{2}x^3} + \sqrt{\frac{1}{4}x^6 - 8x^3} + \sqrt[3]{-\frac{1}{2}x^3} - \sqrt{\frac{1}{4}x^6 - 8x^3}$$

and

So

$$y = \frac{1}{2} \left[ -f(x) \pm \sqrt{-3} \left( \sqrt[3]{-\frac{1}{2} x^3} + \sqrt{\frac{1}{4} x^6 - 8x^3} - \sqrt[3]{-\frac{1}{2} x^3 - \sqrt{\frac{1}{4} x^6 - 8x^3}} \right) \right]$$

(These are the three functions whose graphs are shown in Figure 3.) You can see that the method of implicit differentiation saves an enormous amount of work in cases such as this. Moreover, implicit differentiation works just as easily for equations such as

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

which are *impossible* to solve for y in terms of x.

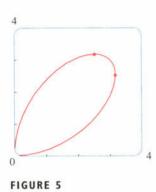
**EXAMPLE 3** Find y' if  $sin(x + y) = y^2 cos x$ .

**SOLUTION** Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = 2yy'\cos x + y^2(-\sin x)$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^{2} \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$
$$y' = \frac{y^{2} \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$



The Norwegian mathematician Niels Abel proved in 1824 that no general formula can be given for the roots of a fifth-degree equation. Later the French mathematician Evariste Galois proved that it is impossible to find a general formula for the roots of an *n*th-degree equation (in terms of algebraic operations on the coefficients) if *n* is any integer larger than 4.

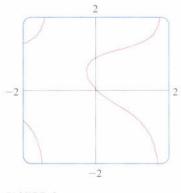




Figure 6, drawn with the implicit-plotting command of a computer algebra system, shows part of the curve  $sin(x + y) = y^2 cos x$ . As a check on our calculation, notice that y' = -1 when x = y = 0 and it appears from the graph that the slope is approximately -1 at the origin.

Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular. In the next example we use implicit differentiation to show that two families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Orthogonal families arise in several areas of physics. For example, the lines of force in an electrostatic field are orthogonal to the lines of constant potential. In thermodynamics, the isotherms (curves of equal temperature) are orthogonal to the flow lines of heat. In aerodynamics, the streamlines (curves of direction of airflow) are orthogonal trajectories of the velocity-equipotential curves.

**EXAMPLE 4** The equation

3

4

$$xy = c$$
  $c \neq$ 

represents a family of hyperbolas. (Different values of the constant c give different hyperbolas. See Figure 7.) The equation

0

$$x^2 - y^2 = k \qquad k \neq 0$$

represents another family of hyperbolas with asymptotes  $y = \pm x$ . Show that every curve in the family (3) is orthogonal to every curve in the family (4); that is, the families are orthogonal trajectories of each other.

**SOLUTION** Implicit differentiation of Equation 3 gives

5 
$$y + x \frac{dy}{dx} = 0$$
 so  $\frac{dy}{dx} = -\frac{y}{x}$ 

Implicit differentiation of Equation 4 gives

$$2x - 2y \frac{dy}{dx} = 0 \quad \text{so} \quad \frac{dy}{dx} = \frac{x}{y}$$

From (5) and (6) we see that at any point of intersection of curves from each family, the slopes of the tangents are negative reciprocals of each other. Therefore, the curves intersect at right angles.

#### Derivatives of Inverse Trigonometric Functions

We can use implicit differentiation to find the derivatives of the inverse trigonometric functions. Recall that

$$y = \sin^{-1}x$$
 means  $\sin y = x$  and  $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ 

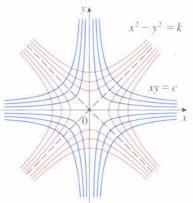


FIGURE 7

Differentiating  $\sin y = x$  implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1$$
 or  $\frac{dy}{dx} = \frac{1}{\cos y}$ 

Now  $\cos y \ge 0$ , since  $-\pi/2 \le y \le \pi/2$ , so

 $\sec^2 y \frac{dy}{dx} = 1$ 

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

 $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$ 

Therefore

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$$

The formula for the derivative of the arctangent function is derived in a similar way. If  $y = \tan^{-1}x$ , then  $\tan y = x$ . Differentiating this latter equation implicitly with respect to x, we have

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

**EXAMPLE 5** Differentiate (a) 
$$y = \frac{1}{\sin^{-1}x}$$
 and (b)  $f(x) = x \tan^{-1} \sqrt{x}$ .

increasing and 
$$f'(x)$$
 is always positive.  
The fact that  $\tan^{-1}x \rightarrow \pm \pi/2$  as  $x \rightarrow \pm \infty$   
is reflected in the fact that  $f'(x) \rightarrow 0$  as  $x \rightarrow \pm \infty$ .  
1.5

Figure 8 shows the graph of  $f(x) = \tan^{-1}x$  and its derivative  $f'(x) = 1/(1 + x^2)$ . Notice that f is

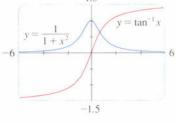


FIGURE 8

SOLUTION

(a)

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2} \frac{d}{dx} (\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

\_

$$f'(x) = \tan^{-1}\sqrt{x} + x \frac{1}{1 + (\sqrt{x})^2} \left(\frac{1}{2}x^{-1/2}\right)$$
$$= \tan^{-1}\sqrt{x} + \frac{\sqrt{x}}{2(1 + x)}$$

(b)



#### 1-2 ■

(a) Find y' by implicit differentiation.

Exercises

- (b) Solve the equation explicitly for y and differentiate to get y' in terms of x.
- (c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).
- **1.**  $x^2 + 3x + xy = 5$  **2.**  $\sqrt{x} + \sqrt{y} = 4$

**3–10** Find dy/dx by implicit differentiation.

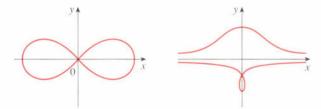
**3.**  $x^2 - xy + y^3 = 8$ **4.**  $y^5 + 3x^2y^2 + 5x^4 = 12$ **5.**  $x^4 + y^4 = 16$ **6.**  $\sqrt{x + y} + \sqrt{xy} = 6$ **7.**  $\cos(x - y) = xe^x$ **8.**  $x \sin y + \cos 2y = \cos y$ **9.**  $xy = \cot(xy)$ **10.**  $x \cos y + y \cos x = 1$ 

11-16 Find an equation of the tangent line to the curve at the given point.

11.  $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ,  $(-5, \frac{9}{4})$  (hyperbola) 12.  $\frac{x^2}{9} + \frac{y^2}{36} = 1$ ,  $(-1, 4\sqrt{2})$  (ellipse) 13.  $y^2 = x^3(2 - x)$  (1,1) (piriform) (astroid) 14.  $x^{2/3} + y^{2/3} = 4$  ( $-3\sqrt{3}, 1$ ) (astroid) (astroid)



**15.**  $2(x^2 + y^2)^2 = 25(x^2 - y^2)$  **16.**  $x^2y^2 = (y + 1)^2(4 - y^2)$ (3,1) (0,-2) (lemniscate) (conchoid of Nicomedes)



17. (a) The curve with equation  $y^2 = 5x^4 - x^2$  is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point (1, 2).

- (b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)
- **18.** (a) The curve with equation  $y^2 = x^3 + 3x^2$  is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point (1, -2).
  - (b) At what points does this curve have a horizontal tangent?
  - (c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.
- **CAS** 19. Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.
  - (a) Graph the curve with equation

 $\wedge$ 

$$y(y^{2} - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the *x*-coordinates of these points.

- (b) Find equations of the tangent lines at the points (0, 1) and (0, 2).
- (c) Find the exact *x*-coordinates of the points in part (a).
- (d) Create even more fanciful curves by modifying the equation in part (a).
- **CAS** 20. (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.

- (b) At how many points does this curve have horizontal tangent lines? Find the *x*-coordinates of these points.
- **21.** Find the points on the lemniscate in Exercise 15 where the tangent is horizontal.
- **22.** Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point  $(x_0, y_0)$  is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

- 23. If x<sup>4</sup> + y<sup>4</sup> = 16, use the following steps to find y".
  (a) Use implicit differentiation to find y'.
  - (b) Use the Quotient Rule to differentiate the expression for y' from part (a). Express your answer in terms of x and y only.

(c) Use the fact that x and y must satisfy the original equation  $x^4 + y^4 = 16$  to simplify your answer to part (b) to the following:

$$y'' = -48 \frac{x^2}{y^7}$$

**24.** If  $x^2 + 6xy + y^2 = 8$ , find y" by implicit differentiation.

**25–30** Find the derivative of the function. Simplify where possible.

- **25.**  $y = \sin^{-1}(x^2)$  **26.**  $y = (\sin^{-1}x)^2$  **27.**  $y = \tan^{-1}(e^x)$  **28.**  $h(x) = \sqrt{1 - x^2} \arcsin x$  **29.**  $H(x) = (1 + x^2) \arctan x$ **30.**  $y = \tan^{-1}(x - \sqrt{1 + x^2})$
- 31-32 Find f'(x). Check that your answer is reasonable by comparing the graphs of f and f'.

**31.**  $f(x) = e^x - x^2 \arctan x$ 

**32.**  $f(x) = x \arcsin(1 - x^2)$ 

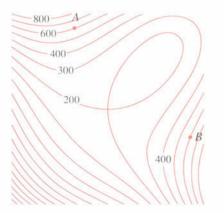
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**33–34** ■ Show that the given curves are orthogonal.

**33.** 
$$2x^2 + y^2 = 3$$
,  $x = y^2$ 

**34.**  $x^2 - y^2 = 5$ ,  $4x^2 + 9y^2 = 72$ 

**35.** Contour lines on a map of a hilly region are curves that join points with the same elevation. A ball rolling down a hill follows a curve of steepest descent, which is orthogonal to the contour lines. Given the contour map of a hill in the figure, sketch the paths of balls that start at positions *A* and *B*.



36. TV weathermen often present maps showing pressure fronts. Such maps display *isobars*—curves along which the air pressure is constant. Consider the family of isobars shown in the figure. Sketch several members of the family of orthogonal trajectories of the isobars. Given the fact that wind blows from regions of high air pressure to regions of low air pressure, what does the orthogonal family represent?



**37-40** Show that the given families of curves are orthogonal trajectories of each other. Sketch both families of curves on the same axes.

**37.**  $x^2 + y^2 = r^2$ , ax + by = 0 **38.**  $x^2 + y^2 = ax$ ,  $x^2 + y^2 = by$  **39.**  $y = cx^2$ ,  $x^2 + 2y^2 = k$ **40.**  $y = ax^3$ ,  $x^2 + 3y^2 = b$ 

Æ

- **41.** Show, using implicit differentiation, that any tangent line at a point *P* to a circle with center *O* is perpendicular to the radius *OP*.
- **42.** Show that the sum of the *x* and *y*-intercepts of any tangent line to the curve  $\sqrt{x} + \sqrt{y} = \sqrt{c}$  is equal to *c*.
- **43.** The equation  $x^2 xy + y^2 = 3$  represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the *x*-axis and show that the tangent lines at these points are parallel.
- **44.** (a) Where does the normal line to the ellipse  $x^2 xy + y^2 = 3$  at the point (-1, 1) intersect the ellipse a second time?
- (b) Illustrate part (a) by graphing the ellipse and the normal line.
- **45.** Find all points on the curve  $x^2y^2 + xy = 2$  where the slope of the tangent line is -1.
- **46.** Find the equations of both the tangent lines to the ellipse  $x^2 + 4y^2 = 36$  that pass through the point (12, 3).
- **47.** (a) Suppose f is a one-to-one differentiable function and its inverse function  $f^{-1}$  is also differentiable. Use implicit differentiation to show that

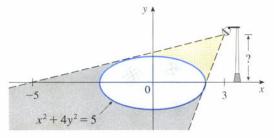
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

(b) If f(4) = 5 and  $f'(4) = \frac{2}{3}$ , find  $(f^{-1})'(5)$ .

- **48.** (a) Show that  $f(x) = 2x + \cos x$  is one-to-one. (b) What is the value of  $f^{-1}(1)$ ?
  - (c) Use the formula from Exercise 47(a) to find  $(f^{-1})'(1)$ .
- **49.** The **Bessel function** of order 0, y = J(x), satisfies the differential equation xy'' + y' + xy = 0 for all values of x and its value at 0 is J(0) = 1.
  - (a) Find J'(0).
  - (b) Use implicit differentiation to find J''(0).
- **50.** The figure shows a lamp located three units to the right of the *y*-axis and a shadow created by the elliptical

region  $x^2 + 4y^2 \le 5$ . If the point (-5, 0) is on the edge of the shadow, how far above the *x*-axis is the lamp located?





## Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions  $y = \log_a x$  and, in particular, the natural logarithmic function  $y = \ln x$ .

1

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

**Proof** Let  $y = \log_a x$ . Then

 $a^y = x$ 

Formula 5 in Section 3.5 says that  $\frac{d}{dx} (a^x) = a^x \ln a$ 

Differentiating this equation implicitly with respect to x, using Formula 5 in Section 3.5, we get

and so

$$a^{y}(\ln a) \frac{dy}{dx} = 1$$

$$\frac{y}{x} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

If we put a = e in Formula 1, then the factor  $\ln a$  on the right side becomes  $\ln e = 1$  and we get the formula for the derivative of the natural logarithmic function  $\log_e x = \ln x$ :

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

d

d

By comparing Formulas 1 and 2 we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when a = e because  $\ln e = 1$ .

## **EXAMPLE I** Differentiate $y = \ln(x^3 + 1)$ .

**SOLUTION** To use the Chain Rule we let  $u = x^3 + 1$ . Then  $y = \ln u$ , so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

3 
$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$
 or  $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$ 

**EXAMPLE 2** Find  $\frac{d}{dx} \ln(\sin x)$ .

SOLUTION Using (3), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

**EXAMPLE 3** Differentiate  $f(x) = \sqrt{\ln x}$ .

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \frac{1}{2} (\ln x)^{-1/2} \frac{d}{dx} (\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

**EXAMPLE 4** Differentiate  $f(x) = \log_{10}(2 + \sin x)$ .

**SOLUTION** Using Formula 1 with a = 10, we have

$$f'(x) = \frac{d}{dx} \log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx} (2 + \sin x)$$
$$= \frac{\cos x}{(2 + \sin x) \ln 10}$$

**EXAMPLE 5** Find  $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$ .

SOLUTION I

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{1}{\frac{x+1}{\sqrt{x-2}}} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}}$$
$$= \frac{\sqrt{x-2}}{\frac{x+1}{x+1}} \frac{\sqrt{x-2} \cdot 1 - (x+1)(\frac{1}{2})(x-2)^{-1/2}}{x-2}$$
$$= \frac{x-2 - \frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right]$$
$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

Figure I shows the graph of the function *f* of Example 5 together with the graph of its derivative. It gives a visual check on our calculation. Notice that f'(x)is large negative when f is rapidly decreasing.

#### FIGURE I

**EXAMPLE 6** Find f'(x) if  $f(x) = \ln |x|$ .

**SOLUTION** Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0\\ \ln(-x) & \text{if } x < 0 \end{cases}$$

Figure 2 shows the graph of the function  $f(x) = \ln |x|$  in Example 6 and its derivative f'(x) = 1/x. Notice that when x is small, the graph of  $y = \ln |x|$  is steep and so f'(x) is large (positive or negative).

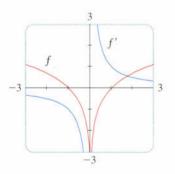


FIGURE 2

4

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0\\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

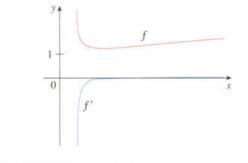
Thus, f'(x) = 1/x for all  $x \neq 0$ .

The result of Example 6 is worth remembering:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

#### Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called logarithmic differentiation.



$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

**EXAMPLE 7** Differentiate

$$y = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$$

**SOLUTION** We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y \left( \frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2} \right)$$
$$= \frac{x^{3/4} \sqrt{x^2 + 1}}{(3x + 2)^5} \left( \frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2} \right)$$

#### **Steps in Logarithmic Differentiation**

- 1. Take natural logarithms of both sides of an equation y = f(x) and use the Laws of Logarithms to simplify.
- 2. Differentiate implicitly with respect to *x*.
- **3.** Solve the resulting equation for y'.

If f(x) < 0 for some values of x, then  $\ln f(x)$  is not defined, but we can write |y| = |f(x)| and use Equation 4. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.1.

**Power Rule** If *n* is any real number and  $f(x) = x^n$ , then

 $f'(x) = nx^{n-1}$ 

**Proof** Let  $y = x^n$  and use logarithmic differentiation:

If x = 0, we can show that f'(0) = 0directly from the definition of a derivative.

$$\ln |y| = \ln |x|^n = n \ln |x|$$
  $x \neq 0$ 

Therefore

Hence

$$\frac{y'}{y} = \frac{r}{x}$$

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

You should distinguish carefully between the Power Rule  $[(x^n)' = nx^{n-1}]$ , where the base is variable and the exponent is constant, and the rule for differentiating

exponential functions  $[(a^x)' = a^x \ln a]$ , where the base is constant and the exponent is variable. In general, there are four cases for exponents and bases:

1. 
$$\frac{d}{dx}(a^b) = 0$$
 (*a* and *b* are constants)  
2.  $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$   
3.  $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$ 

**4.** To find  $(d/dx)[f(x)]^{g(x)}$ , logarithmic differentiation can be used, as in the next example.

**EXAMPLE 8** Differentiate  $y = x^{\sqrt{x}}$ .

n, we have SOLUTION 1 U

> $\frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x} \cdot \frac{1}{x}$  $y' = y \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right) = x^{\sqrt{x}} \left( \frac{\ln x + 2}{2\sqrt{x}} \right)$

**SOLUTION 2** Another method is to write  $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$ :

$$\frac{d}{dx} (x^{\sqrt{x}}) = \frac{d}{dx} (e^{\sqrt{x} \ln x})$$
$$= e^{\sqrt{x} \ln x} \frac{d}{dx} (\sqrt{x} \ln x)$$
$$= x^{\sqrt{x}} \left(\frac{\ln x + 2}{2\sqrt{x}}\right) \qquad \text{(as above)}$$

#### The Number e as a Limit

We have shown that if  $f(x) = \ln x$ , then f'(x) = 1/x. Thus, f'(1) = 1. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$
$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$
$$= \lim_{x \to 0} \ln(1+x)^{1/x} = \ln \left[\lim_{x \to 0} (1+x)^{1/x}\right] \quad \text{(since ln is continuous)}$$

Because f'(1) = 1, we have

$$\ln\left[\lim_{x\to 0} \left(1+x\right)^{1/x}\right] = 1$$



Figure 3 illustrates Example 8 by showing the graphs of  $f(x) = x^{\sqrt{x}}$  and its





Using logarithmic differentiation  

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$y' = 1$$

Using logarithmic differentia  

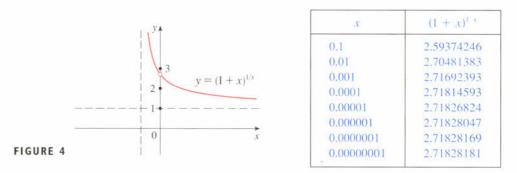
$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

Therefore

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

Formula 5 is illustrated by the graph of the function  $y = (1 + x)^{1/x}$  in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

 $e \approx 2.7182818$ 



If we put n = 1/x in Formula 5, then  $n \to \infty$  as  $x \to 0^+$  and so an alternative expression for *e* is

. \ //

$$e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n$$

3.7

**I.** Explain why the natural logarithmic function  $y = \ln x$  is used much more frequently in calculus than the other logarithmic functions  $y = \log_{a} x$ .

**2–16** ■ Differentiate the function.

**Exercises** 

**2.**  $f(x) = \ln(2 - x)$ 

**3.** 
$$f(\theta) = \ln(\cos \theta)$$
 **4.**  $f(x) = \cos(\ln \theta)$ 

5. 
$$f(x) = \log_3(x^2 - 4)$$

**6.** 
$$f(x) = \log_{10} \left( \frac{x}{x - x} \right)^{1/2}$$

In 1

 $\frac{3u+2}{3u-2}$ 

6

**7.** 
$$g(x) = \ln \frac{a - x}{a + x}$$
  
**8.**  $h(x) = \ln(x + \sqrt{x^2 - 1})$   
**9.**  $F(x) = \ln \sqrt{x}$   
**10.**  $G(x) = \sqrt{\ln x}$ 

9. 
$$F(x) = \ln \sqrt{x}$$

**11.** 
$$f(x) = \sqrt{x} \ln x$$
  
**12.**  $h(y) = \ln(y^3 \sin y)$ 

**13.** 
$$y = \ln |x^3 - x^2|$$
 **14.**  $G(u) =$ 

**15.** 
$$y = \ln(e^{-x} + xe^{-x})$$
 **16.**  $y = \ln(x + \ln x)$ 

**17–18** Find y' and y''.

**17.**  $y = \log_{10} x$ 

**18.**  $y = \ln(\sec x + \tan x)$ 

**19–20** Differentiate f and find its domain.

**19.** 
$$f(x) = x^2 \ln(1 - x^2)$$
 **20.**  $f(x) = \ln \ln \ln x$ 

**21.** If 
$$f(x) = \frac{x}{\ln x}$$
, find  $f'(e)$ .

- **22.** Find an equation of the tangent line to the curve  $y = (\ln x)/x$  at the points (1,0) and (e, 1/e). Illustrate by graphing the curve and its tangent lines.
  - (a) On what interval is f(x) = x ln x decreasing?(b) On what interval is f concave upward?
- **24.** If  $f(x) = \sin x + \ln x$ , find f'(x). Check that your answer is reasonable by comparing the graphs of f and f'.

**25–34** ■ Use logarithmic differentiation to find the derivative of the function.

- **36.** Find y' if  $x^y = y^x$ .
- **37.** Find a formula for  $f^{(n)}(x)$  if  $f(x) = \ln(x 1)$ .

**38.** Find 
$$\frac{d^9}{dx^9} (x^8 \ln x)$$
.

**39.** Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$
**40.** Show that  $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$  for any  $x > 0$ .

## Discovery Project Hyperbolic Functions

Certain combinations of the exponential functions  $e^x$  and  $e^{-x}$  arise so frequently in mathematics and its applications that they deserve to be given special names. This project explores the properties of functions called **hyperbolic functions**. The **hyperbolic sine**, **hyperbolic cosine**, **hyperbolic tangent**, and **hyperbolic secant** functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

The reason for the names of these functions is that they are related to the hyperbola in much the same way that the trigonometric functions are related to the circle.

- 1. (a) Sketch, by hand, the graphs of the functions  $y = \frac{1}{2}e^x$  and  $y = \frac{1}{2}e^{-x}$  on the same axes and use graphical addition to draw the graph of cosh.
- (b) Check the accuracy of your sketch in part (a) by using a graphing calculator or computer to graph y = cosh x. What are the domain and range of this function?
- 2. The most famous application of hyperbolic functions is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as a telephone or power line) is suspended between two points at the same height, then it takes the shape of a curve with equation y = a cosh(x/a) called a *catenary*. (The Latin word *catena* means "chain.") Graph several members of the family of functions y = a cosh(x/a). How does the graph change as a varies?
- **3.** Graph sinh and tanh. Judging from their graphs, which of the functions sinh, cosh, and tanh are even? Which are odd? Use the definitions to prove your assertions.
  - 4. Prove the identity  $\cosh^2 x \sinh^2 x = 1$ .
- **5.** Graph the curve with parametric equations  $x = \cosh t$ ,  $y = \sinh t$ . Can you identify this curve?
  - 6. Prove the identity  $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ .

- **7.** The identities in Problems 4 and 6 are similar to well-known trigonometric identities. Try to discover other hyperbolic identities by using known trigonometric identities as your inspiration.
- **8.** The differentiation formulas for the hyperbolic functions are analogous to those for the trigonometric functions, but the signs are sometimes different.

(a) Show that 
$$\frac{d}{dx}(\sinh x) = \cosh x$$
.

- (b) Discover formulas for the derivatives of  $y = \cosh x$  and  $y = \tanh x$ .
- 9. (a) Explain why sinh is a one-to-one function.
  - (b) Find a formula for the derivative of the inverse hyperbolic sine function  $y = \sinh^{-1}x$ . [*Hint:* How did we find the derivative of  $y = \sin^{-1}x$ ?]
  - (c) Show that  $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$ .
  - (d) Use the result of part (c) to find the derivative of  $\sinh^{-1}x$ . Compare with your answer to part (b).
- 10. (a) Explain why tanh is a one-to-one function.
  - (b) Find a formula for the derivative of the inverse hyperbolic tangent function  $y = \tanh^{-1}x$ .
  - (c) Show that  $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ .
  - (d) Use the result of part (c) to find the derivative of  $tanh^{-1}x$ . Compare with your answer to part (b).
- **II.** At what point on the curve  $y = \cosh x$  does the tangent have slope 1?



## Linear Approximations and Differentials

In Section 2.9 we considered linear approximations to functions, based on the idea that a tangent line lies very close to a graph near the point of tangency. Now that we are equipped with the differentiation rules, we revisit this idea and use graphical methods to decide how good a linear approximation is. We also see how linear approximations are applied in physics.

#### Linear Approximations

2

An equation of the tangent line to the curve y = f(x) at (a, f(a)) is

$$y = f(a) + f'(a)(x - a)$$

So, as in Section 2.9, the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a, and the function

$$L(x) = f(a) + f'(a)(x - a)$$

(whose graph is the tangent line) is called the **linearization** of *f* at *a*. The linear approximation  $f(x) \approx L(x)$  is a good approximation when *x* is near *a* (see Figure 1).

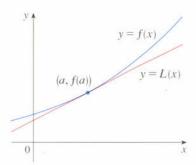


FIGURE I

**EXAMPLE I** Find the linearization of the function  $f(x) = \sqrt{x+3}$  at a = 1 and use it to approximate the numbers  $\sqrt{3.98}$  and  $\sqrt{4.05}$ .

**SOLUTION** The derivative of  $f(x) = (x + 3)^{1/2}$  is

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$$

and so we have f(1) = 2 and  $f'(1) = \frac{1}{4}$ . Putting these values into Equation 2, we see that the linearization is

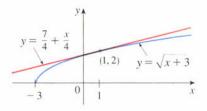
$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (1) is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

In particular, we have:

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$
 and  $\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$ 



The linear approximation in Example 1 is illustrated in Figure 2. You can see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. Of course, a calculator could give us approximations for  $\sqrt{3.98}$  and  $\sqrt{4.05}$ , but the linear approximation gives an approximation over an entire interval.

How good is the approximation that we obtained in Example 1? The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

**EXAMPLE 2** For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

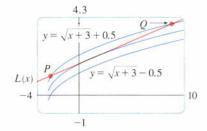
accurate to within 0.5? What about accuracy to within 0.1?

**SOLUTION** Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)\right| < 0.5$$

Equivalently, we could write

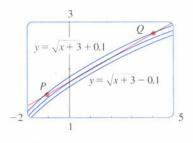
$$\sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5$$



This says that the linear approximation should lie between the curves obtained by shifting the curve  $y = \sqrt{x+3}$  upward and downward by an amount 0.5. Figure 3 shows the tangent line y = (7 + x)/4 intersecting the upper curve

FIGURE 2

FIGURE 3





 $y = \sqrt{x} + 3 + 0.5$  at *P* and *Q*. Zooming in and using the cursor, we estimate that the *x*-coordinate of *P* is about -2.66 and the *x*-coordinate of *Q* is about 8.66. Thus, we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when -2.6 < x < 8.6. (We have rounded to be safe.)

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when -1.1 < x < 3.9.

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation. For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression  $a_T = -g \sin \theta$  for tangential acceleration and then replace  $\sin \theta$  by  $\theta$  with the remark that  $\sin \theta$  is very close to  $\theta$ if  $\theta$  is not too large. [See, for example, Eugene Hecht, *Physics* (Pacific Grove, CA: Brooks/Cole, 1994), p. 413.] You can verify that the linearization of the function  $f(x) = \sin x$  at a = 0 is L(x) = x and so the linear approximation at 0 is

$$\sin x \approx x$$

(see Exercise 11). So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays*. In paraxial (or Gaussian) optics, both  $\sin \theta$  and  $\cos \theta$  are replaced by their linearizations. In other words, the linear approximations

 $\sin\theta \approx \theta$  and  $\cos\theta \approx 1$ 

are used because  $\theta$  is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. [See Eugene Hecht, *Optics*, 2d ed. (Reading, MA: Addison-Wesley, 1987), p. 134.]

In Section 8.9 we will present several other applications of the idea of linear approximations.

#### Differentials

3

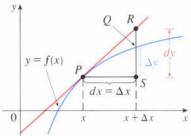
The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If y = f(x), where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

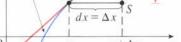
If  $dx \neq 0$ , we can divide both sides of Equation 3 by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

We have seen similar equations before, but now the left side can genuinely be interpreted as a ratio of differentials. So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

dv = f'(x) dx





The geometric meaning of differentials is shown in Figure 5. Let P(x, f(x)) and  $O(x + \Delta x, f(x + \Delta x))$  be points on the graph of f and set  $dx = \Delta x$ . The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line *PR* is the derivative f'(x). Thus, the directed distance from S to R is f'(x) dx = dy. Therefore, dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas  $\Delta y$  represents the amount that the curve y = f(x) rises or falls when x changes by an amount dx.

In the notation of differentials, the linear approximation (1) can be written as

$$f(a + dx) \approx f(a) + dy$$

For instance, for the function  $f(x) = \sqrt{x+3}$  in Example 1, we have

 $dy = f'(x) \, dx = \frac{dx}{2\sqrt{x+3}}$ 

If a = 1 and  $dx = \Delta x = 0.05$ , then

$$dy = \frac{0.05}{2\sqrt{1+3}} = 0.0125$$

and

 $\sqrt{4.05} = f(1.05) \approx f(1) + dv = 2.0125$ 

just as we found in Example 1.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

**EXAMPLE 3** The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

**SOLUTION** If the radius of the sphere is r, then its volume is  $V = \frac{4}{3}\pi r^3$ . If the error in the measured value of r is denoted by  $dr = \Delta r$ , then the corresponding error in the calculated value of V is  $\Delta V$ , which can be approximated by the differential

$$dV = 4\pi r^2 dr$$

When r = 21 and dr = 0.05, this becomes

$$dV = 4\pi(21)^2 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm<sup>3</sup>.

**NOTE** • Although the possible error in Example 3 may appear to be rather large, a better picture of the error is given by the relative error, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3 \frac{dr}{r}$$

FIGURE 5

Exercises ·

Thus, the relative error in the volume is about three times the relative error in the radius. In Example 3 the relative error in the radius is approximately  $dr/r = 0.05/21 \approx 0.0024$  and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as **percentage errors** of 0.24% in the radius and 0.7% in the volume.

# 3.8

**1–4** Find the linearization L(x) of the function at *a*.

1.  $f(x) = x^3$ , a = 1

**2.**  $f(x) = \ln x$ , a = 1

- **3.**  $f(x) = e^{-2x}$ , a = 0
- 4.  $f(x) = \sqrt[3]{x}, a = -8$
- Find the linear approximation of the function f(x) = √1 − x at a = 0 and use it to approximate the numbers √0.9 and √0.99. Illustrate by graphing f and the tangent line.
- 6. Find the linear approximation of the function  $g(x) = \sqrt[3]{1+x}$  at a = 0 and use it to approximate the numbers  $\sqrt[3]{0.95}$  and  $\sqrt[3]{1.1}$ . Illustrate by graphing g and the tangent line.
- **7-10** Verify the given linear approximation at a = 0. Then determine the values of x for which the linear approximation is accurate to within 0.1
  - **7.**  $\sqrt{1+x} \approx 1 + \frac{1}{2}x$  **8.** tan  $x \approx x$

9. 
$$1/(1 + 2x)^4 \approx 1 - 8x$$

**10.** 
$$e^x \approx 1 + x$$

 $\mathcal{A}$ 

- On page 413 of Eugene Hecht, *Physics* (Pacific Grove, CA: Brooks/Cole, 1994), in the course of deriving the formula T = 2π √L/g for the period of a pendulum of length L, the author obtains the equation a<sub>T</sub> = -g sin θ for the tangential acceleration of the bob of the pendulum. He then says, "for small angles, the value of θ in radians is very nearly the value of sin θ; they differ by less than 2% out to about 20°."
  - (a) Verify the linear approximation at 0 for the sine function:

$$\sin x \approx x$$

- (b) Use a graphing device to determine the values of x for which sin x and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.
- 12. Let f be a function such that f(1) = 2 and whose derivative is known to be  $f'(x) = \sqrt{x^3 + 1}$ . [You are not given

a formula for f(x). Don't try to guess one—you won't succeed.]

- (a) Use a linear approximation to estimate the value of f(1.1).
- (b) Do you think the true value of f(1.1) is less than or greater than your estimate? Why?
- **13.** Let  $y = \cos x$ .
  - (a) Find the differential dy.
  - (b) Evaluate dy and  $\Delta y$  if  $x = \pi/6$  and dx = 0.05.
- 14. Let  $y = \sqrt{x}$ .
  - (a) Find the differential dy.
  - (b) Evaluate dy and  $\Delta y$  if x = 1 and  $dx = \Delta x = 1$ .
  - (c) Sketch a diagram like Figure 5 showing the line segments with lengths dx, dy, and  $\Delta y$ .
- 15. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error in computing (a) the volume of the cube and (b) the surface area of the cube.
- **16.** The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
  - (a) Use differentials to estimate the maximum error in the calculated area of the disk.
  - (b) What is the relative error? What is the percentage error?
- Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.
- 18. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 6.6.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

## Laboratory Project

#### Taylor Polynomials

 $\wedge$ 

The tangent line approximation L(x) is the best first-degree (linear) approximation to f(x) near x = a because f(x) and L(x) have the same rate of change (derivative) at a. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation P(x). In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

- (i) P(a) = f(a) (P and f should have the same value at a.)
- (ii) P'(a) = f'(a) (P and f should have the same rate of change at a.)
- (iii) P''(a) = f''(a) (The slopes of *P* and *f* should change at the same rate.)
- 1. Find the quadratic approximation  $P(x) = A + Bx + Cx^2$  to the function  $f(x) = \cos x$  that satisfies conditions (i), (ii), and (iii) with a = 0. Graph *P*, *f*, and the linear approximation L(x) = 1 on a common screen. Comment on how well the functions *P* and *L* approximate *f*.
- **2.** Determine the values of x for which the quadratic approximation f(x) = P(x) in Problem 1 is accurate to within 0.1. [*Hint:* Graph y = P(x),  $y = \cos x 0.1$ , and  $y = \cos x + 0.1$  on a common screen.]
- **3.** To approximate a function f by a quadratic function P near a number a, it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^{2}$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^{2}$$

- 4. Find the quadratic approximation to f(x) = √x + 3 near a = 1. Graph f, the quadratic approximation, and the linear approximation from Example 2 in Section 3.8 on a common screen. What do you conclude?
- 5. Instead of being satisfied with a linear or quadratic approximation to f(x) near x = a, let's try to find better approximations with higher-degree polynomials. We look for an *n*th-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots + c_n(x - a)^n$$

such that  $T_n$  and its first *n* derivatives have the same values at x = a as *f* and its first *n* derivatives. By differentiating repeatedly and setting x = a, show that these conditions are satisfied if  $c_0 = f(a)$ ,  $c_1 = f'(a)$ ,  $c_2 = \frac{1}{2}f''(a)$ , and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where  $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$ . The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the *n*th-degree Taylor polynomial of f centered at a.

**6.** Find the eighth-degree Taylor polynomial centered at a = 0 for the function  $f(x) = \cos x$ . Graph *f* together with the Taylor polynomials  $T_2$ ,  $T_4$ ,  $T_6$ ,  $T_8$  in the viewing rectangle [-5, 5] by [-1.4, 1.4] and comment on how well they approximate *f*.

## Chapter 3 Review

#### CONCEPT CHECK •

- State each of the following differentiation rules both in symbols and in words.
  - (a) The Power Rule
  - (b) The Constant Multiple Rule
  - (c) The Sum Rule
  - (d) The Difference Rule
  - (e) The Product Rule
  - (f) The Quotient Rule
  - (g) The Chain Rule
- 2. State the derivative of each function.
  - (a)  $y = x^n$  (b)  $y = e^x$  (c)  $y = a^x$
  - (d)  $y = \ln x$  (e)  $y = \log_a x$  (f)  $y = \sin x$
  - (g)  $y = \cos x$  (h)  $y = \tan x$  (i)  $y = \csc x$

- (j)  $y = \sec x$  (k)  $y = \cot x$
- (1)  $y = \sin^{-1}x$  (m)  $y = \tan^{-1}x$
- **3.** (a) How is the number *e* defined?
  - (b) Express e as a limit.
  - (c) Why is the natural exponential function  $y = e^x$  used more often in calculus than the other exponential functions  $y = a^x$ ?
  - (d) Why is the natural logarithmic function y = ln x used more often in calculus than the other logarithmic functions y = log<sub>a</sub> x?
- 4. (a) Explain how implicit differentiation works.(b) Explain how logarithmic differentiation works.
- 5. Write an expression for the linearization of f at a.

#### 🔺 TRUE-FALSE QUIZ 🔺 –

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

I. If f and g are differentiable, then

$$\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$$

**2.** If f and g are differentiable, then

$$\frac{d}{dx} \left[ f(x)g(x) \right] = f'(x)g'(x)$$

**3.** If f and g are differentiable, then

$$\frac{d}{dx} \left[ f(g(x)) \right] = f'(g(x))g'(x)$$

**4.** If *f* is differentiable, then  $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$ .

- **5.** If f is differentiable, then  $\frac{d}{dx} f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$ .
- 6. If  $y = e^2$ , then y' = 2e.

7. 
$$\frac{d}{dx}(10^x) = x10^{x-1}$$

**8.** 
$$\frac{d}{dx}(\ln 10) = \frac{1}{10}$$

9. 
$$\frac{d}{dx}(\tan^2 x) = \frac{d}{dx}(\sec^2 x)$$

**0.** 
$$\frac{d}{dx}|x^2 + x| = |2x + 1|$$

11. If 
$$g(x) = x^5$$
, then  $\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = 80$ .

🔶 EXERCISES ﴿

1

I-26 = Calculate y'.

1. 
$$y = (x + 2)^8 (x + 3)^6$$
 2.  $y = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$ 

3. 
$$y = \frac{x}{\sqrt{9 - 4x}}$$
  
5.  $y = \sin(\cos x)$   
4.  $y = \frac{e^x}{1 + x^2}$   
6.  $y = \sin^{-1}(e^x)$ 

7. 
$$y = xe^{-1/x}$$
  
8.  $y = x^r e^{xx}$   
9.  $y = \tan \sqrt{1-x}$   
10.  $y = \frac{1}{\sin(x-\sin x)}$   
11.  $y = x/(8-3x)$   
12.  $y = \ln(\csc 5x)$   
13.  $y = e^{cx}(c\sin x - \cos x)$   
14.  $y = \ln(x^2e^x)$   
15.  $y = e^{e^x}$   
16.  $y = 5^{x\tan x}$   
17.  $x^2y^3 + 3y^2 = x - 4y$   
18.  $x\tan y = y - 1$   
19.  $y = \log_{10}(x^2 - x)$   
20.  $y = e^{\cos x} + \cos(e^x)$   
21.  $y = \ln \sin x - \frac{1}{2} \sin^2 x$   
22.  $y = \arctan(\arcsin \sqrt{x})$   
23.  $y = \sin(\tan \sqrt{1+x^3})$   
24.  $xe^y = y - 1$   
25.  $y = \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7}$   
26.  $y = x^{e^x}$   
27. If  $f(x) = 1/(2x-1)^5$ , find  $f''(0)$ .  
28. Find  $y''$  if  $x^6 + y^6 = 1$ .  
29. If  $f(x) = 2^x$ , find  $f^{(n)}(x)$ .  
30. Find an equation of the tangent to the curve  $\sqrt{x} + \sqrt{y} = 3$  at the point (4, 1).

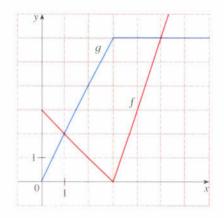
- 31. (a) If f(x) = x√5 x, find f'(x).
  (b) Find equations of the tangent lines to the curve y = x√5 x at the points (1, 2) and (4, 4).
- (c) Illustrate part (b) by graphing the curve and tangent lines.
- (d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

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- **32.** (a) If  $f(x) = 4x \tan x$ ,  $-\pi/2 < x < \pi/2$ , find f' and f''.
- (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.
- **33.** If  $f(x) = xe^{\sin x}$ , find f'(x). Graph f and f' on the same screen and comment.
- **34.** (a) Graph the function  $f(x) = x 2 \sin x$  in the viewing rectangle [0, 8] by [-2, 8].
  - (b) On which interval is the average rate of change larger: [1, 2] or [2, 3]?
  - (c) At which value of x is the instantaneous rate of change larger: x = 2 or x = 5?
  - (d) Check your visual estimates in part (c) by computing f'(x) and comparing the numerical values of f'(2) and f'(5).
  - **35.** Suppose that h(x) = f(x)g(x) and F(x) = f(g(x)), where f(2) = 3, g(2) = 5, g'(2) = 4, f'(2) = -2, and f'(5) = 11. Find (a) h'(2) and (b) F'(2).

**36.** If f and g are the functions whose graphs are shown, let P(x) = f(x)g(x), Q(x) = f(x)/g(x), and C(x) = f(g(x)). Find (a) P'(2), (b) Q'(2), and (c) C'(2).



**37–44** Find f'(x) in terms of g'(x).

<b>37.</b> <i>f</i> ( <i>x</i> )	$= x^2 g$	q(x)			<b>38.</b> $f(x)$	=	$g(x^2)$
<b>39.</b> <i>f</i> ( <i>x</i> )	= [ g(	$[x)]^{2}$		4	<b>40.</b> <i>f</i> ( <i>x</i> )	=	g(g(x))
<b>41.</b> $f(x)$	$= g(\epsilon$	·*)		4	<b>42.</b> $f(x)$	=	$e^{g(x)}$
<b>43.</b> <i>f</i> ( <i>x</i> )	$= \ln$	g(x)		4	<b>44.</b> $f(x)$	-	$g(\ln x)$
a) ( 13	4	1.2	81	1	10		

**45–46** Find h'(x) in terms of f'(x) and g'(x).

**45.** 
$$h(x) = \frac{f(x)g(x)}{f(x) + g(x)}$$
 **46.**  $h(x) = f(g(\sin 4x))$ 

- **47.** At what point on the curve  $y = [\ln(x + 4)]^2$  is the tangent horizontal?
- **48.** (a) Find an equation of the tangent to the curve  $y = e^x$  that is parallel to the line x 4y = 1.
  - (b) Find an equation of the tangent to the curve  $y = e^x$  that passes through the origin.
- **49.** Find the points on the ellipse  $x^2 + 2y^2 = 1$  where the tangent line has slope 1.
- **50.** (a) On what interval is the function  $f(x) = (\ln x)/x$  increasing?
  - (b) On what interval is f concave upward?
- **51.** An equation of motion of the form  $s = Ae^{-ct}\cos(\omega t + \delta)$  represents damped oscillation of an object. Find the velocity and acceleration of the object.
- **52.** A particle moves on a vertical line so that its coordinate at time t is  $y = t^3 12t + 3$ ,  $t \ge 0$ .
  - (a) Find the velocity and acceleration functions.
  - (b) When is the particle moving upward and when is it moving downward?
  - (c) Find the distance that the particle travels in the time interval  $0 \le t \le 3$ .

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 $\mathcal{A}$ 

- (d) Graph the position, velocity, and acceleration functions for  $0 \le t \le 3$ .
- (e) When is the particle speeding up? When is it slowing down?
- **53.** The mass of part of a wire is  $x(1 + \sqrt{x})$  kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when x = 4 m.
- 54. The volume of a right circular cone is  $V = \pi r^2 h/3$ , where r is the radius of the base and h is the height.
  - (a) Find the rate of change of the volume with respect to the height if the radius is constant.
  - (b) Find the rate of change of the volume with respect to the radius if the height is constant.
- **55.** The cost, in dollars, of producing *x* units of a commodity is

 $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$ 

- (a) Find the marginal cost function.
- (b) Find C'(100) and explain its meaning.
- (c) Compare C'(100) with the cost of producing the 101st item.
- (d) For what value of x does C have an inflection point? What is the significance of this value of x?
- **56.** The function  $C(t) = K(e^{-at} e^{-bt})$ , where *a*, *b*, and *K* are positive constants and b > a, is used to model the concentration at time *t* of a drug injected into the bloodstream.
  - (a) Show that  $\lim_{t\to\infty} C(t) = 0$ .
  - (b) Find C'(t), the rate at which the drug is cleared from circulation.
  - (c) When is this rate equal to 0?

- **57.** (a) Find the linearization of  $f(x) = \sqrt[3]{1 + 3x}$  at a = 0. State the corresponding linear approximation and use it to give an approximate value for  $\sqrt[3]{1.03}$ .
- (b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.
  - **58.** A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.
  - 59. Express the limit

$$\lim_{\theta \to \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3}$$

as a derivative and thus evaluate it.

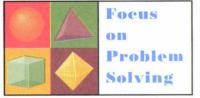
**60.** Find f'(x) if it is known that

$$\frac{d}{dx}\left[f(2x)\right] = x^2$$

61. Evaluate

$$\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$$

**62.** Show that the length of the portion of any tangent line to the astroid  $x^{2/3} + y^{2/3} = a^{2/3}$  cut off by the coordinate axes is constant.



 $3x^2 x^2 \frac{1}{2}x^2$ 

 $v = \ln x$ 

 $0.3x^2$ 

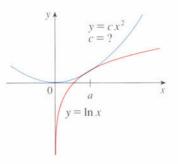
 $0.1x^2$ 

Before you look at the solution of the following example, cover it up and first try to solve the problem yourself. It might help to consult the principles of problem solving on page 87.

**Example** For what values of *c* does the equation  $\ln x = cx^2$  have exactly one solution?

Solution One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: For what values of c does the curve  $y = \ln x$  intersect the curve  $y = cx^2$  in exactly one point?

Let's start by graphing  $y = \ln x$  and  $y = cx^2$  for various values of c. We know that, for  $c \neq 0$ ,  $y = cx^2$  is a parabola that opens upward if c > 0 and downward if c < 0. Figure 1 shows the parabolas  $y = cx^2$  for several positive values of c. Most of them don't intersect  $y = \ln x$  at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.





#### Figure 2

To find that particular value of c, we let a be the x-coordinate of the single point of intersection. In other words,  $\ln a = ca^2$ , so a is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when x = a. That means the curves  $y = \ln x$  and  $y = cx^2$  have the same slope when x = a. Therefore

$$\frac{1}{a} = 2ca$$

Solving the equations  $\ln a = ca^2$  and 1/a = 2ca, we get

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$

Thus,  $a = e^{1/2}$  and

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

For negative values of c we have the situation illustrated in Figure 3: All parabolas  $y = cx^2$  with negative values of c intersect  $y = \ln x$  exactly once.

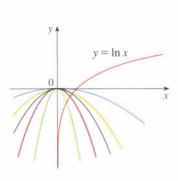


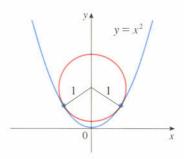
Figure 3

And let's not forget about c = 0: The curve  $y = 0x^2 = 0$  is just the x-axis, which intersects  $y = \ln x$  exactly once.

To summarize, the required values of c are c = 1/(2e) and  $c \le 0$ .

#### **Problems**

1. The figure shows a circle with radius 1 inscribed in the parabola  $y = x^2$ . Find the center of the circle.



- 2. Find the point where the curves  $y = x^3 3x + 4$  and  $y = 3(x^2 x)$  are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.
  - 3. (a) Find the domain of the function  $f(x) = \sqrt{1 \sqrt{2 \sqrt{3 x}}}$ . (b) Find f'(x).
  - (c) Check your work in parts (a) and (b) by graphing f and f' on the same screen.
  - **4.** If f is differentiable at a, where a > 0, evaluate the following limit in terms of f'(a):

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

- 5. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.
  - (a) Find the angular velocity of the connecting rod,  $d\alpha/dt$ , in radians per second, when  $\theta = \pi/3$ .
  - (b) Express the distance x = |OP| in terms of  $\theta$ .
  - (c) Find an expression for the velocity of the pin P in terms of  $\theta$ .
- **6.** Tangent lines  $T_1$  and  $T_2$  are drawn at two points  $P_1$  and  $P_2$  on the parabola  $y = x^2$  and they intersect at a point *P*. Another tangent line *T* is drawn at a point between  $P_1$  and  $P_2$ ; it intersects  $T_1$  at  $Q_1$  and  $T_2$  at  $Q_2$ . Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

7. Show that

A

$$\frac{d^n}{dx^n} e^{ax} \sin bx = r^n e^{ax} \sin(bx + n\theta)$$

where  $r^2 = a^2 + b^2$  and  $\theta = \tan^{-1}(b/a)$ .

8. Evaluate  $\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$ .

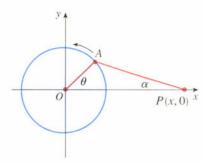


Figure for Problem 5

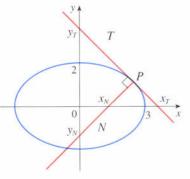
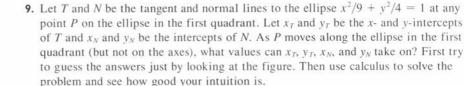


Figure for Problem 9



- 10. If the ellipse of Problem 9 is replaced by a more general ellipse  $x^2/a^2 + y^2/b^2 = 1$ , where a > b > 0, what values can  $x_N$  and  $y_N$  take on? Express your answers in terms of *a*, *b*, *c*, and *e*, where  $c^2 = a^2 b^2$ , and e = c/a is the eccentricity of the ellipse. Interpret your results geometrically.
- **II.** Find the *n*th derivative of the function  $f(x) = x^n/(1 x)$ .
- 12. For which positive numbers a is it true that  $a^x \ge 1 + x$  for all x?
- 13. (a) Use the identity for tan(x y) (see Equation 14b in Appendix C) to show that if two lines  $L_1$  and  $L_2$  intersect at an angle  $\alpha$ , then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where  $m_1$  and  $m_2$  are the slopes of  $L_1$  and  $L_2$ , respectively.

(b) The angle between the curves C<sub>1</sub> and C<sub>2</sub> at a point of intersection P is defined to be the angle between the tangent lines to C<sub>1</sub> and C<sub>2</sub> at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.
(i) v = v<sup>2</sup> and v = (v = 2)<sup>2</sup>
(ii) v<sup>2</sup> = v<sup>2</sup> = 3 and v<sup>2</sup> = 4x + v<sup>2</sup> + 3 = 0

(i) 
$$y = x^2$$
 and  $y = (x - 2)^2$  (ii)  $x^2 - y^2 = 3$  and  $x^2 - 4x + y^2 + 3 = 0$ 

- 14. Let  $P(x_1, y_1)$  be a point on the parabola  $y^2 = 4px$  with focus F(p, 0). Let  $\alpha$  be the angle between the parabola and the line segment *FP* and let  $\beta$  be the angle between the horizontal line  $y = y_1$  and the parabola as in the figure. Prove that  $\alpha = \beta$ . (Thus, by a principle of geometrical optics, light from a source placed at *F* will be reflected along a line parallel to the *x*-axis. This explains why paraboloids, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)
- 15. Suppose that we replace the parabolic mirror of Problem 14 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O. A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that  $\angle PQO = \angle OQR$  (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?
- 16. Given an ellipse  $x^2/a^2 + y^2/b^2 = 1$ , where  $a \neq b$ , find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.
- 17. Find the two points on the curve  $y = x^4 2x^2 x$  that have a common tangent line.
- **18.** Suppose that three points on the parabola  $y = x^2$  have the property that their normal lines intersect at a common point. Show that the sum of their *x*-coordinates is 0.
- 19. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope  $\frac{2}{5}$  intersects some of these circles.

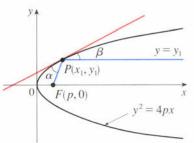


Figure for Problem 14

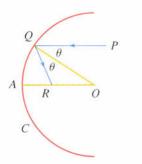
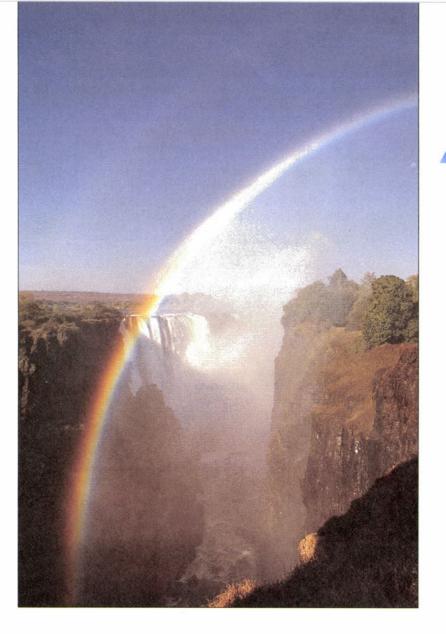


Figure for Problem 15





Applications of Differentiation



• • • These photographs illustrate three of the applications of differentiation considered in this chapter. The formation and location of rainbows is explained in the project on page 282. Exercise 34 on page 320 determines the optimal angle for branching of blood vessels in order to minimize the energy expended by the heart. The project on page 321 investigates the cost of manufacturing cans in order to explain why the smaller ones tend to be relatively tall and thin whereas the bigger ones are usually almost square.





♦ We have already investigated some of the applications of derivatives, but now that we know the differentiation rules we are in a better position to pursue the applications of differentiation in greater depth. We show how to analyze the behavior of families of functions, how to solve related rates problems (how to calculate rates that we can't measure from those that we can), and how to find the maximum or minimum value of a quantity. In particular, we will be able to investigate the optimal shape of a can and to explain the location of rainbows in the sky.



## Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

**EXAMPLE I** Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm<sup>3</sup>/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

**SOLUTION** We start by identifying two things:

the given information:

the rate of increase of the volume of air is 100 cm<sup>3</sup>/s

and the unknown:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically we introduce some suggestive *notation*:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time *t*. The rate of increase of the volume with respect to time is the derivative dV/dt and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

*Given*: 
$$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$
  
*Unknown*:  $\frac{dr}{dt}$  when  $r = 25 \text{ cm}$ 

In order to connect dV/dt and dr/dt we first relate V and r by the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^{3}$$

In order to use the given information, we differentiate each side of this equation with respect to *t*. To differentiate the right side we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

According to the Principles of Problem Solving discussed on page 87, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

The second stage of problem solving is to think of a plan for connecting the given and the unknown. Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

If we put r = 25 and dV/dt = 100 in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2} \ 100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of  $1/(25\pi)$  cm/s.

**EXAMPLE 2** A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

**SOLUTION** We first draw a diagram and label it as in Figure 1. Let x meters be the distance from the bottom of the ladder to the wall and y meters the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time).

We are given that dx/dt = 1 ft/s and we are asked to find dy/dt when x = 6 ft. (See Figure 2.) In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

When x = 6, the Pythagorean Theorem gives y = 8 and so, substituting these values and dx/dt = 1, we have

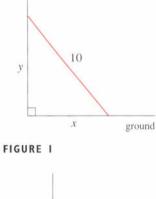
$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

**EXAMPLE 3** A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of  $2 \text{ m}^3/\text{min}$ , find the rate at which the water level is rising when the water is 3 m deep.

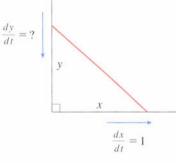
**SOLUTION** We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.

We are given that  $dV/dt = 2 \text{ m}^3/\text{min}$  and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$



wall



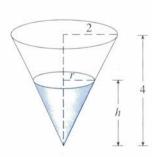


FIGURE 2

FIGURE 3

but it is very useful to express V as a function of h alone. In order to eliminate r we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

SO

$$V = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$$

Now we can differentiate each side with respect to *t*:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

Substituting h = 3 m and dV/dt = 2 m<sup>3</sup>/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi} \approx 0.28 \text{ m/min}$$

Look back: What have we learned from Examples I-3 that will help us solve future problems?

Warning: A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only after the differentiation. (Step 7 follows Step 6.) For instance, in Example 3 we dealt with general values of h until we finally substituted h = 3 at the last stage. (If we had put h = 3 earlier, we would have gotten dV/dt = 0, which is clearly wrong.)

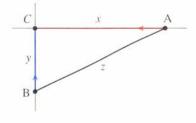


FIGURE 4

**STRATEGY** It is useful to recall some of the problem-solving principles from page 87 and adapt them to related rates in light of our experience in Examples 1–3:

- I. Read the problem carefully.
- 2. Draw a diagram if possible.
- **3.** Introduce notation. Assign symbols to all quantities that are functions of time.
- 4. Express the given information and the required rate in terms of derivatives.
- **5.** Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).
- 6. Use the Chain Rule to differentiate both sides of the equation with respect to *t*.
- **7.** Substitute the given information into the resulting equation and solve for the unknown rate.

The following examples are further illustrations of the strategy.

**EXAMPLE 4** Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

**SOLUTION** We draw Figure 4 where C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

We are given that dx/dt = -50 mi/h and dy/dt = -60 mi/h. (The derivatives are negative because x and y are decreasing.) We are asked to find dz/dt. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t, we have

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$
$$\frac{dz}{dt} = \frac{1}{z} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

When x = 0.3 mi and y = 0.4 mi, the Pythagorean Theorem gives z = 0.5 mi, so

$$\frac{dz}{dt} = \frac{1}{0.5} \left[ 0.3(-50) + 0.4(-60) \right]$$
$$= -78 \text{ mi/h}$$

The cars are approaching each other at a rate of 78 mi/h.

**EXAMPLE 5** • A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

**SOLUTION** We draw Figure 5 and let x be the distance from the point on the path closest to the searchlight to the man. We let  $\theta$  be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that dx/dt = 4 ft/s and are asked to find  $d\theta/dt$  when x = 15. The equation that relates x and  $\theta$  can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$

Differentiating each side with respect to t, we get

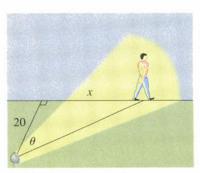
$$\frac{dx}{dt} = 20\sec^2\theta \,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta \,\frac{dx}{dt} = \frac{1}{20}\cos^2\theta(4) = \frac{1}{5}\cos^2\theta$$

When x = 15, the length of the beam is 25, so  $\cos \theta = \frac{4}{5}$  and

$$\frac{d\theta}{dt} = \frac{1}{5} \left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s.



SO



Exercises · · · ·

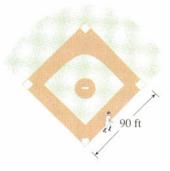


- 1. If V is the volume of a cube with edge length x, find dV/dt in terms of dx/dt.
- 2. If A is the area of a circle with radius r, find dA/dt in terms of dr/dt.

#### 3-6 =

- (a) What quantities are given in the problem?
- (b) What is the unknown?
- (c) Draw a picture of the situation for any time *t*.
- (d) Write an equation that relates the quantities.
- (e) Finish solving the problem.
- **3.** If a snowball melts so that its surface area decreases at a rate of 1 cm<sup>2</sup>/min, find the rate at which the diameter decreases when the diameter is 10 cm.
- **4.** At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 P.M.?
- 5. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.
- **6.** A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?
- 7. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?
- **8.** A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is his shadow on the building decreasing when he is 4 m from the building?
- **9.** A man starts walking north at 4 ft/s from a point *P*. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of *P*. At what rate are the people moving apart 15 min after the woman starts walking?
- **10.** A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
  - (a) At what rate is his distance from second base decreasing when he is halfway to first base?

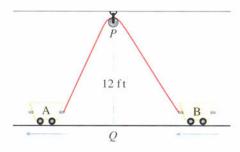
(b) At what rate is his distance from third base increasing at the same moment?



- 11. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm<sup>2</sup>/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm<sup>2</sup>?
- 12. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

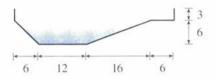


13. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?

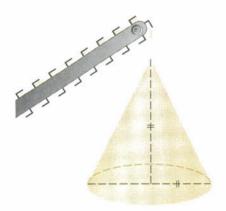


14. Water is leaking out of an inverted conical tank at a rate of 10,000 cm<sup>3</sup>/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

- 15. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m<sup>3</sup>/min, how fast is the water level rising when the water is 30 cm deep?
- 16. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft<sup>3</sup>/min, how fast is the water level rising when the depth at the deepest point is 5 ft?



17. Gravel is being dumped from a conveyor belt at a rate of 30 ft<sup>3</sup>/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



- **18.** A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?
- 19. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is π/3.
- 20. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side

increasing when the angle between the sides of fixed length is  $60^{\circ}$ ?

- **21.** Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure *P* and volume *V* satisfy the equation PV = C, where *C* is a constant. Suppose that at a certain instant the volume is 600 cm<sup>3</sup>, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?
- **22.** When air expands adiabatically (without gaining or losing heat), its pressure *P* and volume *V* are related by the equation  $PV^{1.4} = C$ , where *C* is a constant. Suppose that at a certain instant the volume is 400 cm<sup>3</sup> and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?
- **23.** A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.
  - (a) How fast is the distance from the television camera to the rocket changing at that moment?
  - (b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?
- **24.** A lighthouse is on a small island 3 km away from the nearest point *P* on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from *P*?
- **25.** A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at angle of 30°. At what rate is the distance from the plane to the radar station increasing 1 min later?
- **26.** Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 min?
- **27.** A runner runs around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?
- **28.** The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?



# Maximum and Minimum Values

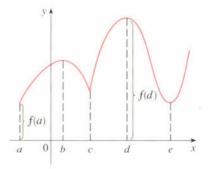
Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function. Let us first explain exactly what we mean by maximum and minimum values.

**Definition** A function f has an **absolute maximum** (or **global maximum**) at c if  $f(c) \ge f(x)$  for all x in D, where D is the domain of f. The number f(c) is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if  $f(c) \le f(x)$  for all x in D and the number f(c) is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Figure 1 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that (d, f(d)) is the highest point on the graph and (a, f(a)) is the lowest point.



#### FIGURE I

Minimum value f(a), maximum value f(d)

In Figure 1, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then f(b) is the largest of those values of f(x) and is called a *local maximum value* of f. Likewise, f(c) is called a *local minimum value* of f because  $f(c) \leq f(x)$  for x near c [in the interval (b, d), for instance]. The function f also has a local minimum at e. In general, we have the following definition.

**2 Definition** A function f has a **local maximum** (or relative maximum) at c if  $f(c) \ge f(x)$  when x is near c. [This means that  $f(c) \ge f(x)$  for all x in some open interval containing c.] Similarly, f has a **local minimum** at c if  $f(c) \le f(x)$  when x is near c.

**EXAMPLE 1** The function  $f(x) = \cos x$  takes on its (local and absolute) maximum value of 1 infinitely many times, since  $\cos 2n\pi = 1$  for any integer nand  $-1 \le \cos x \le 1$  for all x. Likewise,  $\cos(2n + 1)\pi = -1$  is its minimum value, where n is any integer.

**EXAMPLE 2** If  $f(x) = x^2$ , then  $f(x) \ge f(0)$  because  $x^2 \ge 0$  for all x. Therefore, f(0) = 0 is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola  $y = x^2$  (see Figure 2). However, there is no highest point on the parabola and so this function has no maximum value.

**EXAMPLE 3** From the graph of the function  $f(x) = x^3$ , shown in Figure 3, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either.

**EXAMPLE 4** The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \le x \le 4$$

is shown in Figure 4. You can see that f(1) = 5 is a local maximum, whereas the absolute maximum is f(-1) = 37. [This absolute maximum is not a local maximum because it occurs at an endpoint.] Also, f(0) = 0 is a local minimum and f(3) = -27 is both a local and an absolute minimum.

(1, 5)

(-1, 37)

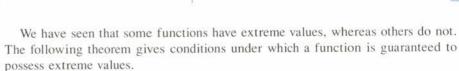
-1

 $y = 3x^4 - 16x^3 + 18x^2$ 

3

4

(3, -27)



**B** The Extreme Value Theorem If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].

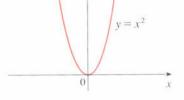
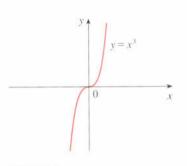


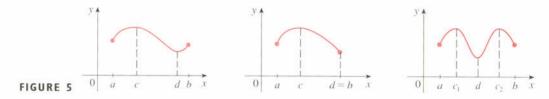
FIGURE 2 Minimum value 0, no maximum



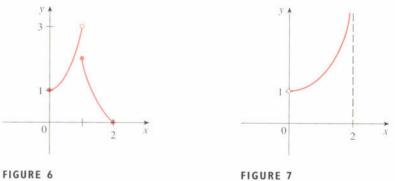


#### FIGURE 4

The Extreme Value Theorem is illustrated in Figure 5. Note that an extreme value can be taken on more than once. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.



Figures 6 and 7 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.



This function has minimum value f(2) = 0, but no maximum value.



The function f whose graph is shown in Figure 6 is defined on the closed interval [0, 2] but has no maximum value. (Notice that the range of f is [0, 3). The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the Extreme Value Theorem because f is not continuous. [Nonetheless, a discontinuous function *could* have maximum and minimum values. See Exercise 13(b).]

The function g shown in Figure 7 is continuous on the open interval (0, 2) but has neither a maximum nor a minimum value. [The range of g is  $(1, \infty)$ . The function takes on arbitrarily large values.] This does not contradict the Extreme Value Theorem because the interval (0, 2) is not closed.

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

Figure 8 shows the graph of a function f with a local maximum at c and a local minimum at d. It appears that at the maximum and minimum points the tangent line is horizontal and therefore has slope 0. We know that the derivative is the slope of the tangent line, so it appears that f'(c) = 0 and f'(d) = 0. The following theorem says that this is always true for differentiable functions.

**4 Fermat's Theorem** If f has a local maximum or minimum at c, and if f'(c) exists, then f'(c) = 0.

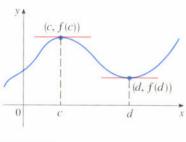
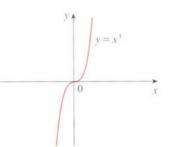


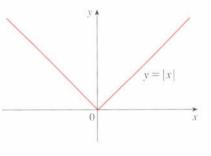
FIGURE 8

Our intuition suggests that Fermat's Theorem is true. A rigorous proof, using the definition of a derivative, is given in Appendix E.

Although Fermat's Theorem is very useful, we have to guard against reading too much into it. If  $f(x) = x^3$ , then  $f'(x) = 3x^2$ , so f'(0) = 0. But f has no maximum or minimum at 0, as you can see from its graph in Figure 9. The fact that f'(0) = 0 simply means that the curve  $y = x^3$  has a horizontal tangent at (0, 0). Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Thus when f'(c) = 0, f doesn't necessarily have a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.)





Fermat's Theorem is named after Pierre Fermat (1601–1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for finding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

# FIGURE 9

If  $f(x) = x^3$ , then f'(0) = 0 but f has no minimum or maximum.

**FIGURE 10** If f(x) = |x|, then f(0) = 0 is a minimum value, but f'(0) does not exist.

We should bear in mind that there may be an extreme value where f'(c) does not exist. For instance, the function f(x) = |x| has its (local and absolute) minimum value at 0 (see Figure 10), but that value cannot be found by setting f'(x) = 0 because, as was shown in Example 5 in Section 2.8, f'(0) does not exist.

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where f'(c) = 0 or where f'(c) does not exist. Such numbers are given a special name.

**Definition** A critical number of a function f is a number c in the domain of f such that either f'(c) = 0 or f'(c) does not exist.

Figure II shows a graph of the function f in Example 5. It supports our answer because there is a horizontal tangent when x = 1.5 and a vertical tangent when x = 0.

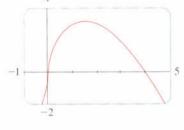


FIGURE II

**EXAMPLE 5** Find the critical numbers of 
$$f(x) = x^{3/5}(4 - x)$$
.

**SOLUTION** The Product Rule gives

$$f'(x) = \frac{3}{5}x^{-2/5}(4-x) + x^{3/5}(-1)$$
$$= \frac{3(4-x) - 5x}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing  $f(x) = 4x^{3/5} - x^{8/5}$ .] Therefore, f'(x) = 0 if 12 - 8x = 0, that is,  $x = \frac{3}{2}$ , and f'(x) does not exist when x = 0. Thus, the critical numbers are  $\frac{3}{2}$  and 0. In terms of critical numbers, Fermat's Theorem can be rephrased as follows (compare Definition 5 with Theorem 4):

If f has a local extremum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is a local extremum [in which case it occurs at a critical number by (6)] or it occurs at an endpoint of the interval. Thus, the following three-step procedure always works.

**The Closed Interval Method** To find the *absolute* maximum and minimum values of a continuous function f on a closed interval [a, b]:

- **I.** Find the values of f at the critical numbers of f in (a, b).
- **2.** Find the values of f at the endpoints of the interval.
- **3.** The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

# EXAMPLE 6

(a) Use a graphing device to estimate the absolute minimum and maximum values of the function  $f(x) = x - 2 \sin x$ ,  $0 \le x \le 2\pi$ .

(b) Use calculus to find the exact minimum and maximum values.

# SOLUTION

(a) Figure 12 shows a graph of f in the viewing rectangle  $[0, 2\pi]$  by [-1, 8]. By moving the cursor close to the maximum point, we see that the y-coordinates do not change very much in the vicinity of the maximum. The absolute maximum value is about 6.97 and it occurs when  $x \approx 5.2$ . Similarly, by moving the cursor close to the minimum point, we see that the absolute minimum value is about -0.68 and it occurs when  $x \approx 1.0$ . It is possible to get more accurate estimates by zooming in toward the maximum and minimum points, but instead let's use calculus.

(b) The function  $f(x) = x - 2 \sin x$  is continuous on  $[0, 2\pi]$ . Since  $f'(x) = 1 - 2 \cos x$ , we have f'(x) = 0 when  $\cos x = \frac{1}{2}$  and this occurs when  $x = \pi/3$  or  $5\pi/3$ . The values of *f* at these critical points are

$$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \approx -0.684853$$

 $f(5\pi/3) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$ 

and

The values of f at the endpoints are

$$f(0) = 0$$
 and  $f(2\pi) = 2\pi \approx 6.28$ 

Comparing these four numbers and using the Closed Interval Method, we see that the absolute minimum value is  $f(\pi/3) = \pi/3 - \sqrt{3}$  and the absolute maximum value is  $f(5\pi/3) = 5\pi/3 + \sqrt{3}$ . The values from part (a) serve as a check on our work.

We can estimate maximum and minimum values very easily using a graphing calculator or a computer with graphing software. But, as Example 6 shows, calculus is needed to find the *exact* values.

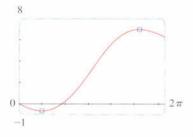
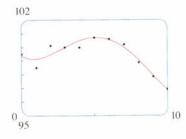




Figure 13 shows the graph of the foodprice index function *I* of Example 7. This model is based on the data points shown.





**EXAMPLE 7** A model for the food-price index (the price of a representative "basket" of foods) between 1984 and 1994 is given by the function

$$I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$$

where t is measured in years since midyear 1984, so  $0 \le t \le 10$ , and I(t) is measured in 1987 dollars and scaled such that I(3) = 100. Estimate the times when food was cheapest and most expensive during the period 1984–1994.

**SOLUTION** We apply the Closed Interval Method to the continuous function I on [0, 10]. Its derivative is

$$I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$$

Since *I'* exists for all *t*, the only critical numbers of *I* occur when I'(t) = 0. We use a root-finder on a computer algebra system (or a graphing device) to find that I'(t) = 0 when  $t \approx -29.7186$ , 0.8231, 5.1309, or 11.0459, but only the second and third roots lie in the interval [0, 10]. The values of *I* at these critical numbers are

 $I(0.8231) \approx 99.09$  and  $I(5.1309) \approx 100.67$ 

The values of I at the endpoints of the interval are

$$I(0) = 99.33$$
  $I(10) \approx 96.86$ 

Comparing these four numbers, we see that food was most expensive at  $t \approx 5.1309$  (corresponding roughly to August, 1989) and cheapest at t = 10 (midyear 1994).

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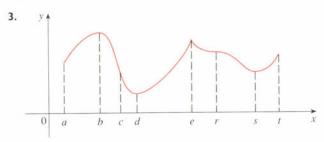


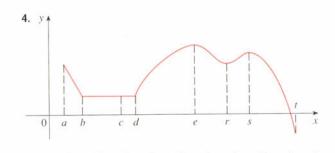
Exercises

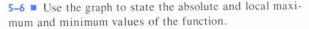
1. Explain the difference between an absolute minimum and a local minimum.

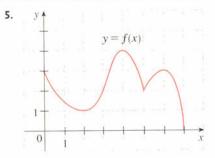
- Suppose f is a continuous function defined on a closed interval [a, b].
  - (a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
  - (b) What steps would you take to find those maximum and minimum values?

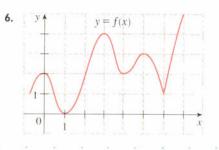
**3-4** For each of the numbers a, b, c, d, e, r, s, and t, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.











**7–10** Sketch the graph of a function f that is continuous on [0, 3] and has the given properties.

- **7.** Absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2
- 8. Absolute maximum at 1, absolute minimum at 2
- **9.** 2 is a critical number, but *f* has no local maximum or minimum
- Absolute minimum at 0, absolute maximum at 2, local maxima at 1 and 2, local minimum at 1.5
- **11.** (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
  - (b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
  - (c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.
- 12. (a) Sketch the graph of a function on [-1, 2] that has an absolute maximum but no local maximum.
  - (b) Sketch the graph of a function on [-1, 2] that has a local maximum but no absolute maximum.
- **13.** (a) Sketch the graph of a function on [-1, 2] that has an absolute maximum but no absolute minimum.
  - (b) Sketch the graph of a function on [-1, 2] that is discontinuous but has both an absolute maximum and an absolute minimum.
- 14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
  - (b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

**15–24** Find the absolute and local maximum and minimum values of f. Begin by sketching its graph by hand. (Use the graphs and transformations of Section 1.2.)

**15.** f(x) = 1 + 2x,  $x \ge -1$  **16.**  $f(x) = 1 - x^2$ ,  $0 < x \le 1$ **17.**  $f(x) = 1 - x^2$ ,  $-2 \le x \le 1$ 

**18.** 
$$f(t) = 1/t, \quad 0 < t < 1$$

**19.**  $f(\theta) = \sin \theta$ ,  $-2\pi \le \theta \le 2\pi$  **20.**  $f(\theta) = \tan \theta$ ,  $-\pi/4 \le \theta < \pi/2$  **21.**  $f(x) = x^5$  **22.**  $f(x) = 2 - x^4$  **23.**  $f(x) = 1 - e^{-x}$ ,  $x \ge 0$  **24.**  $f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 0 \\ 2 - x^2 & \text{if } 0 \le x \le 1 \end{cases}$  **25.**  $f(x) = 4x^3 - 9x^2 - 12x + 3$  **26.**  $f(t) = t^3 + 6t^2 + 3t - 1$  **27.**  $s(t) = t^4 + 4t^3 + 2t^2$  **28.** g(x) = |x + 1| **29.**  $f(x) = \frac{r}{2}$ **20.**  $f(x) = \frac{x + 1}{2}$ 

**30.** 
$$f(z) = \frac{1}{z^2 + z + 1}$$
  
**31.**  $F(x) = x^{4/5}(x - 4)^2$   
**32.**  $G(x) = \sqrt[3]{x^2 - x}$   
**33.**  $f(\theta) = \sin^2(2\theta)$   
**34.**  $g(\theta) = \theta + \sin\theta$   
**35.**  $f(x) = x \ln x$   
**36.**  $f(x) = xe^{2x}$ 

**37–46** Find the absolute maximum and absolute minimum values of f on the given interval.

**37.**  $f(x) = x^2 - 2x + 2$ , [0,3] **38.**  $f(x) = x^3 - 12x + 1$ , [-3,5] **39.**  $f(x) = 3x^5 - 5x^3 - 1$ , [-2,2] **40.**  $f(x) = \sqrt{9 - x^2}$ , [-1,2] **41.**  $f(x) = x^2 + 2/x$ ,  $\left[\frac{1}{2}, 2\right]$  **42.**  $f(x) = \frac{x}{x + 1}$ , [1,2] **43.**  $f(x) = \sin x + \cos x$ , [0,  $\pi/3$ ] **44.**  $f(x) = x - 2\cos x$ , [ $-\pi, \pi$ ] **45.**  $f(x) = xe^{-x}$ , [0,2] **46.**  $f(x) = (\ln x)/x$ , [1,3]

# 47-50 🔳

- (a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.
- (b) Use calculus to find the exact maximum and minimum values.

0

**47.** 
$$f(x) = x^3 - 8x + 1, -3 \le x \le 3$$

**48.** 
$$f(x) = e^{x^3 - x}, \quad -1 \le x \le$$

**49.** 
$$f(x) = x\sqrt{x - x^2}$$

**50.**  $f(x) = (\cos x)/(2 + \sin x), \ 0 \le x \le 2\pi$ 

**51.** Between 0 °C and 30 °C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given

 $V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$ 

Find the temperature at which water has its maximum density.

**52.** An object with weight *W* is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle  $\theta$  with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where  $\mu$  is a positive constant called the *coefficient of* friction and where  $0 \le \theta \le \pi/2$ . Show that F is minimized when  $\tan \theta = \mu$ .

**53.** The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at t = 0 s until the solid rocket boosters were jettisoned at t = 126 s, is given by

 $v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$ 

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

54. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The following table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Livem	Time (s)	Velocity (f) s
Lameb	11	(1)
Begin roll maneuver	10.	185
End roll maneuver	15	310
Firror de to 8903		447
Throttle to 675	3.2	742
	50	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

- (a) Use the methods of Section 1.7 to find the cubic polynomial that best models the velocity of the shuttle for the time interval t ∈ [0, 125]. Then graph this polynomial.
- (b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 s.
- **55.** When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \frac{1}{2}r_0 \le r \le r_0$$

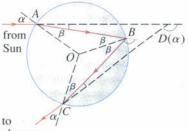
where k is a constant and  $r_0$  is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than  $r_0/2$  is prevented (otherwise the person would suffocate).

- (a) Determine the value of r in the interval  $[r_0/2, r_0]$  at which v has an absolute maximum. How does this compare with experimental evidence?
- (b) What is the absolute maximum value of *v* on the interval?
- (c) Sketch the graph of v on the interval  $[0, r_0]$ .
- **56.** A cubic function is a polynomial of degree 3; that is, it has the form  $f(x) = ax^3 + bx^2 + cx + d$ , where  $a \neq 0$ .
  - (a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.
  - (b) How many local extreme values can a cubic function have?



# The Calculus of Rainbows

Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.



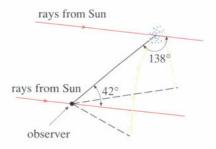


1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that  $\sin \alpha = k \sin \beta$ , where  $\alpha$  is the angle of incidence,  $\beta$  is the angle of refraction, and  $k \approx \frac{4}{3}$  is the index of refraction for water. At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of refraction.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The angle of deviation  $D(\alpha)$  is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

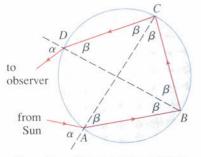
$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$

Show that the minimum value of the deviation is  $D(\alpha) \approx 138^{\circ}$  and occurs when  $\alpha \approx 59.4^{\circ}$ .

The significance of the minimum deviation is that when  $\alpha \approx 59.4^{\circ}$  we have  $D'(\alpha) \approx 0$ , so  $\Delta D/\Delta \alpha \approx 0$ . This means that many rays with  $\alpha \approx 59.4^{\circ}$  become deviated by approximately the same amount. It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The figure shows that the angle of elevation from the observer up to the highest point on the rainbow is  $180^{\circ} - 138^{\circ} = 42^{\circ}$ . (This angle is called the *rainbow angle.*)



2. Problem 1 explains the location of the primary rainbow but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called *dispersion*.) For red light the refractive index is  $k \approx 1.3318$  whereas for violet light it is  $k \approx 1.3435$ . By repeating the calculation of Problem 1 for these values of k, show that the rainbow angle is about 42.3° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.



**3.** Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D (see the figure). This time the deviation angle  $D(\alpha)$  is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D(\alpha) = 2\alpha - 6\beta + 2\pi$$

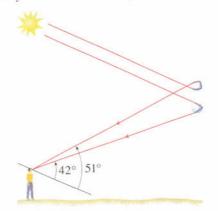
and  $D(\alpha)$  has a minimum value when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{8}}$$

C

Formation of the secondary rainbow

Taking  $k = \frac{4}{3}$ , show that the minimum deviation is about 129° and so the rainbow angle for the secondary rainbow is about 51°, as shown in the figure.



**4.** Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.



# Derivatives and the Shapes of Curves

In Section 2.10 we discussed how the signs of the first and second derivatives f'(x) and f''(x) influence the shape of the graph of *f*. Here we revisit those facts, giving an indication of why they are true and using them, together with the differentation formulas of Chapter 3, to explain the shapes of graphs.

We start with a fact, known as the Mean Value Theorem, that will be useful not only for present purposes but also for explaining why some of the other basic results of calculus are true.

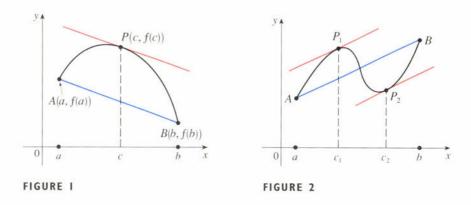
**The Mean Value Theorem** If f is a differentiable function on the interval [a, b], then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

f(b) - f(a) = f'(c)(b - a)

We can see that this theorem is reasonable by interpreting it geometrically. Figures 1 and 2 show the points A(a, f(a)) and B(b, f(b)) on the graphs of two differentiable functions.



The slope of the secant line AB is

$$n_{AB} = \frac{f(b) - f(a)}{b - a}$$

1

which is the same expression as on the right side of Equation 1. Since f'(c) is the slope of the tangent line at the point (c, f(c)), the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point P(c, f(c)) on the graph where the slope of the tangent line is the same as the slope of the secant line *AB*. In other words, there is a point *P* where the tangent line is parallel to the secant line *AB*. It seems clear that there is one such point *P* in Figure 1 and two such points  $P_1$  and  $P_2$  in Figure 2. Because our intuition tells us that the Mean Value Theorem is true, we take it as the starting point for the development of the main facts of calculus. (When calculus is developed from first principles, however, the Mean Value Theorem is proved as a consequence of the axioms that define the real number system.)

**EXAMPLE I** If an object moves in a straight line with position function s = f(t), then the average velocity between t = a and t = b is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at t = c is f'(c). Thus, the Mean Value Theorem tells us that at some time t = c between a and b the instantaneous velocity f'(c) is equal to that average velocity. For instance, if a car traveled 180 km in 2 h, then the speed-ometer must have read 90 km/h at least once.

The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative. Our immediate use of this principle is to prove the basic facts concerning increasing and decreasing functions. (See Exercises 43 and 44 for another use.)

The Mean Value Theorem was first formulated by Joseph-Louis Lagrange (1736-1813), born in Italy of a French father and an Italian mother. He was a child prodigy and became a professor in Turin at the tender age of 19. Lagrange made great contributions to number theory, theory of functions, theory of equations, and analytical and celestial mechanics. In particular, he applied calculus to the analysis of the stability of the solar system. At the invitation of Frederick the Great, he succeeded Euler at the Berlin Academy and, when Frederick died, Lagrange accepted King Louis XVI's invitation to Paris, where he was given apartments in the Louvre. He was a kind and quiet man, though, living only for science.

# Increasing and Decreasing Functions

In Section 1.1 we defined increasing functions and decreasing functions and in Section 2.10 we observed from graphs that a function with a positive derivative is increasing. We now deduce this fact from the Mean Value Theorem.

Let's abbreviate the name of this test to the  $\ensuremath{\mathsf{I/D}}$  Test.

#### Increasing/Decreasing Test

(a) If f'(x) > 0 on an interval, then f is increasing on that interval.
(b) If f'(x) < 0 on an interval, then f is decreasing on that interval.</li>

## Proof

(a) Let  $x_1$  and  $x_2$  be any two numbers in the interval with  $x_1 < x_2$ . Because we are given that f'(x) > 0, we know that f is differentiable on  $[x_1, x_2]$ . So, by the Mean Value Theorem there is a number c between  $x_1$  and  $x_2$  such that

# 3

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now f'(c) > 0 by assumption and  $x_2 - x_1 > 0$  because  $x_1 < x_2$ . Thus, the right side of Equation 3 is positive, and so

$$f(x_2) - f(x_1) > 0$$
 or  $f(x_1) < f(x_2)$ 

This shows that *f* is increasing.

Part (b) is proved similarly.

**EXAMPLE 2** Find where the function  $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$  is increasing and where it is decreasing.

SOLUTION 
$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

To use the I/D Test we have to know where f'(x) > 0 and where f'(x) < 0. This depends on the signs of the three factors of f'(x), namely, 12x, x - 2, and x + 1. We divide the real line into intervals whose endpoints are the critical numbers -1, 0, and 2 and arrange our work in a chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test. For instance, f'(x) < 0 for 0 < x < 2, so f is decreasing on (0, 2). (It would also be true to say that f is decreasing on the closed interval [0, 2].)

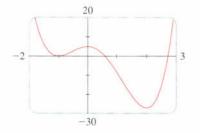
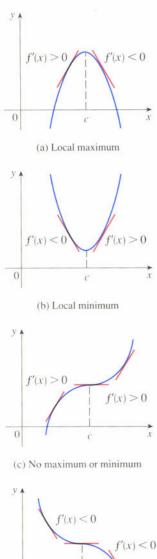




FIGURE 3

The graph of f shown in Figure 3 confirms the information in the chart.

٠



(d) No maximum or minimum

FIGURE 4

Recall from Section 4.2 that if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem), but not every critical number gives rise to a maximum or a minimum. We therefore need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

You can see from Figure 3 that f(0) = 5 is a local maximum value of f because f increases on (-1, 0) and decreases on (0, 2). Or, in terms of derivatives, f'(x) > 0 for -1 < x < 0 and f'(x) < 0 for 0 < x < 2. In other words, the sign of f'(x) changes from positive to negative at 0. This observation is the basis of the following test.

**The First Derivative Test** Suppose that *c* is a critical number of a continuous function *f*.

- (a) If f' changes from positive to negative at c, then f has a local maximum at c.
- (b) If f' changes from negative to positive at c, then f has a local minimum at c.
- (c) If f' does not change sign at c, (that is, f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of f'(x) changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 4.

**EXAMPLE 3** Find the local minimum and maximum values of the function f in Example 2.

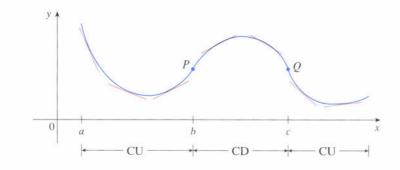
**SOLUTION** From the chart in the solution to Example 2 we see that f'(x) changes from negative to positive at -1, so f(-1) = 0 is a local minimum value by the First Derivative Test. Similarly, f' changes from negative to positive at 2, so f(2) = -27 is also a local minimum value. As previously noted, f(0) = 5 is a local maximum value because f'(x) changes from positive to negative at 0.

# Concavity

Let us recall the definition of concavity from Section 2.10.

A function (or its graph) is called **concave upward** on an interval I if f' is an increasing function on I. It is called **concave downward** on I if f' is decreasing on I.

Notice in Figure 5 that the slopes of the tangent lines increase from left to right on the interval (a, b), so f' is increasing and f is concave upward (abbreviated CU) on (a, b). [It can be proved that this is equivalent to saying that the graph of f lies above all of its tangent lines on (a, b).] Similarly, the slopes of the tangent lines decrease from left to right on (b, c), so f' is decreasing and f is concave downward (CD) on (b, c).





A point where a curve changes its direction of concavity is called an **inflection point**. The curve in Figure 5 changes from concave upward to concave downward at P and from concave downward to concave upward at Q, so both P and Q are inflection points.

Because f'' = (f')', we know that if f''(x) is positive, then f' is an increasing function and so f is concave upward. Similarly, if f''(x) is negative, then f' is decreasing and f is concave downward. Thus, we have the following test for concavity.

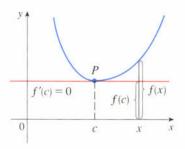
#### **Concavity Test**

- (a) If f''(x) > 0 for all x in I, then the graph of f is concave upward on I.
- (b) If f''(x) < 0 for all x in I, then the graph of f is concave downward on I.

In view of the Concavity Test, there is a point of inflection at any point where the second derivative changes sign. A consequence of the Concavity Test is the following test for maximum and minimum values.

The Second Derivative Test	Suppose $f''$ is continuous near $c$ .
(a) If $f'(c) = 0$ and $f''(c) > 0$	> 0, then $f$ has a local minimum at $c$ .
(b) If $f'(c) = 0$ and $f''(c) < 0$	< 0, then f has a local maximum at c.

For instance, part (a) is true because f''(x) > 0 near c and so f is concave upward near c. This means that the graph of f lies *above* its horizontal tangent at c and so f has a local minimum at c. (See Figure 6).



**FIGURE 6** f''(c) > 0, concave upward

**EXAMPLE 4** Discuss the curve  $y = x^4 - 4x^3$  with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

**SOLUTION** If  $f(x) = x^4 - 4x^3$ , then

$$f'(x) = 4x^{3} - 12x^{2} = 4x^{2}(x - 3)$$
$$f''(x) = 12x^{2} - 24x = 12x(x - 2)$$

To find the critical numbers we set f'(x) = 0 and obtain x = 0 and x = 3. To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since f'(3) = 0 and f''(3) > 0, f(3) = -27 is a local minimum. Since f''(0) = 0, the Second Derivative Test gives no information about the critical number 0. But since f'(x) < 0 for x < 0 and also for 0 < x < 3, the First Derivative Test tells us that *f* does not have a local maximum or minimum at 0. [In fact, the expression for f'(x) shows that *f* decreases to the left of 3 and increases to the right of 3.]

Since f''(x) = 0 when x = 0 or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	f''(x) = 12x(x - 2)	CON-0.15
(-z, 0)		apsoad
(0,2)		dess normal.
$(2, \times)$		apwaaih

The point (0,0) is an inflection point since the curve changes from concave upward to concave downward there. Also (2, -16) is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 7.

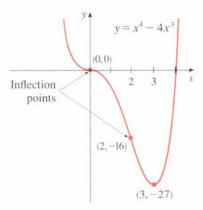
**NOTE** • The Second Derivative Test is inconclusive when f''(c) = 0. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 4). This test also fails when f''(c) does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

**EXAMPLE 5** Sketch the graph of the function  $f(x) = x^{2/3}(6 - x)^{1/3}$ .

**SOLUTION** Calculation of the first two derivatives gives

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since f'(x) = 0 when x = 4 and f'(x) does not exist when x = 0 or x = 6, the critical numbers are 0, 4, and 6.





Use the differentiation rules to check these calculations.

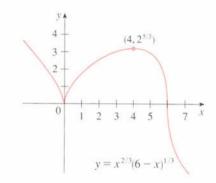
Interval	4 = x	$x^{1/3}$	$(6 - x)^{2/3}$	f'(x)	ſ
x < 0	+		+	-	decreasing on $(-\infty, 0)$
0 < x < 4	+	+	+	+	increasing on (0,4)
$4 \le x \le 6$	<u>144</u> )	+	+	-	decreasing on (4,6)
$x \ge 6$		+	+	-	decreasing on (6,∞)

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, f(0) = 0 is a local minimum. Since f' changes from positive to negative at 4,  $f(4) = 2^{5/3}$  is a local maximum. The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist there.)

Looking at the expression for f''(x) and noting that  $x^{4/3} \ge 0$  for all x, we have f''(x) < 0 for x < 0 and for 0 < x < 6 and f''(x) > 0 for x > 6. So f is concave downward on  $(-\infty, 0)$  and (0, 6) and concave upward on  $(6, \infty)$ , and the only inflection point is (6, 0). The graph is sketched in Figure 8. Note that the curve has vertical tangents at (0, 0) and (6, 0) because  $|f'(x)| \to \infty$  as  $x \to 0$  and as  $x \to 6$ .

Try reproducing the graph in Figure 8 with a graphing calculator or computer. Some machines produce the complete graph, some produce only the portion to the right of the *y*-axis, and some produce only the portion between x = 0 and x = 6. For an explanation and cure, see Example 7 in Section 1.3. An equivalent expression that gives the correct graph is

$$y = (x^2)^{1/3} \cdot \frac{6-x}{|6-x|} |6-x|^{1/3}$$



#### FIGURE 8

**EXAMPLE 6** Use the first and second derivatives of  $f(x) = e^{1/x}$ , together with asymptotes, to sketch its graph.

**SOLUTION** Notice that the domain of f is  $\{x \mid x \neq 0\}$ , so we check for vertical asymptotes by computing the left and right limits as  $x \to 0$ . As  $x \to 0^+$ , we know that  $t = 1/x \to \infty$ , so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that x = 0 is a vertical asymptote. As  $x \to 0^-$ , we have  $t = 1/x \to -\infty$ , so

$$\lim_{x \to 0^{-}} e^{1/x} = \lim_{t \to -\infty} e^{t} = 0$$

As  $x \to \pm \infty$ , we have  $1/x \to 0$  and so

$$\lim_{x \to \pm \infty} e^{1/x} = e^0 = 1$$

This shows that y = 1 is a horizontal asymptote.

Now let's compute the derivative. The Chain Rule gives

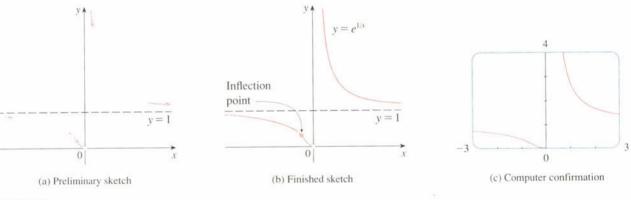
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since  $e^{1/x} > 0$  and  $x^2 > 0$  for all  $x \neq 0$ , we have f'(x) < 0 for all  $x \neq 0$ . Thus, f is decreasing on  $(-\infty, 0)$  and on  $(0, \infty)$ . There is no critical number, so the function has no maximum or minimum. The second derivative is

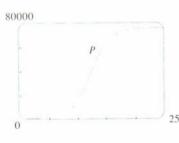
$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x+1)}{x^4}$$

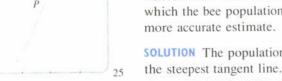
Since  $e^{1/x} > 0$  and  $x^4 > 0$ , we have f''(x) > 0 when  $x > -\frac{1}{2}$  ( $x \neq 0$ ) and f''(x) < 0 when  $x < -\frac{1}{2}$ . So the curve is concave downward on  $(-\infty, -\frac{1}{2})$  and concave upward on  $(-\frac{1}{2}, 0)$  and on  $(0, \infty)$ . The inflection point is  $(-\frac{1}{2}, e^{-2})$ .

To sketch the graph of f we first draw the horizontal asymptote y = 1 (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 9(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both  $(-\infty, 0)$  and  $(0, \infty)$ . Notice that we have indicated that  $f(x) \rightarrow 0$  as  $x \rightarrow 0^-$  even though f(0) does not exist. In Figure 9(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 9(c) we check our work with a graphing device.









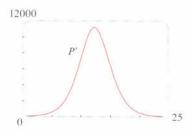


**EXAMPLE 7** • A population of honeybees raised in an apiary started with 50 bees at time t = 0 and was modeled by the function

$$P(t) = \frac{75,200}{1 + 1503e^{-0.5932t}}$$

where *t* is the time in weeks,  $0 \le t \le 25$ . Use a graph to estimate the time at which the bee population was growing fastest. Then use derivatives to give a more accurate estimate.

**SOLUTION** The population grows fastest when the population curve y = P(t) has the steepest tangent line. From the graph of *P* in Figure 10, we estimate that the steepest tangent occurs when  $t \approx 12$ , so the bee population was growing most rapidly after about 12 weeks.





For a better estimate we calculate the derivative P'(t), which is the rate of increase of the bee population:

$$P'(t) = \frac{67,046,785.92e^{-0.5932t}}{(1+1503e^{-0.5932t})^2}$$

We graph P' in Figure 11 and observe that P' has its maximum value when  $t \approx 12.3$ .

To get a still better estimate we note that f' has its maximum value when f' changes from increasing to decreasing. This happens when f changes from concave upward to concave downward, so we ask a CAS to compute the second derivative:

$$P''(t) \approx \frac{119555093144e^{-1.1864t}}{(1+1503e^{-0.5932t})^3} - \frac{39772153e^{-0.5932t}}{(1+1503e^{-0.5932t})^2}$$

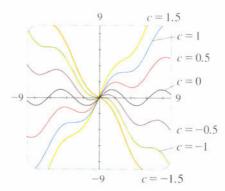
We could plot this function to see where it changes from positive to negative, but instead let's have the CAS solve the equation P''(t) = 0. It gives the answer  $t \approx 12.3318$ .

Our final example is concerned with *families* of functions. This means that the functions in the family are related to each other by a formula that contains one or more arbitrary constants. Each value of the constant gives rise to a member of the family and the idea is to see how the graph of the function changes as the constant changes.

**EXAMPLE 8** Investigate the family of functions given by  $f(x) = cx + \sin x$ . What features do the members of this family have in common? How do they differ?

**SOLUTION** The derivative is  $f'(x) = c + \cos x$ . If c > 1, then f'(x) > 0 for all x (since  $\cos x \ge -1$ ), so f is always increasing. If c = 1, then f'(x) = 0 when x is an odd multiple of  $\pi$ , but f just has horizontal tangents there and is still an increasing function. Similarly, if  $c \le -1$ , then f is always decreasing. If -1 < c < 1, then the equation  $c + \cos x = 0$  has infinitely many solutions  $[x = 2n\pi \pm \cos^{-1}(-c)]$  and f has infinitely many minima and maxima.

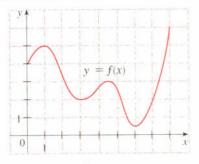
The second derivative is  $f''(x) = -\sin x$ , which is negative when  $0 < x < \pi$ and, in general, when  $2n\pi < x < (2n + 1)\pi$ , where *n* is any integer. Thus, *all* members of the family are concave downward on  $(0, \pi)$ ,  $(2\pi, 3\pi)$ ,... and concave upward on  $(\pi, 2\pi)$ ,  $(3\pi, 4\pi)$ ,.... This is illustrated by several members of the family in Figure 12.



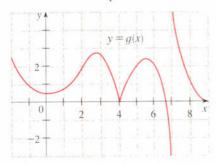


Exercises · · · ·

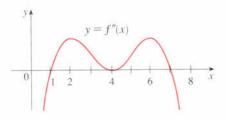
1. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval [0,8].



- 2. From the given graph of g, state
  - (a) the largest open intervals on which g is concave upward,
  - (b) the largest open intervals on which g is concave downward, and
  - (c) the coordinates of the points of inflection.



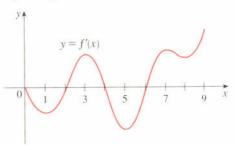
- **3.** (a) How do you determine where a function is increasing or decreasing?
  - (b) How do you determine where a curve is concave upward or concave downward?
  - (c) How do you locate inflection points?
- 4. (a) State the First Derivative Test.
  - (b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?
- The graph of the second derivative f" of a function f is shown. State the x-coordinates of the inflection points of f. Give reasons for your answers.



- **6.** The graph of the first derivative f' of a function f is shown.
  - (a) On what intervals is f increasing? Explain.

and a second second second second second

- (b) At what values of x does f have a local maximum or minimum? Explain.
- (c) On what intervals is *f* concave upward or concave downward? Explain.
- (d) What are the x-coordinates of the inflection points of f? Why?



#### 7-12 🔳

- (a) Find the intervals on which *f* is increasing or decreasing.
- (b) Find the local maximum and minimum values of f.
- (c) Find the intervals of concavity and the inflection points.

7. 
$$f(x) = x^6 + 192x + 17$$
 8.  $f(x) = 2 \sin x + \sin^2 x$ 

 9.  $y = xe^x$ 
 10.  $y = x^2e^x$ 

 11.  $y = (\ln x)/\sqrt{x}$ 
 12.  $y = x \ln x$ 

#### 13-20 ■

- (a) Find the intervals on which *f* is increasing or decreasing.
- (b) Find the local maximum and minimum values of f.
- (c) Find the intervals of concavity and the inflection points.
- (d) Use the information from parts (a), (b), and (c) to sketch the graph of *f*. Check your work with a graphing device.

**13.** 
$$f(x) = 1 - 3x + 5x^2 - x^3$$

14. 
$$f(x) = x^4 - 6x^4$$

**15.**  $f(x) = (x^2 - 1)^3$ 

**16.** 
$$f(x) = x\sqrt{x+1}$$

- 17.  $f(x) = x^{1/3}(x + 3)^{2/3}$
- **18.**  $f(x) = 2x + \cot x$ ,  $0 < x < \pi$

**19.** 
$$f(x) = 2\cos x + \sin^2 x$$

**20.**  $f(x) = \ln(1 + x^2)$ 

#### 21-26

- (a) Find the vertical and horizontal asymptotes.
- (b) Find the intervals of increase or decrease.
- (c) Find the local maximum and minimum values.
- (d) Find the intervals of concavity and the inflection points.
- (e) Use the information from parts (a)-(d) to sketch the graph of *f*.

**21.** 
$$f(x) = \frac{1+x^2}{1-x^2}$$
  
**22.**  $f(x) = \frac{x}{(x-1)^2}$   
**23.**  $f(x) = \sqrt{x^2+1} - x$   
**24.**  $f(x) = x \tan x$ ,  $-\pi/2 < x < \pi/2$   
**25.**  $f(x) = e^{-1/(x+1)}$ 

**26.**  $f(x) = \ln(\tan^2 x)$ 

#### 27-28

- (a) Use a graph of f to give a rough estimate of the intervals of concavity and the coordinates of the points of inflection.
- (b) Use a graph of f'' to give better estimates.

**27.** 
$$f(x) = 3x^5 - 40x^3 + 30x^2$$

**28.**  $f(x) = 2\cos x + \sin 2x$ ,  $0 \le x \le 2\pi$ 

# 29-30 🔳

- (a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values,
- (b) Estimate the value of *x* at which *f* increases most rapidly. Then find the exact value.

**29.** 
$$f(x) = \frac{x+1}{\sqrt{x^2+1}}$$
 **30.**  $f(x) = x^2 e^{-x}$ 

**31.** For the period from 1980 to 1994, the percentage of households in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{75}{1 + 74e^{-0.6t}}$$

where the time t is measured in years since midyear 1980, so  $0 \le t \le 14$ . Use a graph to estimate the time at which the number of VCRs was increasing most rapidly. Then use derivatives to give a more accurate estimate.

**32.** The family of bell-shaped curves

$$y = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant  $\mu$  is called the *mean* and the positive constant  $\sigma$  is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor  $1/(\sigma \sqrt{2\pi})$  and let's analyze the special case where  $\mu = 0$ . So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

- (a) Find the asymptote, maximum value, and inflection points of *f*.
- (b) What role does  $\sigma$  play in the shape of the curve?
- (c) Illustrate by graphing four members of this family on the same screen.

**CAS** 33–34 Estimate the intervals of concavity to one decimal place by using a computer algebra system to compute and graph *f*".

**33.** 
$$f(x) = \frac{x^3 - 10x + 5}{\sqrt{x^2 + 4}}$$
  
**34.** 
$$f(x) = \frac{(x+1)^3(x^2 + 5)}{(x^3 + 1)(x^2 + 4)}$$

- **35.** Find a cubic function  $f(x) = ax^3 + bx^2 + cx + d$  that has a local maximum value of 3 at -2 and a local minimum value of 0 at 1.
- **36.** For what values of the numbers *a* and *b* does the function

$$f(x) = axe^{bx}$$

have the maximum value f(2) = 1?

**37–40** ■ Assume that all of the functions are twice differentiable.

- **37.** If f and g are concave upward on I, show that f + g is concave upward on I.
- **38.** If *f* is positive and concave upward on *I*, show that the function  $g(x) = [f(x)]^2$  is concave upward on *I*.
- **39.** If *f* and *g* are positive increasing concave upward functions on *I*, show that the product function *fg* is concave upward on *I*.
- 40. Suppose f and g are both concave upward on (-∞,∞). Under what condition on f will the composite function h(x) = f(g(x)) be concave upward?
- **41.** Show that  $\tan x > x$  for  $0 < x < \pi/2$ . [*Hint:* Show that  $f(x) = \tan x x$  is increasing on  $(0, \pi/2)$ .]
- **42.** (a) Show that  $e^x \ge 1 + x$  for  $x \ge 0$ .
  - (b) Deduce that  $e^x \ge 1 + x + \frac{1}{2}x^2$  for  $x \ge 0$ .
  - (c) Use mathematical induction to prove that for  $x \ge 0$ and any positive integer *n*,

$$e^x \ge 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

**43.** Suppose that f(0) = -3 and  $f'(x) \le 5$  for all values of x. The inequality gives a restriction on the rate of growth of f, which then imposes a restriction on the

possible values of f. Use the Mean Value Theorem to determine how large f(4) can possibly be.

- **44.** Suppose that  $1 \le f'(x) \le 4$  for all *x* in [2, 5]. Show that  $3 \le f(5) f(2) \le 12$ .
- **45.** Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same velocity. [*Hint:* Consider f(t) = g(t) h(t) where g and h are the position functions of the two runners.]
- **46.** At 2:00 P.M. a car's speedometer reads 30 mi/h. At 2:10 P.M. it reads 50 mi/h. Show that at some time

between 2:00 and 2:10 the acceleration is exactly  $120 \text{ mi/h}^2$ .

- **47.** Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three *x*-intercepts  $x_1$ ,  $x_2$ , and  $x_3$ , show that the *x*-coordinate of the inflection point is  $(x_1 + x_2 + x_3)/3$ .
- 48. For what values of c does the polynomial P(x) = x<sup>4</sup> + cx<sup>3</sup> + x<sup>2</sup> have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

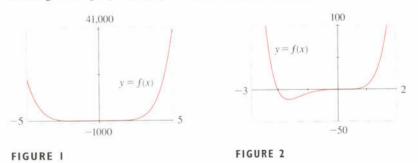


# Graphing with Calculus and Calculators

If you have not already read Section 1.3, you should do so now. In particular, it explains how to avoid some of the pitfalls of using graphing devices by choosing appropriate viewing rectangles. When we graph with technology, the theme is the *interaction* between calculus and graphing devices. We start with a graph produced by a graphing calculator or computer and then we refine it. We use calculus to make sure that we reveal all the important aspects of the curve.

**EXAMPLE 1** Graph the polynomial  $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$ . Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

**SOLUTION** If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed. Figure 1 shows the plot from one such device if we specify that  $-5 \le x \le 5$ . Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for  $y = 2x^6$ , it is obviously hiding some finer detail. So we change to the viewing rectangle [-3, 2] by [-50, 100] shown in Figure 2.

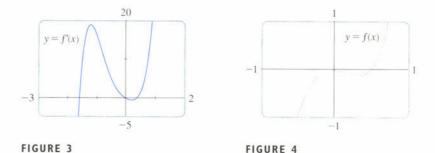


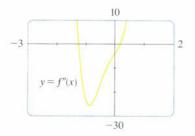
From this graph it appears that there is an absolute minimum value of about -15.33 when  $x \approx -1.62$  (by using the cursor) and *f* is decreasing on  $(-\infty, -1.62)$  and increasing on  $(-1.62, \infty)$ . Also there appears to be a horizontal tangent at the origin and inflection points when x = 0 and when *x* is somewhere between -2 and -1.

Now let's try to confirm these impressions using calculus. We differentiate and get

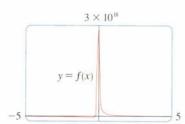
$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x \qquad f''(x) = 60x^4 + 60x^3 + 18x - 4$$

When we graph f' in Figure 3 we see that f'(x) changes from negative to positive when  $x \approx -1.62$ ; this confirms (by the First Derivative Test) the minimum value that we found earlier. But, perhaps to our surprise, we also notice that f'(x) changes from positive to negative when x = 0 and from negative to positive when  $x \approx 0.35$ . This means that f has a local maximum at 0 and a local minimum when  $x \approx 0.35$ , but these were hidden in Figure 2. Indeed, if we now zoom in toward the origin in Figure 4, we see what we missed before: a local maximum value of 0 when x = 0 and a local minimum value of about -0.1 when  $x \approx 0.35$ .

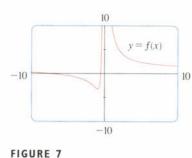












What about concavity and inflection points? From Figures 2 and 4 there appear to be inflection points when x is a little to the left of -1 and when x is a little to the right of 0. But it is difficult to determine inflection points from the graph of f, so we graph the second derivative f" in Figure 5. We see that f" changes from positive to negative when  $x \approx -1.23$  and from negative to positive when  $x \approx 0.19$ . So, correct to two decimal places, f is concave upward on  $(-\infty, -1.23)$  and  $(0.19, \infty)$  and concave downward on (-1.23, 0.19). The inflection points are (-1.23, -10.18) and (0.19, -0.05).

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture.

**EXAMPLE 2** Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

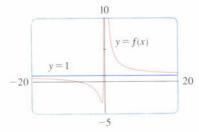
in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

**SOLUTION** Figure 6, produced by a computer with automatic scaling, is a disaster. Some graphing calculators use [-10, 10] by [-10, 10] as the default viewing rectangle, so let's try it. We get the graph shown in Figure 7; it's a major improvement.

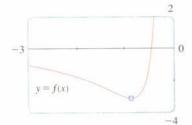
The y-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x \to 0} \frac{x^2 + 7x + 3}{x^2} = \infty$$

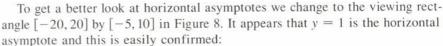
Figure 7 also allows us to estimate the *x*-intercepts: about -0.5 and -6.5. The exact values are obtained by using the quadratic formula to solve the equation  $x^2 + 7x + 3 = 0$ ; we get  $x = (-7 \pm \sqrt{37})/2$ .











$$\lim_{x \to \pm \infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x \to \pm \infty} \left( 1 + \frac{7}{x} + \frac{3}{x^2} \right) = 1$$

To estimate the minimum value we zoom in to the viewing rectangle [-3, 0] by [-4, 2] in Figure 9. The cursor indicates that the absolute minimum value is about -3.1 when  $x \approx -0.9$  and we see that the function decreases on  $(-\infty, -0.9)$  and  $(0, \infty)$  and increases on (-0.9, 0). The exact values are obtained by differentiating:

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x+6}{x^3}$$

This shows that f'(x) > 0 when  $-\frac{6}{7} < x < 0$  and f'(x) < 0 when  $x < -\frac{6}{7}$  and when x > 0. The exact minimum value is  $f(-\frac{6}{7}) = -\frac{37}{12} \approx -3.08$ .

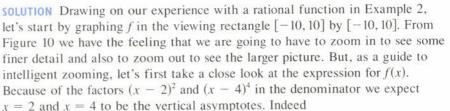
Figure 9 also shows that an inflection point occurs somewhere between x = -1 and x = -2. We could estimate it much more accurately using the graph of the second derivative, but in this case it is just as easy to find exact values. Since

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = 2\frac{7x+9}{x^4}$$

we see that f''(x) > 0 when  $x > -\frac{9}{7}$  ( $x \neq 0$ ). So *f* is concave upward on  $\left(-\frac{9}{7}, 0\right)$  and  $(0, \infty)$  and concave downward on  $\left(-\infty, -\frac{9}{7}\right)$ . The inflection point is  $\left(-\frac{9}{7}, -\frac{71}{27}\right)$ .

The analysis using the first two derivatives shows that Figures 7 and 8 display all the major aspects of the curve.

**EXAMPLE 3** Graph the function 
$$f(x) = \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$$
.

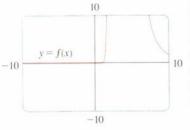


$$\lim_{x \to 2} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty \quad \text{and} \quad \lim_{x \to 4} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty$$

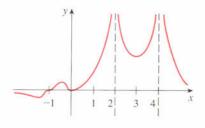
To find the horizontal asymptotes we divide numerator and denominator by  $x^6$ :

$$\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \frac{\frac{1}{x}\left(1+\frac{1}{x}\right)^3}{\left(1-\frac{2}{x}\right)^2\left(1-\frac{4}{x}\right)^4} \to 0 \quad \text{as} \quad x \to \pm \infty$$

so the x-axis is the horizontal asymptote.







y = f(x)

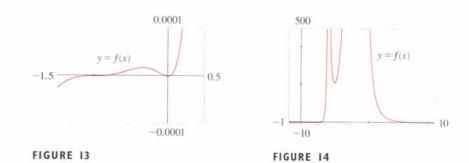
0.05

1

-0.05

It is also very useful to consider the behavior of the graph near the *x*-intercepts. Since  $x^2$  is positive, f(x) does not change sign at 0 and so its graph doesn't cross the *x*-axis at 0. But, because of the factor  $(x + 1)^3$ , the graph does cross the *x*-axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.



We can read from these graphs that the absolute minimum is about -0.02and occurs when  $x \approx -20$ . There is also a local maximum  $\approx 0.00002$  when  $x \approx -0.3$  and a local minimum  $\approx 211$  when  $x \approx 2.5$ . These graphs also show two inflection points near -5 and -1 and two between -1 and 0. To estimate the inflection points closely we would need to graph f'', but to compute f'' by hand is an unreasonable chore. If you have a computer algebra system, then it is easy (see Exercise 13).

We have seen that, for this particular function, *three* graphs (Figures 12, 13, and 14) are necessary to convey all the useful information. The only way to display all these features of the function on a single graph is to draw it by hand. Despite the exaggerations and distortions, Figure 11 does manage to summarize the essential nature of the function.

**EXAMPLE 4** Graph the function  $f(x) = \sin(x + \sin 2x)$ . For  $0 \le x \le \pi$ , locate all maximum and minimum values, intervals of increase and decrease, and inflection points correct to one decimal place.

**SOLUTION** We first note that *f* is periodic with period  $2\pi$ . Also, *f* is odd and  $|f(x)| \leq 1$  for all *x*. So the choice of a viewing rectangle is not a problem for this function: we start with  $[0, \pi]$  by [-1.1, 1.1] (see Figure 15). It appears that there are three local maximum values and two local minimum values in that window.

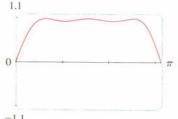


FIGURE 12

FIGURE II

-100

# The family of functions

 $f(x) = \sin(x + \sin cx)$ 

where c is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency (sin cx). The case where c = 2 is studied in Example 4. Exercise 15 explores another special case.

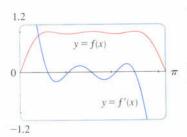
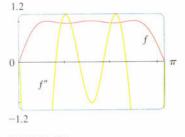
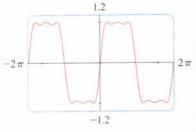


FIGURE 16







To confirm this and locate them more accurately, we calculate that

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2\cos 2x)$$

and graph both f and f' in Figure 16. Using zoom-in and the First Derivative Test, we find the following values to one decimal place.

Intervals of increase:	(0, 0.6), (1.0, 1.6), (2.1, 2.5)
Intervals of decrease:	$(0.6, 1.0), (1.6, 2.1), (2.5, \pi)$
Local maximum values:	$f(0.6) \approx 1, f(1.6) \approx 1, f(2.5) \approx 1$
Local minimum values:	$f(1.0) \approx 0.94, f(2.1) \approx 0.94$

The second derivative is

$$f''(x) = -(1 + 2\cos 2x)^2 \sin(x + \sin 2x) - 4\sin 2x \cos(x + \sin 2x)$$

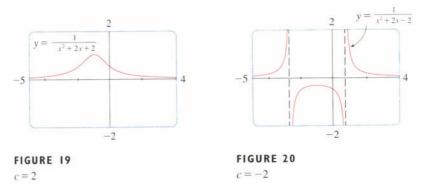
Graphing both f and f'' in Figure 17, we obtain the following approximate values:

Concave upward on: (0.8, 1.3), (1.8, 2.3)Concave downward on:  $(0, 0.8), (1.3, 1.8), (2.3, \pi)$ Inflection points: (0, 0), (0.8, 0.975), (1.3, 0.97), (1.8, 0.97), (2.3, 0.975)

Having checked that Figure 15 does indeed represent f accurately for  $0 \le x \le \pi$ , we can state that the extended graph in Figure 18 represents f  $2\pi$  accurately for  $-2\pi \le x \le 2\pi$ .

**EXAMPLE 5** How does the graph of  $f(x) = 1/(x^2 + 2x + c)$  vary as c varies?

**SOLUTION** The graphs in Figures 19 and 20 (the special cases c = 2 and c = -2) show two very different-looking curves.



Before drawing any more graphs, let's see what members of this family have in common. Since

$$\lim_{x \to \pm \infty} \frac{1}{x^2 + 2x + c} = 0$$

for any value of c, they all have the x-axis as a horizontal asymptote. A vertical asymptote will occur when  $x^2 + 2x + c = 0$ . Solving this quadratic equation, we get  $x = -1 \pm \sqrt{1 - c}$ . When c > 1, there is no vertical asymptote (as in

FIGURE 18

Figure 19). When c = 1 the graph has a single vertical asymptote x = -1 because

$$\lim_{x \to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{1}{(x + 1)^2} = \infty$$

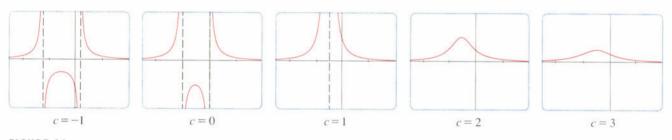
When c < 1 there are two vertical asymptotes:  $x = -1 + \sqrt{1 - c}$  and  $x = -1 - \sqrt{1 - c}$  (as in Figure 20).

Now we compute the derivative:

$$f'(x) = -\frac{2x+2}{(x^2+2x+c)^2}$$

This shows that f'(x) = 0 when x = -1 (if  $c \neq 1$ ), f'(x) > 0 when x < -1, and f'(x) < 0 when x > -1. For  $c \ge 1$  this means that f increases on  $(-\infty, -1)$ and decreases on  $(-1, \infty)$ . For c > 1, there is an absolute maximum value f(-1) = 1/(c - 1). For c < 1, f(-1) = 1/(c - 1) is a local maximum value and the intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed in the viewing rectangle [-5, 4] by [-2, 2].



# FIGURE 21

The family of functions  $f(x) = \frac{1}{x^2 + 2x + c}$  As predicted, c = 1 is the value at which a transition takes place from two vertical asymptotes to one, and then to none. As *c* increases from 1, we see that the maximum point becomes lower; this is explained by the fact that  $1/(c-1) \rightarrow 0$  as  $c \rightarrow \infty$ . As *c* decreases from 1, the vertical asymptotes become more widely separated because the distance between them is  $2\sqrt{1-c}$ , which becomes large as  $c \rightarrow -\infty$ . Again, the maximum point approaches the *x*-axis because  $1/(c-1) \rightarrow 0$  as  $c \rightarrow -\infty$ .

There is clearly no inflection point when  $c \le 1$ . For c > 1 we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when  $x = -1 \pm \sqrt{3(c-1)}/3$ . So the inflection points become more spread out as *c* increases and this seems plausible from the last two parts of Figure 21.

In Section 1.4 we used graphing devices to graph parametric curves and in Section 3.5 we found tangents to parametric curves. But, as our final example shows, we are now in a position to use calculus to ensure that a parameter interval or a viewing rectangle will reveal all the important aspects of a curve. **EXAMPLE 6** Graph the curve with parametric equations

$$x(t) = t^{2} + t + 1$$
  $y(t) = 3t^{4} - 8t^{3} - 18t^{2} + 25t^{4}$ 

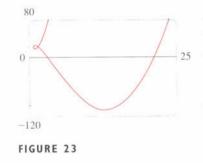
in a viewing rectangle that displays the important features of the curve. Find the coordinates of the interesting points on the curve.

**SOLUTION** Figure 22 shows the graph of this curve in the viewing rectangle [0, 4] by [-20, 60]. Zooming in toward the point *P* where the curve intersects itself, we estimate that the coordinates of *P* are (1.50, 22.25). We estimate the highest point on the loop has coordinates (1, 25), the lowest point (1, 18), and the leftmost point (0.75, 21.7). To be sure that we have discovered all the interesting aspects of the curve, however, we need to use calculus. From Equation 7 in Section 3.5, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^3 - 24t^2 - 36t}{2t + 1}$$

The vertical tangent occurs when dx/dt = 2t + 1 = 0, that is,  $t = -\frac{1}{2}$ . So the exact coordinates of the leftmost point of the loop are  $x(-\frac{1}{2}) = 0.75$  and  $y(-\frac{1}{2}) = 21.6875$ . Also,

$$\frac{dy}{dt} = 12t(t^2 - 2t - 3) = 12t(t + 1)(t - 3)$$



Exercises · ·

and so horizontal tangents occur when t = 0, -1, and 3. The top of the loop corresponds to t = -1 and, indeed, its coordinates are x(-1) = 1 and y(-1) = 18. Similarly, the coordinates of the bottom of the loop are exactly what we estimated: x(0) = 1 and y(0) = 25. But what about the parameter value t = 3? The corresponding point on the curve has coordinates x(3) = 13 and y(3) = -110. Figure 23 shows the graph of the curve in the viewing rectangle [0, 25] by [-120, 80]. This shows that the point (13, -110) is the lowest point on the curve. We can now be confident that there are no hidden maximum or minimum points.

I-6 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. 
$$f(x) = 4x^4 - 7x^2 + 4x + 6$$
  
2.  $f(x) = 8x^5 + 45x^4 + 80x^3 + 90x^2 + 200x$   
3.  $f(x) = \sqrt[3]{x^2 - 3x - 5}$   
4.  $f(x) = \frac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2}$   
5.  $f(x) = x^2 \sin x, -7 \le x \le 7$   
6.  $f(x) = \sin x + \frac{1}{3} \sin 3x$ 

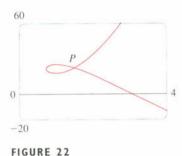
**7–8** Produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points, and use calculus to find these quantities exactly.

**7.** 
$$f(x) = 8x^3 - 3x^2 - 10$$
 **8.**  $f(x) = x\sqrt{9 - x^2}$ 

**9–10** Produce a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

**9.** 
$$f(x) = e^{x^3 - x}$$
 **10.**  $f(x) = e^{\cos x}$ 

**11–12** Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a



guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

**11.** 
$$f(x) = \frac{(x+4)(x-3)^2}{x^4(x-1)}$$
 **12.**  $f(x) = \frac{10x(x-1)^4}{(x-2)^3(x+1)^2}$ 

- **13.** If f is the function considered in Example 3, use a computer algebra system to calculate f' and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate f" and use it to estimate the intervals of concavity and inflection points.
- **CASE 14.** If f is the function of Exercise 12, find f' and f'' and use their graphs to estimate the intervals of increase and decrease and the concavity of f.
  - **15.** In Example 4 we considered a member of the family of functions  $f(x) = \sin(x + \sin cx)$  that occur in FM synthesis. Here we investigate the function with c = 3. Start by graphing *f* in the viewing rectangle  $[0, \pi]$  by [-1.2, 1.2]. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of *f'* very carefully. In fact, it helps to look at the graph of *f''* at the same time. Find all the maximum and minimum values and inflection points. Then graph *f* in the viewing rectangle  $[-2\pi, 2\pi]$  by [-1.2, 1.2] and comment on symmetry.
  - 16. Use a graph to estimate the coordinates of the leftmost point on the curve  $x = t^4 t^2$ ,  $y = t + \ln t$ . Then use calculus to find the exact coordinates.

**17–18** Graph the curve in a viewing rectangle that displays all the important aspects of the curve. At what points does the curve have vertical or horizontal tangents?

**17.** 
$$x = t^4 - 2t^3 - 2t^2$$
,  $y = t^3 - t$   
**18.**  $x = t^4 + 4t^3 - 8t^2$ ,  $y = 2t^2 - t$ 

- 19. Investigate the family of curves given by the parametric equations  $x = t^3 ct$ ,  $y = t^2$ . In particular, determine the values of c for which there is a loop and find the point where the curve intersects itself. What happens to the loop as c increases? Find the coordinates of the leftmost and rightmost points of the loop.
- **20.** The family of functions  $f(t) = C(e^{-at} e^{-bt})$ , where *a*, *b*, and *C* are positive numbers and b > a, has been used to model the concentration of a drug injected into the blood at time t = 0. Graph several members of this family. What do they have in common? For fixed values of *C* and *a*, discover graphically what happens as *b* increases. Then use calculus to prove what you have discovered.

**21–25** Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

21. 
$$f(x) = \frac{cx}{1 + c^2 x^2}$$
  
22.  $f(x) = \ln(x^2 + c)$   
23.  $f(x) = e^{-c/x^2}$   
24.  $f(x) = \frac{1}{(1 - x^2)^2 + cx^2}$   
25.  $f(x) = x^4 + cx^2$ 

**26.** Investigate the family of curves given by the equation  $f(x) = x^4 + cx^2 + x$ . Start by determining the transitional value of *c* at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of *c* at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.



# Indeterminate Forms and L'Hospital's Rule • • • • •

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when x = 1, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

But we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits) to (1) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and  $\frac{0}{0}$  is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both  $f(x) \to 0$  and  $g(x) \to 0$  as  $x \to a$ , then this limit may or may not exist and is called an **indeterminate form of type**  $\frac{0}{0}$ . We met some limits of this type in Chapter 2. For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as l'Hospital's Rule, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

$$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It is not obvious how to evaluate this limit because both numerator and denominator become large as  $x \to \infty$ . There is a struggle between numerator and denominator. If the numerator wins, the limit will be  $\infty$ ; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer may be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both  $f(x) \to \infty$  (or  $-\infty$ ) and  $g(x) \to \infty$  (or  $-\infty$ ), then the limit may or may not exist and is called an **indeterminate form of type**  $\infty/\infty$ . We saw in Section 2.5 that this type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of x that occurs. For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as (2), but l'Hospital's Rule also applies to this type of indeterminate form.

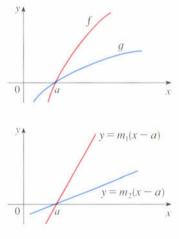


FIGURE I

Figure I suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g, each of which approaches 0 as  $x \rightarrow a$ . If we were to zoom in toward the point (a, 0), the graphs would start to look almost linear. But if the functions were actually linear, as in the second graph, then their ratio would be

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

 $\lim q(x) = \pm \infty$ 

**L'Hospital's Rule** Suppose f and g are differentiable and  $g'(x) \neq 0$  near a (except possibly at a). Suppose that

$$\lim f(x) = 0 \qquad \text{and} \qquad \lim g(x) = 0$$

and

or that

(In other words, we have an indeterminate form of type  $\frac{0}{0}$  or  $\infty/\infty$ .) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is  $\infty$  or  $-\infty$ ).

 $\lim f(x) = \pm \infty$ 

**NOTE I** • L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

**NOTE 2** • L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, " $x \to a$ " can be replaced by any of the following symbols:  $x \to a^+$ ,  $x \to a^-$ ,  $x \to \infty$ ,  $x \to -\infty$ .

**NOTE 3** • For the special case in which f(a) = g(a) = 0, f' and g' are continuous, and  $g'(a) \neq 0$ , it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}$$
$$= \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$
$$= \lim_{x \to a} \frac{f(x)}{g(x)}$$

The general version of l'Hospital's Rule is more difficult; its proof can be found in more advanced books.

**EXAMPLE 1** Find 
$$\lim_{x \to 1} \frac{\ln x}{x-1}$$
.

1

**SOLUTION** Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \to 1} (x - 1) = 0$$

we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx} (\ln x)}{\frac{d}{dx} (x - 1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1}$$
$$= \lim_{x \to 1} \frac{1}{x} = 1$$

**EXAMPLE 2** Calculate 
$$\lim_{x \to \infty} \frac{e^x}{x^2}$$
.

**SOLUTION** We have  $\lim_{x\to\infty} e^x = \infty$  and  $\lim_{x\to\infty} x^2 = \infty$ , so l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since  $e^x \to \infty$  and  $2x \to \infty$  as  $x \to \infty$ , the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

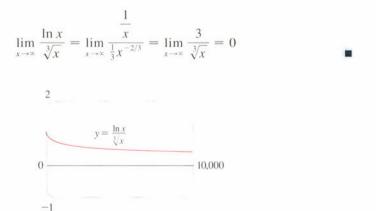
$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

**EXAMPLE 3** Calculate  $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$ .

**SOLUTION** Since  $\ln x \to \infty$  and  $\sqrt[3]{x} \to \infty$  as  $x \to \infty$ , l'Hospital's Rule applies:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type  $\frac{0}{0}$ . But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:



The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 45.

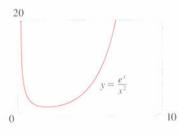


FIGURE 2

The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it is not surprising that this ratio approaches 0 as  $x \to \infty$ . See also Exercise 46.

**EXAMPLE 4** Find  $\lim_{x \to 0} \frac{\tan x - x}{x^3}$ . (See Exercise 18 in Section 2.2.)

**SOLUTION** Noting that both  $\tan x - x \to 0$  and  $x^3 \to 0$  as  $x \to 0$ , we use l'Hospital's Rule:

 $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$ 

Since the limit on the right side is still indeterminate of type  $\frac{0}{0}$ , we apply l'Hospital's Rule again:

 $\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$ 

Again both numerator and denominator approach 0, so a third application of l'Hospital's Rule is necessary. Putting together all three steps, we get

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$$
$$= \lim_{x \to 0} \frac{4 \sec^2 x \tan^2 x + 2 \sec^4 x}{6} = \frac{2}{6} = \frac{1}{3}$$

**EXAMPLE 5** Find 
$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$$
.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

 $\oslash$ 

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is *wrong!* Although the numerator  $\sin x \to 0$  as  $x \to \pi^-$ , notice that the denominator  $(1 - \cos x)$  does not approach 0, so l'Hospital's Rule cannot be applied here.

The required limit is, in fact, easy to find because the function is continuous and the denominator is nonzero at  $\pi$ :

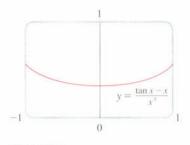
$$\lim_{x \to \pi^{-}} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

Example 5 shows what can go wrong if you use l'Hospital's Rule without thinking. Other limits *can* be found using l'Hospital's Rule but are more easily found by other methods. (See Examples 3 and 5 in Section 2.3, Example 5 in Section 2.5, and the discussion at the beginning of this section.) So when evaluating any limit, you should consider other methods before using l'Hospital's Rule.

# Indeterminate Products

If  $\lim_{x\to a} f(x) = 0$  and  $\lim_{x\to a} g(x) = \infty$  (or  $-\infty$ ), then it is not clear what the value of  $\lim_{x\to a} f(x)g(x)$ , if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be  $\infty$  (or  $-\infty$ ). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit

The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because  $\tan x$  is close to x when x is small. See Exercise 18(d) in Section 2.2.





is called an **indeterminate form of type 0**  $\cdot \infty$ . We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g}$$
 or  $fg = \frac{g}{1/f}$ 

This converts the given limit into an indeterminate form of type  $\frac{0}{0}$  or  $\infty/\infty$  so that we can use l'Hospital's Rule.

**EXAMPLE 6** Evaluate  $\lim_{x\to 0^+} x \ln x$ . Use the knowledge of this limit, together with information from derivatives, to sketch the curve  $y = x \ln x$ .

**SOLUTION** The given limit is indeterminate because, as  $x \to 0^+$ , the first factor (x) approaches 0 while the second factor (ln x) approaches  $-\infty$ . Writing x = 1/(1/x), we have  $1/x \to \infty$  as  $x \to 0^+$ , so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$
$$= \lim_{x \to 0^+} (-x) = 0$$

$$y = x \ln x$$

FIGURE 5

If 
$$f(x) = x \ln x$$
, then

$$f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

so f'(x) = 0 when  $\ln x = -1$ , which means that  $x = e^{-1}$ . In fact, f'(x) > 0 when  $x > e^{-1}$  and f'(x) < 0 when  $x < e^{-1}$ , so f is increasing on  $(1/e, \infty)$  and decreasing on (0, 1/e). Thus, by the First Derivative Test, f(1/e) = -1/e is a local (and absolute) minimum. Also, f''(x) = 1/x > 0, so f is concave upward on  $(0, \infty)$ . We use this information, together with the crucial knowledge that  $\lim_{x\to 0^+} f(x) = 0$ , to sketch the curve in Figure 5.

# Indeterminate Differences

If  $\lim_{x\to a} f(x) = \infty$  and  $\lim_{x\to a} g(x) = \infty$ , then the limit

$$\lim_{x \to a} \left[ f(x) - g(x) \right]$$

is called an **indeterminate form of type**  $\infty - \infty$ . Again there is a contest between *f* and *g*. Will the answer be  $\infty$  (*f* wins) or will it be  $-\infty$  (*g* wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type  $\frac{0}{0}$  or  $\infty/\infty$ .

**EXAMPLE 7** Compute  $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$ .

**SOLUTION** First notice that sec  $x \to \infty$  and  $\tan x \to \infty$  as  $x \to (\pi/2)^-$ , so the limit

is indeterminate. Here we use a common denominator:

$$\lim_{x \to (\pi/2)^{-}} (\sec x - \tan x) = \lim_{x \to (\pi/2)^{-}} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$
$$= \lim_{x \to (\pi/2)^{-}} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^{-}} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because  $1 - \sin x \to 0$  and  $\cos x \to 0$  as  $x \to (\pi/2)^-$ .

 $\lim \left[ f(x) \right]^{g(x)}$ 

Indeterminate Powers

Several indeterminate forms arise from the limit

**1.** 
$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0 \quad \text{type } 0^0$$
  
**2.** 
$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} g(x) = 0 \quad \text{type } \infty^0$$
  
**3.** 
$$\lim_{x \to a} f(x) = 1 \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty \quad \text{type } 1^\infty$$

Each of these three cases can be treated either by taking the natural logarithm:

let 
$$y = [f(x)]^{g(x)}$$
, then  $\ln y = g(x) \ln f(x)$ 

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product  $g(x) \ln f(x)$ , which is of type  $0 \cdot \infty$ .

**EXAMPLE 8** Calculate  $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$ .

**SOLUTION** First notice that as  $x \to 0^+$ , we have  $1 + \sin 4x \to 1$  and  $\cot x \to \infty$ , so the given limit is indeterminate. Let

$$v = (1 + \sin 4x)^{\cot x}$$

Then

$$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$
$$= \lim_{x \to 0^+} \frac{\frac{4\cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of ln y, but what we want is the limit of y. To find this we use the fact that  $y = e^{\ln y}$ :

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

The graph of the function  $y = x^{x}$ , x > 0, is shown in Figure 6. Notice that although  $0^0$  is not defined, the values of the function approach I as  $x \to 0^+$ . This confirms the result of Example 9.

2 -1 2 **EXAMPLE 9** Find lim  $x^x$ .

Therefore

**SOLUTION** Notice that this limit is indeterminate since  $0^x = 0$  for any x > 0 but  $x^0 = 1$  for any  $x \neq 0$ . We could proceed as in Example 8 or by writing the function as an exponential:

$$x^{x} = (e^{\ln x})^{x} = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

FIGURE 6

1-30 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, use it. If l'Hospital's Rule doesn't apply, explain why.

1.  $\lim_{x \to 2} \frac{x-2}{x^2-4}$ **2.**  $\lim_{x \to 1} \frac{x^a - 1}{x^b - 1}$  $3. \lim_{x \to 0} \frac{e^x - 1}{\sin x}$ 4.  $\lim_{x \to 0} \frac{\sin mx}{\sin nx}$ **5.** lim <u>tan x</u>  $\mathbf{6.} \lim_{x \to \pi} \frac{\tan x}{x}$  $x \to 0$   $x + \sin x$ 8.  $\lim_{x \to 0} \frac{6^x - 2^x}{x}$ 7.  $\lim_{x \to \infty} \frac{e^x}{x^3}$ 9.  $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$ 10.  $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ 12.  $\lim_{x \to \infty} \frac{\ln(1 + e^x)}{5x}$ 11.  $\lim_{x \to \infty} \frac{\ln \ln x}{\sqrt{x}}$ **13.**  $\lim_{x \to 0} \frac{\tan^{-1}(2x)}{3x}$ 14.  $\lim \frac{\sin x}{x}$  $x \to 0 e^x$ **15.**  $\lim \sqrt{x} \ln x$ **16.** lim *xe<sup>x</sup>*  $x \rightarrow 0^+$  $\chi \rightarrow -\chi$ **17.**  $\lim e^{-x} \ln x$ **18.** lim  $\sec 7x \cos 3x$  $x \rightarrow (\pi/2)^{-}$  $\chi \longrightarrow \infty$ **19.**  $\lim x^3 e^{-x^2}$ **20.**  $\lim (x - \pi) \cot x$  $\chi \longrightarrow \infty$  $x \rightarrow \pi$ **21.**  $\lim_{x \to 0} \left( \frac{1}{x} - \csc x \right)$ **22.**  $\lim (\csc x - \cot x)$ 

**24.**  $\lim_{x \to 1} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right)$ **23.** lim  $(xe^{1/x} - x)$  $\chi \longrightarrow \infty$ **25.**  $\lim x^{\sin x}$ **26.**  $\lim_{x \to \infty} (\sin x)^{\tan x}$  $x \rightarrow 0^{+}$  $x \rightarrow 0^+$ **28.**  $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^t$ **27.** lim  $(1 - 2x)^{1/x}$  $x \rightarrow 0$ **30.**  $\lim (e^x + x)^{1/x}$ **29.**  $\lim_{x \to \infty} (-\ln x)^x$  $x \rightarrow 0^{+}$  $\chi \rightarrow \infty$ 

🚰 31–32 🔳 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

**31.**  $\lim x [\ln(x + 5) - \ln x]$  **32.**  $\lim (\tan x)^{\tan 2x}$  $\chi \rightarrow \infty$  $x \rightarrow \pi/4$ 

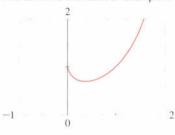
🚰 33–34 🔳 Illustrate l'Hospital's Rule by graphing both f(x)/g(x) and f'(x)/g'(x) near x = 0 to see that these ratios have the same limit as  $x \to 0$ . Also, calculate the exact value of the limit.

**33.**  $f(x) = e^x - 1$ ,  $g(x) = x^3 + 4x$ 

**34.**  $f(x) = 2x \sin x$ ,  $g(x) = \sec x - 1$ 

**35–38** ■ Use l'Hospital's Rule to help find the asymptotes of f. Then use them, together with information from f'and f'', to sketch the graph of f. Check your work with a graphing device.

<b>35.</b> $f(x) = xe^{-x}$	<b>36.</b> $f(x) = e^{x}/x$				
<b>37.</b> $f(x) = (\ln x)/x$	<b>38.</b> $f(x) = xe^{-x^2}$				
ha dai't as ha pai	<ol> <li>a per las las las</li> </ol>				



Exercises

₩ 39-40

- (a) Graph the function.
- (b) Use l'Hospital's Rule to explain the behavior as  $x \to 0$ .
- (c) Estimate the minimum value and intervals of concavity. Then use calculus to find the exact values.

**39.** 
$$f(x) = x^2 \ln x$$
 **40.**  $f(x) = xe^{1/x}$ 

#### CAS 41-42

- (a) Graph the function.
- (b) Explain the shape of the graph by computing the limit as  $x \to 0^+$  or as  $x \to \infty$ .
- (c) Estimate the maximum and minimum values and then use calculus to find the exact values.
- (d) Use a graph of f" to estimate the x-coordinates of the inflection points.

**41.**  $f(x) = x^{1/x}$  **42.**  $f(x) = (\sin x)^{\sin x}$ 

- **43.** Investigate the family of curves given by  $f(x) = xe^{-cx}$ , where c is a real number. Start by computing the limits as  $x \to \pm \infty$ . Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.
- 44. Investigate the family of curves given by f(x) = x<sup>n</sup>e<sup>-x</sup>, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.
  - 45. Prove that

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$$

for any integer n. This shows that the exponential function approaches infinity faster than any power of x.

46. Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number p > 0. This shows that the logarithmic function approaches  $\infty$  more slowly than any power of x.

**47.** If an initial amount  $A_0$  of money is invested at an interest rate *i* compounded *n* times a year, the value of the investment after *t* years is

$$A = A_0 \left( 1 + \frac{i}{n} \right)^m$$

If we let  $n \to \infty$ , we refer to the *continuous compound-ing* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after *n* years is

$$A = A_0 e^i$$

**48.** If an object with mass *m* is dropped from rest, one model for its speed *v* after *t* seconds, taking air resistance into account, is

$$v = \frac{mg}{c} \left(1 - e^{-ct/m}\right)$$

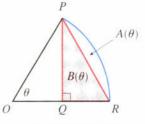
where g is the acceleration due to gravity and c is a positive constant. (In Chapter 7 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object.)

- (a) Calculate  $\lim_{t\to\infty} v$ . What is the meaning of this limit?
- (b) For fixed *t*, use l'Hospital's Rule to calculate lim<sub>m→x</sub> v. What can you conclude about the speed of a very heavy falling object?
- **49.** The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus textbook ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where a > 0. (At that time it was common to write aa instead of  $a^2$ .) Solve this problem.

**50.** The figure shows a sector of a circle with central angle  $\theta$ . Let  $A(\theta)$  be the area of the segment between the chord *PR* and the arc *PR*. Let  $B(\theta)$  be the area of the triangle *PQR*. Find  $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$ .



**51.** If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

🊰 52. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0\\ 1 & \text{if } x = 0 \end{cases}$$

- (a) Show that f is continuous at 0.
- (b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point (0, 1) on the graph of f.
- (c) Show that *f* is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?

# Writing Project

## The Origins of L'Hospital's Rule

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook *Analyse des Infiniment Petits*, but the rule was discovered in 1694 by the Swiss mathematician John Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries. The details, including a translation of l'Hospital's letter to Bernoulli proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by providing brief biographical details of both men (the dictionary edited by Gillispie [2] is a good source) and outline the business deal between them. Then give l'Hospital's statement of his rule, which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notice that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms of differentials. Compare their statement with the version of l'Hospital's Rule given in Section 4.5 and show that the two statements are essentially the same.

- 1. Howard Eves, In Mathematical Circles (Volume 2: Quadrants III and IV) (Boston: Prindle, Weber and Schmidt, 1969), pp. 20–22.
- C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII.
- 3. Victor Katz, A History of Mathematics: An Introduction (New York: Harper-Collins, 1993), p. 484.
- 4. D. J. Struik, ed., A Sourcebook in Mathematics, 1200–1800 (Princeton, NJ: Princeton University Press, 1969), pp. 315–316.



## Optimization Problems

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. Fermat's Principle in optics states that light follows the path that takes the least time. In this section and the next we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed on page 87 and adapt them to this situation:

#### STEPS IN SOLVING OPTIMIZATION PROBLEMS

**I. Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?

**2. Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

**3.** Introduce Notation Assign a symbol to the quantity that is to be maximized or minimized (let us call it Q for now.) Also select symbols  $(a, b, c, \ldots, x, y)$  for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus, Q will be expressed as a function of *one* variable x, say, Q = f(x). Write the domain of this function.

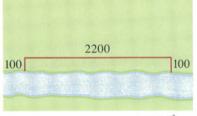
6. Use the methods of Sections 4.2 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.2 can be used.

**EXAMPLE I** A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Understand the problem
Analogy: Try special cases

Draw diagrams

**SOLUTION** In order to get a feeling for what is happening in this problem let's experiment with some special cases. Figure 1 (not to scale) shows three possible ways of laying out the 2400 ft of fencing. We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.





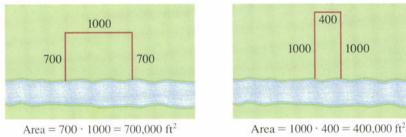
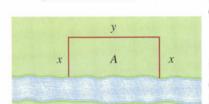


FIGURE I

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in feet). Then we express A in terms of x and y:

A = xy

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus



Introduce notation

$$2x + y = 2400$$

FIGURE 2

From this equation we have y = 2400 - 2x, which gives

$$4 = x(2400 - 2x) = 2400x - 2x^2$$

Note that  $x \ge 0$  and  $x \le 1200$  (otherwise A < 0). So the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \le x \le 1200$$

The derivative is A'(x) = 2400 - 4x, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives x = 600. The maximum value of *A* must occur either at this critical number or at an endpoint of the interval. Since A(0) = 0, A(600) = 720,000, and A(1200) = 0, the Closed Interval Method gives the maximum value as A(600) = 720,000.

[Alternatively, we could have observed that A''(x) = -4 < 0 for all x, so A is always concave downward and the local maximum at x = 600 must be an absolute maximum.]

Thus, the rectangular field should be 600 ft deep and 1200 ft wide.

**EXAMPLE 2** A can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

**SOLUTION** Draw the diagram as in Figure 3 where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides), which is

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h we use the fact that the volume is given as 1 L, which we take to be 1000 cm<sup>3</sup>. Thus

$$\pi r^2 h = 1000$$

which gives  $h = 1000/(\pi r^2)$ . Substitution of this into the expression for A gives

$$A = 2\pi r^{2} + 2\pi r \left(\frac{1000}{\pi r^{2}}\right) = 2\pi r^{2} + \frac{2000}{r}$$

Therefore, the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

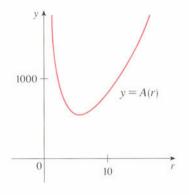
$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then A'(r) = 0 when  $\pi r^3 = 500$ , so the only critical number is  $r = \sqrt[3]{500/\pi}$ .

Since the domain of A is  $(0, \infty)$ , we cannot use the argument of Example 1 concerning endpoints. But we can observe that A'(r) < 0 for  $r < \sqrt[3]{500/\pi}$  and



FIGURE 3





A'(r) > 0 for  $r > \sqrt[3]{500/\pi}$ , so A is decreasing for all r to the left of the critical number and increasing for all r to the right. Thus,  $r = \sqrt[3]{500/\pi}$  must give rise to an *absolute* minimum.

[Alternatively, we could argue that  $A(r) \to \infty$  as  $r \to 0^+$  and  $A(r) \to \infty$  as  $r \to \infty$ , so there must be a minimum value of A(r), which must occur at the critical number. See Figure 4.]

The value of h corresponding to  $r = \sqrt[3]{500/\pi}$  is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be  $\sqrt[3]{500/\pi}$  cm and the height should be equal to twice the radius, namely, the diameter.

**NOTE I** • The argument used in Example 2 to justify the absolute minimum is a variant of the First Derivative Test (which applies only to *local* maximum or minimum values) and is stated here for future reference.

**First Derivative Test for Absolute Maximum or Minimum Values** Suppose that c is a critical number of a continuous function f defined on an interval.

- (a) If f'(x) > 0 for all x < c and f'(x) < 0 for all x > c, then f(c) is the absolute maximum value of f.
- (b) If f'(x) < 0 for all x < c and f'(x) > 0 for all x > c, then f(c) is the absolute minimum value of f.

**NOTE 2** • An alternative method for solving optimization problems is to use implicit differentiation. Let's look at Example 2 again to illustrate the method. We work with the same equations

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 100$$

but instead of eliminating h, we differentiate both equations implicitly with respect to r:

$$A' = 4\pi r + 2\pi h + 2\pi r h' \qquad 2\pi r h + \pi r^2 h' = 0$$

The minimum occurs at a critical number, so we set A' = 0, simplify, and arrive at the equations

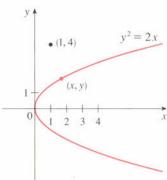
2r + h + rh' = 0 2h + rh' = 0

and subtraction gives 2r - h = 0, or h = 2r.

**EXAMPLE 3** Find the point on the parabola  $y^2 = 2x$  that is closest to the point (1, 4).

**SOLUTION** The distance between the point (1, 4) and the point (x, y) is

$$d = \sqrt{(x - 1)^2 + (y - 4)^2}$$







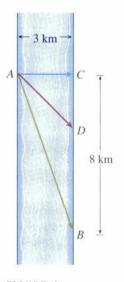


FIGURE 6

(see Figure 5). But if (x, y) lies on the parabola, then  $x = y^2/2$ , so the expression for *d* becomes

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y - 4)^2}$$

(Alternatively, we could have substituted  $y = \sqrt{2x}$  to get d in terms of x alone.) Instead of minimizing d, we minimize its square:

$$d^{2} = f(y) = \left(\frac{y^{2}}{2} - 1\right)^{2} + (y - 4)^{2}$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of  $d^2$ , but  $d^2$  is easier to work with.) Differentiating, we obtain

$$f'(y) = 2\left(\frac{y^2}{2} - 1\right)y + 2(y - 4) = y^3 - 8$$

so f'(y) = 0 when y = 2. Observe that f'(y) < 0 when y < 2 and f'(y) > 0 when y > 2, so by the First Derivative Test for Absolute Extrema, the absolute minimum occurs when y = 2. (Or we could simply say that because of the geometric nature of the problem, it is obvious that there is a closest point but not a farthest point.) The corresponding value of x is  $x = y^2/2 = 2$ . Thus, the point on  $y^2 = 2x$  closest to (1, 4) is (2, 2).

**EXAMPLE 4** A man is at point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 6). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row at 6 km/h and run at 8 km/h, where should he land to reach B as soon as possible?

**SOLUTION** Let x be the distance from C to D. Then the running distance is |DB| = 8 - x and the Pythagorean Theorem gives the rowing distance as  $|AD| = \sqrt{x^2 + 9}$ . We assume the speed of the water is 0 km/h and use the equation

time = 
$$\frac{\text{distance}}{\text{rate}}$$

Then the rowing time is  $\sqrt{x^2 + 9}/6$  and the running time is (8 - x)/8, so the total time *T* as a function of *x* is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function *T* is [0, 8]. Notice that if x = 0 he rows to *C* and if x = 8 he rows directly to *B*. The derivative of *T* is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that  $x \ge 0$ , we have

$$T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9}$$
$$\iff 16x^2 = 9(x^2 + 9) \iff 7x^2 = 81$$
$$\iff x = \frac{9}{\sqrt{7}}$$

The only critical number is  $x = 9/\sqrt{7}$ . To see whether the minimum occurs at this critical number or at an endpoint of the domain [0,8], we evaluate *T* at all three points:

$$T(0) = 1.5$$
  $T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33$   $T(8) = \frac{\sqrt{73}}{6} \approx 1.42$ 

Since the smallest of these values of T occurs when  $x = 9/\sqrt{7}$ , the absolute minimum value of T must occur there. Figure 7 illustrates this calculation by showing the graph of T.

Thus, the man should land the boat at a point  $9/\sqrt{7}$  km ( $\approx 3.4$  km) down-stream from his starting point.

**EXAMPLE 5** Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

**SOLUTION I** Let us take the semicircle to be the upper half of the circle  $x^2 + y^2 = r^2$  with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the *x*-axis as shown in Figure 8.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths 2x and y, so its area is

$$4 = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle  $x^2 + y^2 = r^2$  and so  $y = \sqrt{r^2 - x^2}$ . Thus

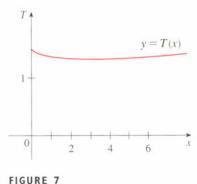
$$A = 2x\sqrt{r^2 - x^2}$$

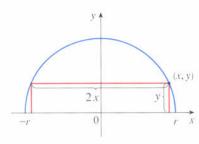
The domain of this function is  $0 \le x \le r$ . Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when  $2x^2 = r^2$ , that is,  $x = r/\sqrt{2}$  (since  $x \ge 0$ ). This value of x gives a maximum value of A since A(0) = 0 and A(r) = 0. Therefore the area of the largest inscribed rectangle is

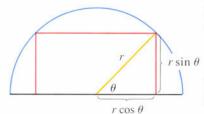
$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$







#### 316 CHAPTER 4 APPLICATIONS OF DIFFERENTIATION



**SOLUTION 2** A simpler solution is possible if we think of using an angle as a variable. Let  $\theta$  be the angle shown in Figure 9. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta) = r^2\sin 2\theta$$

We know that  $\sin 2\theta$  has a maximum value of 1 and it occurs when  $2\theta = \pi/2$ . So  $A(\theta)$  has a maximum value of  $r^2$  and it occurs when  $\theta = \pi/4$ .

Notice that this trigonometric solution does not involve differentiation. In fact we didn't need to use calculus at all.



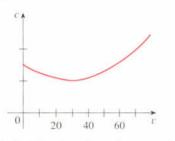
FIGURE 9

Exercises · · · · ·

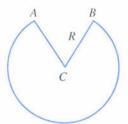
- 1. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
  - (a) Draw several diagrams illustrating the situation, some with short wide pens and some with long narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
  - (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
  - (c) Write an expression for the total area.
  - (d) Use the given information to write an equation that relates the variables.
  - (e) Use part (d) to write the total area as a function of one variable.
  - (f) Finish solving the problem and compare the answer with your estimate in part (a).
- 2. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
  - (a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
  - (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
  - (c) Write an expression for the volume.
  - (d) Use the given information to write an equation that relates the variables.
  - (e) Use part (d) to write the volume as a function of one variable.
  - (f) Finish solving the problem and compare the answer with your estimate in part (a).

- **3.** If 1200 cm<sup>2</sup> of material is available to make a box with a square base and an open top, find the largest possible volume of the box.
- **4.** A box with a square base and open top must have a volume of 32,000 cm<sup>3</sup>. Find the dimensions of the box that minimize the amount of material used.
- 5. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
  - (b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.
- 6. A rectangular storage container with an open top is to have a volume of 10 m<sup>3</sup>. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.
- 7. Find the point on the line y = 2x 3 that is closest to the origin.
- 8. Find the points on the hyperbola  $y^2 x^2 = 4$  that are closest to the point (2,0).
- **9.** Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius *r*.
- **10.** Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius *r*.
- Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.
- **12.** A right circular cylinder is inscribed in a sphere of radius *r*. Find the largest possible volume of such a cylinder.
- **13.** A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus, the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

14. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that c(v) is minimized for this car when v ≈ 30 mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons per mile. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



- 15. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?
- **16.** A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?
- **17.** A conical drinking cup is made from a circular piece of paper of radius *R* by cutting out a sector and joining the edges *CA* and *CB*. Find the maximum capacity of such a cup.



18. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to  $v^3$ . It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u (u < v), then the time required to swim a distance L is L/(v - u) and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v-u}$$

where a is the proportionality constant.

- (a) Determine the value of v that minimizes E.
- (b) Sketch the graph of E.

**Note:** This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

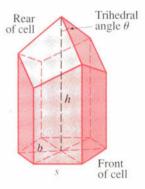
19. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle  $\theta$  is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area *S* is given by

$$S = 6sh - \frac{3}{2}s^2\cot\theta + (3s^2\sqrt{3}/2)\csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.

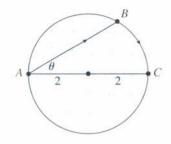
- (a) Calculate  $dS/d\theta$ .
- (b) What angle should the bees prefer?
- (c) Determine the minimum surface area of the cell (in terms of s and h).

**Note:** Actual measurements of the angle  $\theta$  in behives have been made, and the measures of these angles seldom differ from the calculated value by more than  $2^\circ$ .

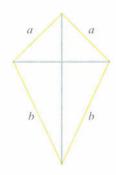


- 20. A boat leaves a dock at 2:00 P.M. and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 P.M. At what time were the two boats closest together?
- **21.** The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?
- **22.** A woman at a point *A* on the shore of a circular lake with radius 2 mi wants to arrive at the point *C*

diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. At what angle  $\theta$  to the diameter should she row?



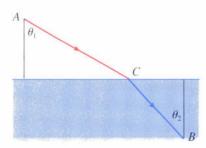
- **23.** Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.
- **CAS** 24. The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



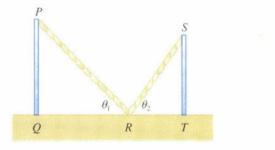
**25.** Let  $v_1$  be the velocity of light in air and  $v_2$  the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point *A* in the air to a point *B* in the water by a path *ACB* that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

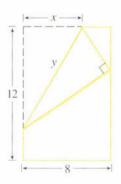
where  $\theta_1$  (the angle of incidence) and  $\theta_2$  (the angle of refraction) are as shown in the figure. This equation is known as **Snell's Law**.



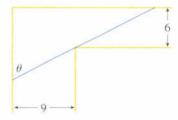
**26.** Two vertical poles *PQ* and *ST* are secured by a rope *PRS* strung from the top of the first pole to a point *R* on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when  $\theta_1 = \theta_2$ .



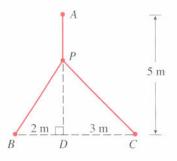
**27.** The upper left-hand corner of a piece of paper 8 in. wide by 12 in. long is folded over to the right-hand edge as in the figure. How would you fold it so as to minimize the length of the fold? In other words, how would you choose *x* to minimize *y*?



28. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



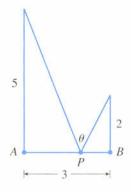
29. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B, and C is minimized (see the figure). Express L as a function of x = |AP| and use the graphs of L and dL/dx to estimate the minimum value.



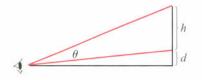
**30.** A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle *θ*. How should *θ* be chosen so that the gutter will carry the maximum amount of water?



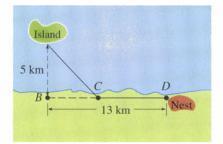
**31.** Where should the point *P* be chosen on the line segment *AB* so as to maximize the angle  $\theta$ ?



32. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



- **33.** Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point *B* on a straight shoreline, flies to a point *C* on the shoreline, and then flies along the shoreline to its nesting area *D*. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points *B* and *D* are 13 km apart.
  - (a) In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?
  - (b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.
  - (c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?
  - (d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from *B*, how many times more energy does it take a bird to fly over water than land?

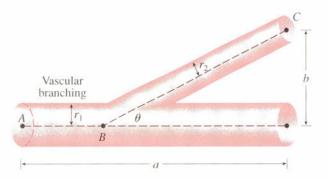


**34.** The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C \frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity

of the blood. (Poiseuille established this law experimentally but it also follows from Equation 2 in Section 6.6.) The figure shows a main blood vessel with radius  $r_1$  branching at an angle  $\theta$  into a smaller vessel with radius  $r_2$ .



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path *ABC* is

$$R = C \left( \frac{a - b \cot \theta}{r_1^4} + \frac{b \csc \theta}{r_2^4} \right)$$

where *a* and *b* are the distances shown in the figure. (b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

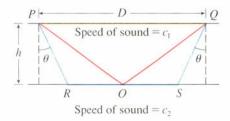
(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.



**35.** The speeds of sound  $c_1$  in an upper layer and  $c_2$  in a lower layer of rock and the thickness *h* of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point *P* and the transmitted signals are

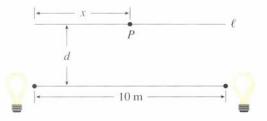
recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes  $T_1$  seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking  $T_2$  seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes  $T_3$ seconds to reach Q.

- (a) Express  $T_1$ ,  $T_2$ , and  $T_3$  in terms of D, h,  $c_1$ ,  $c_2$ , and  $\theta$ .
- (b) Show that  $T_2$  is a minimum when  $\sin \theta = c_1/c_2$ .
- (c) Suppose that D = 1 km,  $T_1 = 0.26 \text{ s}$ ,  $T_2 = 0.32 \text{ s}$ ,  $T_3 = 0.34 \text{ s}$ . Find  $c_1, c_2$ , and h.

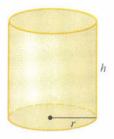


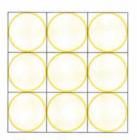
**Note:** Geophysicists use this technique when studying the structure of the earth's crust, whether searching for oil or examining fault lines.

- **36.** Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line  $\ell$  parallel to the line joining the light sources and at a distance of d meters from it (see the figure). We want to locate P on  $\ell$  so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.
  - (a) Find an expression for the intensity I(x) at the point *P*.
  - (b) If d = 5 m, use graphs of I(x) and I'(x) to show that the intensity is minimized when x = 5 m, that is, when P is at the midpoint of ℓ.
  - (c) If *d* = 10 m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.
  - (d) Somewhere between d = 5 m and d = 10 m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d.

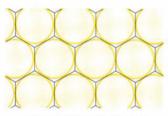


# Applied Project





Discs cut from squares



Discs cut from hexagons

#### The Shape of a Can

In this project we investigate the most economical shape for a can. We first interpret this to mean that the volume V of a cylindrical can is given and we need to find the height h and radius r that minimize the cost of the metal to make the can (see the figure). If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We solved this problem in Example 2 in Section 4.6 and we found that h = 2r, that is, the height should be the same as the diameter. But if you go to your cupboard or your supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/rvaries from 2 up to about 3.8. Let's see if we can explain this phenomenon.

1. The material for the cans is cut from sheets of metal. The cylindrical sides are formed by bending rectangles; these rectangles are cut from the sheet with little or no waste. But if the top and bottom discs are cut from squares of side 2r (as in the figure), this leaves considerable waste metal, which may be recycled but has little or no value to the can makers. If this is the case, show that the amount of metal used is minimized when

$$\frac{h}{r} = \frac{8}{\pi} \approx 2.55$$

**2.** A more efficient packing of the discs is obtained by dividing the metal sheet into hexagons and cutting the circular lids and bases from the hexagons (see the figure). Show that if this strategy is adopted, then

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21$$

**3.** The values of h/r that we found in Problems 1 and 2 are a little closer to the ones that actually occur on supermarket shelves, but they still don't account for everything. If we look more closely at some real cans, we see that the lid and the base are formed from discs with radius larger than r that are bent over the ends of the can. If we allow for this we would increase h/r. More significantly, in addition to the cost of the metal we need to incorporate the manufacturing of the can into the cost. Let's assume that most of the expense is incurred in joining the sides to the rims of the cans. If we cut the discs from hexagons as in Problem 2, then the total cost is proportional to

$$4\sqrt{3}r^2 + 2\pi rh + k(4\pi r + h)$$

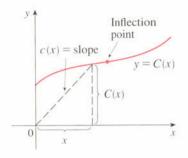
where k is the reciprocal of the length that can be joined for the cost of one unit area of metal. Show that this expression is minimized when

$$\frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$$

- 4. Plot <sup>3</sup>√V/k as a function of x = h/r and use your graph to argue that when a can is large or joining is cheap, we should make h/r approximately 2.21 (as in Problem 2). But when the can is small or joining is costly, h/r should be substantially larger.
  - 5. Our analysis shows that large cans should be almost square but small cans should be tall and thin. Take a look at the relative shapes of the cans in a supermarket. Is our conclusion usually true in practice? Are there exceptions? Can you suggest reasons why small cans are not always tall and thin?



#### **Applications to Economics**







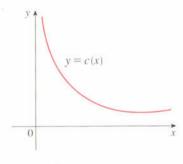


FIGURE 2 Average cost function

See Example 8 in Section 3.3 for an explanation of why it is reasonable to model a cor function b, a polynomial. In Section 3.3 we introduced the idea of marginal cost. Recall that if C(x), the cost function, is the cost of producing x units of a certain product, then the marginal cost is the rate of change of C with respect to x. In other words, the marginal cost function is the derivative, C'(x), of the cost function.

The graph of a typical cost function is shown in Figure 1. The marginal cost C'(x) is the slope of the tangent to the cost curve at (x, C(x)). Notice that the cost curve is initially concave downward (the marginal cost is decreasing) because of economies of scale (more efficient use of the fixed costs of production). But eventually there is an inflection point and the cost curve becomes concave upward (the marginal cost is increasing) perhaps because of overtime costs or the inefficiencies of a large-scale operation.

The average cost function

$$c(x) = \frac{C(x)}{x}$$

represents the cost per unit when x units are produced. We sketch a typical average cost function in Figure 2 by noting that C(x)/x is the slope of the line that joins the origin to the point (x, C(x)) in Figure 1. It appears that there will be an absolute minimum. To find it we locate the critical point of c by using the Quotient Rule to differentiate Equation 1:

$$c'(x) = \frac{xC'(x) - C(x)}{x^2}$$

Now c'(x) = 0 when xC'(x) - C(x) = 0 and this gives

$$C'(x) = \frac{C(x)}{x} = c(x)$$

Therefore:

1

If the average cost is a minimum, then marginal cost = average cost

This principle is plausible because if our marginal cost is smaller than our average cost, then we should produce more, thereby lowering our average cost. Similarly, if our marginal cost is larger than our average cost, then we should produce less in order to lower our average cost.

**EXAMPLE I** A company estimates that the cost (in dollars) of producing x items is  $C(x) = 2600 + 2x + 0.001x^2$ .

(a) Find the cost, average cost, and marginal cost of producing 1000 items, 2000 items, and 3000 items.

(b) At what production level will the average cost be lowest, and what is this minimum average cost?

#### SOLUTION

(a) The average cost function is

$$c(x) = \frac{C(x)}{x} = \frac{2600}{x} + 2 + 0.001x$$

The marginal cost function is

$$C'(x) = 2 + 0.002x$$

	We use these expressions to fill in the table at the left, giving the cost, average
	cost, and marginal cost (in dollars, or dollars per item, rounded to the nearest
6	cent).

(b) To minimize the average cost we must have

marginal cost = average cost

$$C'(x) = c(x)$$

 $2 + 0.002x = \frac{2600}{x} + 2 + 0.001x$ 

This equation simplifies to

$$0.001x = \frac{2600}{x}$$

SO

$$x = \sqrt{2,600,000} \approx 1612$$
  
hat this production level actually gives a minimum, we note that

 $x^2 = \frac{2600}{0.001} = 2,600,000$ 

3000 To see that this production level actually gives a minimum, we note that  $c''(x) = 5200/x^3 > 0$ , so c is concave upward on its entire domain. The minimum average cost is

$$c(1612) = \frac{2600}{1612} + 2 + 0.001(1612) =$$
\$5.22/item

Now let's consider marketing. Let p(x) be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price func-tion**) and we would expect it to be a decreasing function of x. If x units are sold and the price per unit is p(x), then the total revenue is

$$R(x) = xp(x)$$

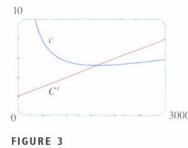
and R is called the **revenue function** (or **sales function**). The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

x	C(x)	c(x)	C'(x)	
1000	5,600.00	5.60	4.00	
2000	10,600.00	5.30	6.00	
3000	17,600.00	5.87	8.00	

Figure 3 shows the graphs of the marginal cost function C' and average cost function c in Example I. Notice that c has its minimum value when the two graphs intersect.



and P is called the profit function. The marginal profit function is P', the derivative of the profit function. In order to maximize profit we look for the critical numbers of P, that is, the numbers where the marginal profit is 0. But if

$$P'(x) = R'(x) - C'(x) = 0$$

R'(x) = C'(x)

then

Therefore:

To ensure that this condition gives a maximum we could use the Second Derivative Test. Note that

P''(x) = R''(x) - C''(x) < 0

$$\mathsf{R}''(x) < C''(x)$$

and this condition says that the rate of increase of marginal revenue is less than the rate of increase of marginal cost. Thus the profit will be a maximum when

$$R'(x) = C'(x)$$
 and  $R''(x) < C''(x)$ 

**EXAMPLE 2** Determine the production level that will maximize the profit for a company with cost and demand functions

$$C(x) = 84 + 1.26x - 0.01x^{2} + 0.00007x^{3} \qquad p(x) = 3.5 - 0.01x$$

**SOLUTION** The revenue function is

$$R(x) = xp(x) = 3.5x - 0.01x^2$$

so the marginal revenue function is

$$R'(x) = 3.5 - 0.02x$$

and the marginal cost function is

$$C'(x) = 1.26 - 0.02x + 0.00021x^2$$

Thus, marginal revenue is equal to marginal cost when

$$3.5 - 0.02x = 1.26 - 0.02x + 0.00021x^2$$

Solving, we get

$$x = \sqrt{\frac{2.24}{0.00021}} \approx 103$$

160 To check that this gives a maximum we compute the second derivatives:

$$R''(x) = -0.02 \qquad C''(x) = -0.02 + 0.00042x$$

Figure 4 shows the graphs of the revenue and cost functions in Example 2. The company makes a profit when R > C and the profit is a maximum when  $x \approx 103$ . Notice that the curves have parallel tangents at this production level because marginal revenue equals marginal cost.

32 0

FIGURE 4

Thus, R''(x) < C''(x) for all x > 0. Therefore, a production level of 103 units will maximize the profit.

**EXAMPLE 3** A store has been selling 200 compact disc players a week at \$350 each. A market survey indicates that for each \$10 rebate offered to the buyers, the number of sets sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

**SOLUTION** If x is the number of CD players sold per week, then the weekly increase in sales is x - 200. For each increase of 20 players sold, the price is decreased by \$10. So for each additional player sold the decrease in price will be  $\frac{1}{20} \times 10$  and the demand function is

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \frac{1}{2}x^2$$

Since R'(x) = 450 - x, we see that R'(x) = 0 when x = 450. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \frac{1}{2}(450) = 225$$

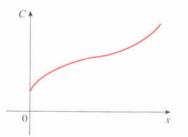
and the rebate is 350 - 225 = 125. Therefore, to maximize revenue the store should offer a rebate of \$125.

**1.** A manufacturer keeps precise records of the cost C(x) of producing *x* items and produces the graph of the cost function shown in the figure.

(a) Explain why C(0) > 0.

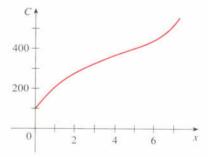
Exercises

- (b) What is the significance of the inflection point?
- (c) Use the graph of *C* to sketch the graph of the marginal cost function.



- 2. The graph of a cost function C is given.
  - (a) Draw a careful sketch of the marginal cost function.
  - (b) Use the geometric interpretation of the average cost c(x) as a slope (see Figure 1) to draw a careful sketch of the average cost function.

(c) Estimate the value of x for which c(x) is a minimum. How are the average cost and the marginal cost related at that value of x?



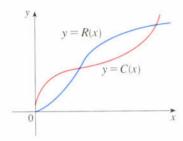
3. The average cost of producing *x* units of a commodity is

$$c(x) = 21.4 - 0.002x$$

Find the marginal cost at a production level of 1000 units. In practical terms, what is the meaning of your answer?

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- **4.** The figure shows graphs of the cost and revenue functions reported by a manufacturer.
  - (a) Identify on the graph the value of x for which the profit is maximized.
  - (b) Sketch a graph of the profit function.
  - (c) Sketch a graph of the marginal profit function.



**5–6** For each cost function (given in dollars), find (a) the cost, average cost, and marginal cost at a production level of 1000 units; (b) the production level that will minimize the average cost; and (c) the minimum average cost.

5. 
$$C(x) = 1600 + 8x + 0.01x$$

6. 
$$C(x) = 2\sqrt{x} + x^2/8000$$

**7−8** A cost function is given.

- (a) Find the average cost and marginal cost functions.
- (b) Use graphs of the functions in part (a) to estimate the production level that minimizes the average cost.
- (c) Use calculus to find the minimum average cost.
- (d) Find the minimum value of the marginal cost.

7.  $C(x) = 3700 + 5x - 0.04x^2 + 0.0003x^3$ 

8.  $C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$ 

**9–10** ■ For the given cost and demand functions, find the production level that will maximize profit.

**9.** 
$$C(x) = 680 + 4x + 0.01x^2$$
,  $p(x) = 12 - x/500$ 

- **10.**  $C(x) = 10,000 + 28x 0.01x^2 + 0.002x^3$ , p(x) = 90 - 0.02x
- **11.** The cost, in dollars, of producing x yards of a certain fabric is

$$C(x) = 1200 + 12x - 0.1x^{2} + 0.0005x^{3}$$

and the company finds that if it sells *x* yards, it can charge

$$p(x) = 29 - 0.00021x$$

dollars per yard for the fabric.

(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.

- (b) Use calculus to find the production level for maximum profit.
- 12. An aircraft manufacturer wants to determine the best selling price for a new airplane. The company estimates that the initial cost of designing the airplane and setting up the factories in which to build it will be 500 million dollars, and that the additional cost of manufacturing each plane can be modeled by the function  $m(x) = 20x 5x^{3/4} + 0.01x^2$ , where x is the number of aircraft produced and m is the manufacturing cost, in millions of dollars. The company estimates that if it charges a price p (in millions of dollars) for each plane, it will be able to sell x(p) = 320 7.7p planes.
  - (a) Find the cost, demand, and revenue functions.
  - (b) Find the production level and the associated selling price of the aircraft that maximizes profit.
- 13. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000.(a) Find the demand function, assuming that it is
  - linear.(b) How should ticket prices be set to maximize
- revenue?14. During the summer months Terry makes and sells neck-laces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that he lost two sales
  - per day. (a) Find the demand function, assuming that it is linear.
  - (b) If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?
- 15. A manufacturer has been selling 1000 television sets a week at \$450 each. A market survey indicates that for each \$10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.
  - (a) Find the demand function.
  - (b) How large a rebate should the company offer the buyer in order to maximize its revenue?
  - (c) If its weekly cost function is C(x) = 68,000 + 150x, how should it set the size of the rebate in order to maximize its profit?
- 16. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$400 per month. A market survey suggests that, on the average, one additional unit will remain vacant for each \$5 increase in rent. What rent should the manager charge to maximize revenue?



#### Newton's Method

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation

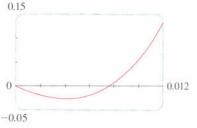
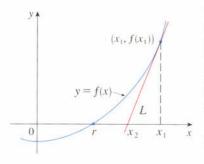




FIGURE 2

Try to solve Equation 1 using the numerical rootfinder on your calculator or computer. Some machines are not able to solve it. Others are successful but require you to specify a starting point for the search.



 $48x(1+x)^{60} - (1+x)^{60} + 1 = 0$ 

(The details are explained in Exercise 29.) How would you solve such an equation?

For a quadratic equation  $ax^2 + bx + c = 0$  there is a well-known formula for the roots. For third- and fourth-degree equations there are also formulas for the roots but they are extremely complicated. If *f* is a polynomial of degree 5 or higher, there is no such formula (see the note on page 242). Likewise, there is no formula that will enable us to find the exact roots of a transcendental equation such as  $\cos x = x$ .

We can find an *approximate* solution to Equation 1 by plotting the left side of the equation. Using a graphing device, and after experimenting with viewing rectangles, we produce the graph in Figure 1.

We see that in addition to the solution x = 0, which doesn't interest us, there is a solution between 0.007 and 0.008. Zooming in shows that the root is approximately 0.0076. If we need more accuracy we could zoom in repeatedly, but that becomes tiresome. A faster alternative is to use a numerical rootfinder on a calculator or computer algebra system. If we do so, we find that the root, correct to nine decimal places, is 0.007628603.

How do those numerical rootfinders work? They use a variety of methods, but most of them make some use of **Newton's method**, also called the **Newton-Raphson method**. We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

The geometry behind Newton's method is shown in Figure 2, where the root that we are trying to find is labeled r. We start with a first approximation  $x_1$ , which is obtained by guessing, or from a rough sketch of the graph of f, or from a computer-generated graph of f. Consider the tangent line L to the curve y = f(x) at the point  $(x_1, f(x_1))$  and look at the *x*-intercept of L, labeled  $x_2$ . The idea behind Newton's method is that the tangent line is close to the curve and so its *x*-intercept,  $x_2$ , is close to the *x*-intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its *x*-intercept.

To find a formula for  $x_2$  in terms of  $x_1$  we use the fact that the slope of L is  $f'(x_1)$ , so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is  $x_2$ , we set y = 0 and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If  $f'(x_1) \neq 0$ , we can solve this equation for  $x_2$ :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use  $x_2$  as a second approximation to r.

2

Next we repeat this procedure with  $x_1$  replaced by  $x_2$ , using the tangent line at  $(x_2, f(x_2))$ . This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process we obtain a sequence of approximations  $x_1, x_2, x_3, x_4, \ldots$  as shown in Figure 3. In general, if the *n*th approximation is  $x_n$  and  $f'(x_n) \neq 0$ , then the next approximation is given by

 $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ 

If the numbers  $x_n$  become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write

$$\lim_{n\to\infty} x_n = r$$

Although the sequence of successive approximations converges to the desired root for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge. For example, consider the situation shown in Figure 4. You can see that  $x_2$  is a worse approximation than  $x_1$ . This is likely to be the case when  $f'(x_1)$  is close to 0. It might even happen that an approximation (such as  $x_3$  in Figure 4) falls outside the domain of f. Then Newton's method fails and a better initial approximation  $x_1$  should be chosen. See Exercises 21–23 for specific examples in which Newton's method works very slowly or does not work at all.

**EXAMPLE I** Starting with  $x_1 = 2$ , find the third approximation  $x_3$  to the root of the equation  $x^3 - 2x - 5 = 0$ .

SOLUTION We apply Newton's method with

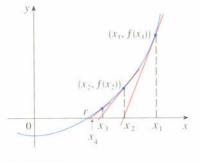
$$f(x) = x^3 - 2x - 5$$
 and  $f'(x) = 3x^2 - 2$ 

Newton himself used this equation to illustrate his method and he chose  $x_1 = 2$  after some experimentation because f(1) = -6, f(2) = -1, and f(3) = 16. Equation 2 becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

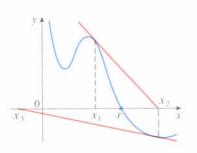
With n = 1 we have

$$x_{2} = x_{1} - \frac{x_{1}^{3} - 2x_{1} - 5}{3x_{1}^{2} - 2}$$
$$= 2 - \frac{2^{3} - 2(2) - 5}{3(2)^{2} - 2} = 2.1$$





Sequences were briefly introduced in A Preview of Calculus on page 6. A more thorough discussion starts in Section 8.I.





Then with n = 2 we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$
  
= 2.1 -  $\frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$ 

It turns out that this third approximation  $x_3 \approx 2.0946$  is accurate to four decimal places.

Suppose that we want to achieve a given accuracy, say to eight decimal places, using Newton's method. How do we know when to stop? The rule of thumb that is generally used is that we can stop when successive approximations  $x_n$  and  $x_{n+1}$  agree to eight decimal places. (A precise statement concerning accuracy in Newton's method will be given in Exercises 8.9.)

Notice that the procedure in going from n to n + 1 is the same for all values of n. (It is called an *iterative* process.) This means that Newton's method is particularly convenient for use with a programmable calculator or a computer.

**EXAMPLE 2** Use Newton's method to find  $\sqrt[6]{2}$  correct to eight decimal places.

**SOLUTION** First we observe that finding  $\sqrt[6]{2}$  is equivalent to finding the positive root of the equation

$$x^6 - 2 = 0$$

so we take  $f(x) = x^6 - 2$ . Then  $f'(x) = 6x^5$  and Formula 2 (Newton's method) becomes

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

If we choose  $x_1 = 1$  as the initial approximation, then we obtain

 $x_2 \approx 1.16666667$   $x_3 \approx 1.12644368$   $x_4 \approx 1.12249707$   $x_5 \approx 1.12246205$  $x_6 \approx 1.12246205$ 

Since  $x_5$  and  $x_6$  agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

.

to eight decimal places.

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**EXAMPLE 3** Find, correct to six decimal places, the root of the equation  $\cos x = x$ .

**SOLUTION** We first rewrite the equation in standard form:

$$\cos x - x = 0$$

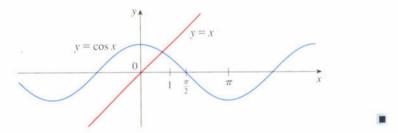
Therefore, we let  $f(x) = \cos x - x$ . Then  $f'(x) = -\sin x - 1$ , so Formula 2 becomes

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

In order to guess a suitable value for  $x_1$  we sketch the graphs of  $y = \cos x$  and y = x in Figure 5. It appears that they intersect at a point whose *x*-coordinate is somewhat less than 1, so let's take  $x_1 = 1$  as a convenient first approximation. Then

 $x_2 \approx 0.75036387$   $x_3 \approx 0.73911289$   $x_4 \approx 0.73908513$  $x_5 \approx 0.73908513$ 

Since  $x_4$  and  $x_5$  agree to six decimal places (eight, in fact), we conclude that the root of the equation, correct to six decimal places, is 0.739085.



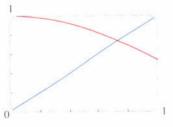


FIGURE 5

Instead of using the rough sketch in Figure 5 to get a starting approximation for Newton's method in Example 3, we could have used the more accurate graph that a calculator or computer provides. Figure 6 suggests that we use  $x_1 = 0.75$  as the initial approximation. Then Newton's method gives

$$x_2 \approx 0.73911114$$
  
 $x_3 \approx 0.73908513$   
 $x_4 \approx 0.73908513$ 

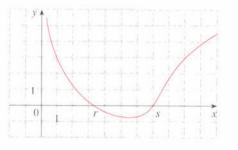
and so we obtain the same answer as before, but with one fewer step. In general, it is often efficient to use a computer and Newton's method in tandem—the graphing device to get started and Newton's method to finish.

FIGURE 6

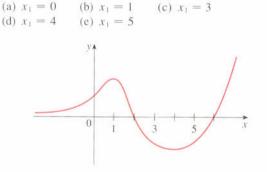
14.  $\ln(4 - r^2) = r$ 

# 4.8

1. The figure shows the graph of a function *f*. Suppose that Newton's method is used to approximate the root *r* of the equation f(x) = 0 with initial approximation  $x_1 = 1$ . Draw the tangent lines that are used to find  $x_2$  and  $x_3$ , and estimate the numerical values of  $x_2$  and  $x_3$ .



- **2.** Follow the instructions for Exercise 1 but use  $x_1 = 9$  as the starting approximation for finding the root *s*.
- **3.** Suppose the line y = 5x 4 is tangent to the curve y = f(x) when x = 3. If Newton's method is used to locate a root of the equation f(x) = 0 and the initial approximation is  $x_1 = 3$ , find the second approximation  $x_2$ .
- **4.** For each initial approximation, determine graphically what happens if Newton's method is used for the function whose graph is shown.



**5–6** Use Newton's method with the given initial approximation  $x_1$  to find  $x_3$ , the third approximation to the root of the given equation. (Give your answer to four decimal places.)

- **5.**  $x^3 + x + 1 = 0$ ,  $x_1 = -1$ **6.**  $x^7 - 100 = 0$ ,  $x_1 = 2$
- . . . . .

**7–8** ■ Use Newton's method to approximate the given number correct to eight decimal places.

**7.**  $\sqrt[4]{22}$  **8.**  $\sqrt[10]{100}$ 

**9–10** Use Newton's method to approximate the indicated root of the equation correct to six decimal places.

- **9.** The positive root of  $2 \sin x = x$
- 10. The root of  $\tan x = x$  in the interval  $(\pi/2, 3\pi/2)$
- II-18 Use Newton's method to find all the roots of the equation correct to eight decimal places. Start by drawing a graph to find initial approximations.
  - **11.**  $x^3 = 4x 1$  **12.**  $e^{-x^2} = x^2 x$
  - **13.**  $2\cos x = 2 x$

- **15.**  $x^4 + 3x^3 x 10 = 0$
- **16.**  $x^9 x^6 + 2x^4 + 5x 14 = 0$
- 17.  $\sqrt{x^2 x + 1} = 2\sin \pi x$
- **18.**  $\cos(x^2 + 1) = x^3$
- **19.** (a) Apply Newton's method to the equation  $x^2 a = 0$  to derive the following square-root algorithm (used by the ancient Babylonians to compute  $\sqrt{a}$ ):

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right)$$

- (b) Use part (a) to compute  $\sqrt{1000}$  correct to six decimal places.
- **20.** (a) Apply Newton's method to the equation 1/x a = 0 to derive the following reciprocal algorithm:

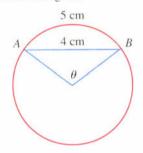
$$x_{n+1} = 2x_n - ax_n^2$$

(This algorithm enables a computer to find reciprocals without actually dividing.)

- (b) Use part (a) to compute 1/1.6984 correct to six decimal places.
- **21.** Explain why Newton's method does not work for finding the root of the equation  $x^3 3x + 6 = 0$  if the initial approximation is chosen to be  $x_1 = 1$ .
- **22.** (a) Use Newton's method with  $x_1 = 1$  to find the root of the equation  $x^3 x = 1$  correct to six decimal places.
  - (b) Solve the equation in part (a) using  $x_{\perp} = 0.6$  as the initial approximation.
  - (c) Solve the equation in part (a) using  $x_1 = 0.57$ . (You definitely need a programmable calculator for this part.)
  - (d) Graph  $f(x) = x^3 x 1$  and its tangent lines at  $x_1 = 1, 0.6, and 0.57$  to explain why Newton's method is so sensitive to the value of the initial approximation.

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- **23.** Explain why Newton's method fails when applied to the equation  $\sqrt[3]{x} = 0$  with any initial approximation  $x_1 \neq 0$ . Illustrate your explanation with a sketch.
- 24. Use Newton's method to find the absolute minimum value of the function  $f(x) = x^2 + \sin x$  correct to four decimal places.
- **25.** Use Newton's method to find the coordinates of the inflection point of the curve  $y = e^{\cos x}$  correct to six decimal places.
- **26.** Of the infinitely many lines that are tangent to the curve  $y = -\sin x$  and pass through the origin, there is one that has the largest slope. Use Newton's method to find the slope of that line correct to six decimal places.
- **27.** A grain silo consists of a cylindrical main section, with height 30 ft, and a hemispherical roof. In order to achieve a total volume of 15,000 ft<sup>3</sup> (including the part inside the roof section), what would the radius of the silo have to be?
- **28.** In the figure, the length of the chord *AB* is 4 cm and the length of the arc *AB* is 5 cm. Find the central angle  $\theta$ , in radians, correct to four decimal places. Then give the answer to the nearest degree.



**29.** A car dealer sells a new car for \$18,000. He also offers to sell the same car for payments of \$375 per month for five years. What monthly interest rate is this dealer charging?

To solve this problem you will need to use the formula for the present value A of an annuity consisting of n equal payments of size R with interest rate i per time period:

$$A = \frac{R}{i} \left[ 1 - (1+i)^{-n} \right]$$

Replacing i by x, show that

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

Use Newton's method to solve this equation.

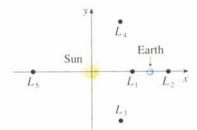
•30. The figure shows the Sun located at the origin and Earth at the point (1,0). (The unit here is the distance between the centers of Earth and the Sun, called an *astronomical unit:* 1 AU ≈ 1.496 × 10<sup>8</sup> km.) There are five locations  $L_1$ ,  $L_2$ ,  $L_3$ ,  $L_4$ , and  $L_5$  in this plane of rotation of Earth about the Sun where a satellite remains motionless with respect to Earth because the gravitational attractions of Earth and the Sun acting on the satellite balance each other. These locations are called *libration points*. (A solar research satellite has been placed at one of these libration points.) If  $m_1$  is the mass of the Sun,  $m_2$  is the mass of Earth, and  $r = m_2/(m_1 + m_2)$ , it turns out that the x-coordinate of  $L_1$  is the unique root of the fifth-degree equation

$$p(x) = x^{5} - (2 + r)x^{4} + (1 + 2r)x^{3} - (1 - r)x^{2}$$
$$+ 2(1 - r)x + r - 1 = 0$$

and the x-coordinate of  $L_2$  is the root of the equation

 $p(x) - 2rx^2 = 0$ 

Using the value  $r \approx 3.04042 \times 10^{-6}$ , find the locations of the libration points (a)  $L_1$  and (b)  $L_2$ .





#### 

A physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time. In each case, the problem is to find a function whose derivative is a known function. A function *F* is called an **antiderivative** of *f* on an interval *I* if F'(x) = f(x) for all *x* in *I*.

In Section 2.10 we introduced the idea of an antiderivative and we learned how to sketch the graph of an antiderivative of f if we are given the graph of f. Now that we know the differentiation formulas, we are in a position to find explicit expressions for antiderivatives. For instance, let  $f(x) = x^2$ . It is not difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if  $F(x) = \frac{1}{3}x^3$ , then  $F'(x) = x^2 = f(x)$ . But the function  $G(x) = \frac{1}{3}x^3 + 100$  also satisfies  $G'(x) = x^2$ . Therefore, both F and G are antiderivatives of f. Indeed, any function of the form  $H(x) = \frac{1}{3}x^3 + C$ , where C is a constant, is an antiderivative of f. The following theorem says that f has no other antiderivative. A proof of Theorem 1, using the Mean Value Theorem is outlined in Exercise 43.

**Theorem** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Going back to the function  $f(x) = x^2$ , we see that the general antiderivative of f is  $x^3/3 + C$ . By assigning specific values to the constant C we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1).

**EXAMPLE I** Find the most general antiderivative of each of the following functions:

(a) 
$$f(x) = \sin x$$
 (b)  $f(x) = 1/x$  (c)  $f(x) = x^n$ ,  $n \neq -1$ 

#### SOLUTION

(a) If  $F(x) = -\cos x$ , then  $F'(x) = \sin x$ , so an antiderivative of sine is -cosine.

By Theorem 1, the most general antiderivative is  $G(x) = -\cos x + C$ .

(b) Recall from Section 3.7 that

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

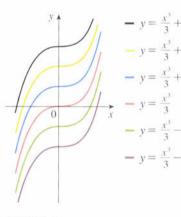
So on the interval  $(0, \infty)$  the general antiderivative of 1/x is  $\ln x + C$ . We also learned that

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

for all  $x \neq 0$ . Theorem 1 then tells us that the general antiderivative of f(x) = 1/x is  $\ln |x| + C$  on any interval that doesn't contain 0. In particular, this is true on each of the intervals  $(-\infty, 0)$  and  $(0, \infty)$ .

(c) We use the Power Rule to discover an antiderivative of  $x^n$ . In fact, if  $n \neq -1$ , then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$



#### FIGURE I

Members of the family of antiderivatives of  $f(x) = x^2$ 

Thus, the general antiderivative of  $f(x) = x^n$  is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for  $n \ge 0$  since then  $f(x) = x^n$  is defined on the interval  $(-\infty, \infty)$ . If *n* is negative (but  $n \ne -1$ ), it is valid on any interval that doesn't contain 0.

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives. Each formula in the table is true because the derivative of the function in the right column appears in the left column. In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation F' = f, G' = g.)

#### 2 Table of Antidifferentiation Formulas

To obtain the most general antiderivative (on an interval) from the particular ones in Table 2 we have to add a constant, as in Example 1.

Function	Particular antiderivative		
cf(x)	cF(x)		
f(x) + g(x)	F(x) + G(x)		
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$		
I/x	$\ln  x $		
e '	e *		
cosx	sin x		
sin x	$-\cos x$		
sec <sup>2</sup> x	tan x		
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$		
$\frac{1}{1+x^2}$	$\tan^{-1}x$		

#### **EXAMPLE 2** Find all functions g such that

$$g'(x) = 4\sin x - 3x^5 + 6\sqrt[4]{x^3}$$

SOLUTION We want to find an antiderivative of

$$f(x) = q'(x) = 4\sin x - 3x^5 + 6x^{3/4}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$g(x) = 4(-\cos x) - 3\frac{x^6}{6} + 6\frac{x^{7/4}}{\frac{7}{4}} + C$$
$$= -4\cos x - \frac{x^6}{2} + \frac{24}{7}x^{7/4} + C$$

#### SECTION 4.9 ANTIDERIVATIVES 335

In applications of calculus it is very common to have a situation as in Example 2, where it is required to find a function, given knowledge about its derivatives. An equation that involves the derivatives of a function is called a **differential equation**. These will be studied in some detail in Chapter 7, but for the present we can solve some elementary differential equations. The general solution of a differential equation involves an arbitrary constant (or constants) as in Example 2. However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

**EXAMPLE 3** Find f if 
$$f'(x) = e^x + 20(1 + x^2)^{-1}$$
 and  $f(0) = -2$ .

SOLUTION The general antiderivative of

$$f'(x) = e^{x} + \frac{20}{1 + x^{2}}$$
$$f(x) = e^{x} + 20 \tan^{-1} x + C$$

To determine C we use the fact that f(0) = -2:

$$f(0) = e^0 + 20\tan^{-1}0 + C = -2$$

Thus, we have C = -2 - 1 = -3, so the particular solution is

$$f(x) = e^x + 20\tan^{-1}x - 3$$

**EXAMPLE 4** Find f if 
$$f''(x) = 12x^2 + 6x - 4$$
,  $f(0) = 4$ , and  $f(1) = 1$ .

**SOLUTION** The general antiderivative of  $f''(x) = 12x^2 + 6x - 4$  is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that f(0) = 4 and f(1) = 1. Since f(0) = 0 + D = 4, we have D = 4. Since

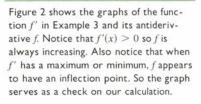
$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have C = -3. Therefore, the required function is

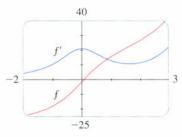
$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

**EXAMPLE 5** If  $f(x) = \sqrt{1 + x^3} - x$ , sketch the graph of the antiderivative F that satisfies the initial condition F(-1) = 0.

**SOLUTION** We could try all day to think of a formula for an antiderivative of f and still be unsuccessful. A second possibility would be to draw the graph of f first and then use it to graph F as in Example 4 in Section 2.10. That would work, but instead let's create a more accurate graph by using what is called a **direction field**.

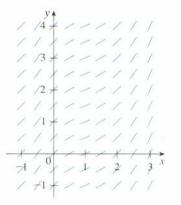


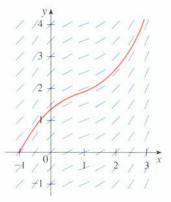
is





Since f(0) = 1, the graph of *F* has slope 1 when x = 0. So we draw several short tangent segments with slope 1, all centered at x = 0. We do the same for several other values of *x* and the result is shown in Figure 3. It is called a direction field because each segment indicates the direction in which the curve y = F(x) proceeds at that point.





#### FIGURE 3

A direction field for  $f(x) = \sqrt{1 + x^3 - x}$ . The slope of the line segments above x = a is f(a).

**FIGURE 4** The graph of an antiderivative follows the direction field.

Now we use the direction field to sketch the graph of F. Because of the initial condition F(-1) = 0, we start at the point (-1, 0) and draw the graph so that it follows the directions of the tangent segments. The result is pictured in Figure 4. Any other antiderivative would be obtained by shifting the graph of f upward or downward.

#### **Rectilinear Motion**

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function s = f(t), then the velocity function is v(t) = s'(t). This means that the position function is an antiderivative of the velocity function. Likewise, the acceleration function is a(t) = v'(t), so the velocity function is an antiderivative of the acceleration. If the acceleration and the initial values s(0) and v(0) are known, then the position function can be found by antidifferentiating twice.

**EXAMPLE 6** • A particle moves in a straight line and has acceleration given by a(t) = 6t + 4. Its initial velocity is v(0) = -6 cm/s and its initial displacement is s(0) = 9 cm. Find its position function s(t).

**SOLUTION** Since v'(t) = a(t) = 6t + 4, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that v(0) = C. But we are given that v(0) = -6, so C = -6 and

$$v(t) = 3t^2 + 4t - 6$$

Since v(t) = s'(t), s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives s(0) = D. We are given that s(0) = 9, so D = 9 and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by q. For motion close to the earth we may assume that g is constant, its value being about 9.8 m/s<sup>2</sup> (or 32 ft/s<sup>2</sup>).

**EXAMPLE 7** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

SOLUTION The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is s(t) and the velocity v(t) is decreasing. Therefore, the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -32$$

Taking antiderivatives, we have

$$v(t) = -32t + C$$

To determine C we use the given information that v(0) = 48. This gives 48 = 0 + C, so

$$v(t) = -32t + 48$$

The maximum height is reached when v(t) = 0, that is, after 1.5 s. Since s'(t) = v(t), we antidifferentiate again and obtain

$$s(t) = -16t^2 + 48t + D$$

Using the fact that s(0) = 432, we have 432 = 0 + D, and so

$$s(t) = -16t^2 + 48t + 432$$

The expression for s(t) is valid until the ball hits the ground. This happens when s(t) = 0, that is, when

$$-16t^2 + 48t + 432 = 0$$

 $t^2 - 3t - 27 = 0$ 

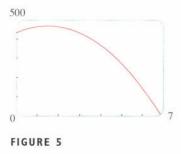
or, equivalently,

$$t = \frac{3 \pm 3\sqrt{13}}{2}$$

2 + 2 /12

We reject the solution with the minus sign since it gives a negative value for t. Therefore, the ball hits the ground after  $3(1 + \sqrt{13})/2 \approx 6.9$  s.

Figure 5 shows the position function of the ball in Example 7. The graph corroborates the conclusions we reached: The ball reaches its maximum height after 1.5 s and hits the ground after 6.9 s.



Exercises

4.9

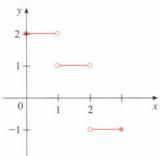
**I**−**I**0 ■ Find the most general antiderivative of the function. Check your answer by differentiation.

1. 
$$f(x) = 12x^{2} + 6x - 5$$
  
2.  $f(x) = 6x^{9} - 4x^{7} + 3x^{2} + 1$   
3.  $f(x) = 6/x^{5}$   
4.  $f(x) = \sqrt[3]{x^{2}} - \sqrt{x^{3}}$   
5.  $g(t) = (t^{3} + 2t^{2})/\sqrt{t}$   
6.  $h(x) = \sin x - 2\cos x$   
7.  $f(t) = \sec^{2}t + t^{2}$   
8.  $f(\theta) = e^{\theta} + \sec \theta \tan \theta$   
9.  $f(x) = 2x + 5(1 - x^{2})^{-1/2}$   
10.  $f(x) = \frac{x^{2} + x + 1}{x}$ 

Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

11. 
$$f(x) = 5x^4 - 2x^5$$
,  $F(0) = 4$   
12.  $f(x) = 4 - 3(1 + x^2)^{-1}$ ,  $F(1) = 0$   
13-20 Find  $f(x)$ .  
13.  $f''(x) = x^2 + x^3$   
14.  $f'''(x) = \sqrt{x}$   
15.  $f'(x) = 2/x$ ,  $x < 0$ ,  $f(-1) = 7$   
16.  $f'(x) = 1 + 1/x^2$ ,  $x > 0$ ,  $f(1) = 1$   
17.  $f''(x) = 6x + 6$ ,  $f(0) = 4$ ,  $f(1) = 3$   
18.  $f''(x) = 12x^2 - 6x + 2$ ,  $f(0) = 1$ ,  $f(2) = 11$   
19.  $f''(x) = 3e^x + 5 \sin x$ ,  $f(0) = 1$ ,  $f'(0) = 2$   
20.  $f''(x) = 3e^x + 5 \sin x$ ,  $f(0) = 1$ ,  $f'(0) = 2$ 

- **21.** Given that the graph of f passes through the point (1, 6) and that the slope of its tangent line at (x, f(x)) is 2x + 1, find f(2).
- **22.** Find a function f such that  $f'(x) = x^3$  and the line x + y = 0 is tangent to the graph of f.
- **23.** The graph of f' is shown. Sketch the graph of f if f is continuous and f(0) = -1.



**24.** (a) Use a graphing device to graph  $f(x) = e^x - 2x$ .

- (b) Starting with the graph in part (a), sketch a rough graph of the antiderivative F that satisfies F(0) = 1.
- (c) Use the rules of this section to find an expression for F(x).
- (d) Graph F using the expression in part (c). Compare with your sketch in part (b).

**25–26** A direction field is given for a function. Use it to draw the antiderivative F that satisfies F(0) = -2.

25. 
$$y^{*}$$
  
 $2 + / / / - \land \land - / | | |$   
 $1 + / / / - \land \land - / | | |$   
 $1 + / / / - \land \land - / | | |$   
 $0 / / / + \land \land - / | | |$   
 $0 / / / + \land \land - / | | |$   
 $0 / / / + \land \land - / | | |$   
 $0 / / / + \land \land - / | | |$   
 $0 / / / + \land \land - / | | |$ 

. . . . . . .

**27–28** Use a direction field to graph the antiderivative that satisfies F(0) = 0.

**27.** 
$$f(x) = \frac{\sin x}{x}, \ 0 < x < 2\pi$$

**28.** 
$$f(x) = x \tan x, \ -\pi/2 < x < \pi/2$$

**29.** A function is defined by the following experimental data. Use a direction field to sketch the graph of its antiderivative if the initial condition is F(0) = 0.

X	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
f'(x)	0	0.2	0.5	0.8	1.0	0.6	0.2	0	-0.1

- **30.** (a) Draw a direction field for the function  $f(x) = 1/x^2$  and use it to sketch several members of the family of antiderivatives.
  - (b) Compute the general antiderivative explicitly and sketch several particular antiderivatives. Compare with your sketch in part (a).
- **31.** A particle moves along a straight line with velocity function  $v(t) = 3\sqrt{t}$  and its initial displacement is s(1) = 5 m. Find its position function s(t).
- **32.** A particle moves with acceleration function  $a(t) = \cos t + \sin t$ . Its initial velocity is v(0) = 5 cm/s and its initial displacement is s(0) = 0 cm. Find its position after t seconds.
- 33. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
  - (a) Find the distance of the stone above ground level at time *t*.
  - (b) How long does it take the stone to reach the ground?
  - (c) With what velocity does it strike the ground?
  - (d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?
- **34.** Show that for motion in a straight line with constant acceleration a, initial velocity  $v_0$ , and initial displacement  $s_0$ , the displacement after time t is

$$s = \frac{1}{2}at^2 + v_0t + s_0$$

**35.** An object is projected upward with initial velocity  $v_0$  meters per second from a point  $s_0$  meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

- **36.** Two balls are thrown upward from the edge of the cliff in Example 7. The first is thrown with a speed of 48 ft/s and the second is thrown 1 s later with a speed of 24 ft/s. Do the balls ever pass each other?
- **37.** A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?
- **38.** A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 40 ft/s<sup>2</sup>. What is the distance covered before the car comes to a stop?
- **39.** What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 s?
- **40.** A car braked with a constant deceleration of 40 ft/s<sup>2</sup>, producing skid marks measuring 160 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?
- **41.** A company estimates that the marginal cost (in dollars per item) of producing x items is 1.92 0.002x. If the

cost of producing one item is \$562, find the cost of producing 100 items.

- 42. The linear density of a rod of length 1 m is given by  $\rho(x) = 1/\sqrt{x}$  in grams per centimeter, where x is measured in centimeters from one end of the rod. Find the mass of the rod.
- **43.** To prove Theorem 1, let F and G be any two antiderivatives of f on I and let H = G - F.
  - (a) If  $x_1$  and  $x_2$  are any two numbers in I with  $x_1 < x_2$ , apply the Mean Value Theorem on the interval  $[x_1, x_2]$  to show that  $H(x_1) = H(x_2)$ . Why does this show that H is a constant function?
  - (b) Deduce Theorem 1 from the result of part (a).
- **44.** Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \le t \le 10\\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?

- **45.** A high-speed "bullet" train accelerates and decelerates at the rate of 4 ft/s<sup>2</sup>. Its maximum cruising speed is 90 mi/h.
  - (a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?
  - (b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?
  - (c) Find the minimum time that the train takes to travel between two consecutive stations that are 45 miles apart.
  - (d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?
- **46.** A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is a(t) = 60t at which time the fuel is exhausted and it becomes a freely "falling" body. After 17 seconds, the rocket's parachute opens, and the (downward) velocity slows linearly to -18 ft/s in 5 s. The rocket then "floats" to the ground at that rate.
  - (a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.
  - (b) At what time does the rocket reach its maximum height and what is that height?
  - (c) At what time does the rocket land?



## Chapter 4 Review

- **1.** Explain the difference between an absolute maximum and a local maximum.
- (a) What does the Extreme Value Theorem say?(b) Explain how the Closed Interval Method works.
- 3. (a) State Fermat's Theorem.(b) Define a critical number of *f*.
- **4.** State the Mean Value Theorem and give a geometrical interpretation.
- 5. (a) State the Increasing/Decreasing Test.(b) State the Concavity Test.
- 6. (a) State the First Derivative Test.
  - (b) State the Second Derivative Test.
  - (c) What are the relative advantages and disadvantages of these tests?
- 7. (a) What does l'Hospital's Rule say?
  - (b) How can you use l'Hospital's Rule if you have a

product f(x)g(x) where  $f(x) \to 0$  and  $g(x) \to \infty$ as  $x \to q$ ?

- (c) How can you use l'Hospital's Rule if you have a difference f(x) − g(x) where f(x) → ∞ and g(x) → ∞ as x → a?
- (d) How can you use l'Hospital's Rule if you have a power  $[f(x)]^{g(x)}$  where  $f(x) \to 0$  and  $g(x) \to 0$  as  $x \to a$ ?
- 8. (a) Given an initial approximation x<sub>1</sub> to a root of the equation f(x) = 0, explain geometrically, with a diagram, how the second approximation x<sub>2</sub> in Newton's method is obtained.
  - (b) Write an expression for  $x_2$  in terms of  $x_1$ ,  $f(x_1)$ , and  $f'(x_1)$ .
  - (c) Write an expression for x<sub>n+1</sub> in terms of x<sub>n</sub>, f(x<sub>n</sub>), and f'(x<sub>n</sub>).
  - (d) Under what circumstances is Newton's method likely to fail or to work very slowly?

#### 🔺 TRUE-FALSE QUIZ 🔺

CONCEPT CHECK •

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

- 1. If f'(c) = 0, then f has a local maximum or minimum at c.
- **2.** If f has an absolute minimum value at c, then f'(c) = 0.
- If f is continuous on (a, b), then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in (a, b).
- **4.** If *f* is differentiable and f(-1) = f(1), then there is a number *c* such that |c| < 1 and f'(c) = 0.

- **5.** If f'(x) < 0 for 1 < x < 6, then *f* is decreasing on (1, 6).
- 6. If f''(2) = 0, then (2, f(2)) is an inflection point of the curve y = f(x).
- 7. If f'(x) = g'(x) for 0 < x < 1, then f(x) = g(x) for 0 < x < 1.
- 8. There exists a function f such that f(1) = -2, f(3) = 0, and f'(x) > 1 for all x.
- **9.** There exists a function f such that f(x) > 0, f'(x) < 0, and f''(x) > 0 for all x.
- **10.** There exists a function f such that f(x) < 0, f'(x) < 0, and f''(x) > 0 for all x.

🔶 EXERCISES 🧇

**1**−4 ■ Find the local and absolute maximum and minimum values of the function on the given interval.

1. 
$$f(x) = x^3 - 12x + 5$$
, [-5,3]

**2.** 
$$f(x) = \sqrt{x^2 + 4x + 8}$$
, [-3,0]

**3.** 
$$f(x) = x - \sqrt{2} \sin x$$
,  $[0, \pi]$ 

**4.** 
$$f(x) = x^2 e^{-x}$$
, [0,3]

#### 5-12 🔳

- (a) Find the vertical and horizontal asymptotes, if any.
- (b) Find the intervals of increase or decrease.
- (c) Find the local maximum and minimum values.
- (d) Find the intervals of concavity and the inflection points.
- (e) Use the information from parts (a)-(d) to sketch the graph of *f*. Check your work with a graphing device.

5.  $f(x) = x^4 - 3x^3 + 3x^2 - x$ 

6. 
$$f(x) = \frac{1}{x(x-3)^2}$$
  
7.  $f(x) = x + \sqrt{1-x}$   
8.  $y = 4x - \tan x, -\pi/2 < x < \pi/2$   
9.  $y = \sin^{-1}(1/x)$   
10.  $y = e^{2x-x^2}$   
11.  $y = e^x + e^{-3x}$   
12.  $y = \ln(x^2 - 1)$ 

Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f" to estimate the intervals of increase and decrease, maximum and minimum values, intervals of concavity, and inflection points. In Exercise 13 use calculus to find these quantities exactly.

**13.** 
$$f(x) = \frac{x^2 - 1}{x^3}$$
  
**14.**  $f(x) = \frac{\sqrt[3]{x}}{1 - x}$   
**15.**  $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$   
**16.**  $f(x) = \sin x \cos^2 x, \ 0 \le x \le 2\pi$ 

- **17.** Graph  $f(x) = e^{-1/x^2}$  in a viewing rectangle that shows all the main aspects of this function. Estimate the inflection points. Then use calculus to find them exactly.
- **CAS** 18. (a) Graph the function  $f(x) = 1/(1 + e^{1/x})$ .
  - (b) Explain the shape of the graph by computing the limits of f(x) as x approaches ∞, -∞, 0<sup>+</sup>, and 0<sup>-</sup>.
  - (c) Use the graph of *f* to estimate the coordinates of the inflection points.
  - (d) Use your CAS to compute and graph f''.
  - (e) Use the graph in part (d) to estimate the inflection point more accurately.
- **19.** If  $f(x) = \arctan(\cos(3 \arcsin x))$ , use the graphs of f, f', and f'' to estimate the *x*-coordinates of the maximum and minimum points and inflection points of f.
- **20.** If  $f(x) = \ln(2x + x \sin x)$ , use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval (0, 15].
- Investigate the family of functions f(x) = ln(sin x + C). What features do the members of this family have in common? How do they differ? For which values of C is f continuous on (-∞, ∞)? For which values of C does f have no graph at all? What happens as C → ∞?
- **22.** Investigate the family of functions  $f(x) = cxe^{-cx^2}$ . What happens to the maximum and minimum points and the inflection points as *c* changes? Illustrate your conclusions by graphing several members of the family.
  - **23.** For what values of the constants *a* and *b* is (1, 6) a point of inflection of the curve  $y = x^3 + ax^2 + bx + 1$ ?

- **24.** Let  $g(x) = f(x^2)$ , where f is twice differentiable for all x, f'(x) > 0 for all  $x \neq 0$ , and f is concave downward on  $(-\infty, 0)$  and concave upward on  $(0, \infty)$ .
  - (a) At what numbers does *g* have a maximum or minimum value?
  - (b) Discuss the concavity of g.
- **25–32** Evaluate the limit.
- 25.  $\lim_{x \to \pi} \frac{\sin x}{x^2 \pi^2}$ 26.  $\lim_{x \to 0} \frac{e^{ax} e^{bx}}{x}$ 27.  $\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x}$ 28.  $\lim_{x \to 0} \frac{1 + \sin x \cos x}{1 \sin x \cos x}$ 29.  $\lim_{x \to 0} \frac{\ln(1 x) + x + \frac{1}{2}x^2}{x^3}$ 30.  $\lim_{x \to \pi^2} \left(\frac{\pi}{2} x\right) \tan x$ 31.  $\lim_{x \to 0} (\csc^2 x x^{-2})$ 32.  $\lim_{x \to 1} x^{1/(1-x)}$
- **33.** The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is  $\pi/6$ ?
- **34.** A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of  $2 \text{ cm}^3/\text{s}$ , how fast is the water level rising when the water is 5 cm deep?
- **35.** A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?
- **36.** A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?

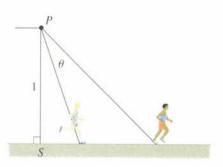


**37.** Show that the shortest distance from the point  $(x_1, y_1)$  to the straight line Ax + By + C = 0 is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

- **38.** Find the point on the hyperbola xy = 8 that is closest to the point (3, 0).
- **39.** Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius *r*.
- **40.** Find the volume of the largest circular cone that can be inscribed in a sphere of radius *r*.

- **41.** In  $\triangle ABC$ , *D* lies on *AB*, |CD| = 5 cm, |AD| = 4 cm, |BD| = 4 cm, and  $CD \perp AB$ . Where should a point *P* be chosen on *CD* so that the sum |PA| + |PB| + |PC| is a minimum? What if |CD| = 2 cm?
- **42.** An observer stands at a point *P*, one unit away from a track. Two runners start at the point *S* in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight  $\theta$  between the runners. [*Hint:* Maximize tan  $\theta$ .]



43. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

- **44.** A metal storage tank with volume *V* is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?
- **45.** A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar that the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?
- **46.** A manufacturer determines that the cost of making *x* units of a commodity is

 $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ 

and the demand function is

$$p(x) = 48.2 - 0.03x$$

- (a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.
- (b) Use calculus to find the production level for maximum profit.
- (c) Estimate the production level that minimizes the average cost.

- **47.** Use Newton's method to find the absolute minimum value of the function  $f(x) = x^6 + 2x^2 8x + 3$  correct to six decimal places.
- **48.** Use Newton's method to find all roots of the equation  $6 \cos x = x$  correct to six decimal places.

**49–50** ■ Find the most general antiderivative of the function.

**49.**  $f(x) = e^x - (1/x)$  **50.**  $g(t) = (1 + t)/\sqrt{t}$  **51.**  $f'(x) = 2/(1 + x^2), f(0) = -1$  **52.**  $f'(x) = 1 + 2\sin x - \cos x, f(0) = 3$  **53.**  $f''(x) = x^3 + x, f(0) = -1, f'(0) = 1$ **54.**  $f''(x) = x^4 - 4x^2 + 3x - 2, f(0) = 0, f(1) = 1$ 

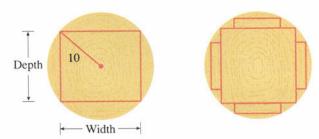
- **55.** (a) If  $f(x) = 0.1e^x + \sin x$ ,  $-4 \le x \le 4$ , use a graph of f to sketch a rough graph of the antiderivative F of f that satisfies F(0) = 0.
  - (b) Find an expression for F(x).
  - (c) Graph F using the expression in part (b). Compare with your sketch in part (a).
  - 56. Sketch the graph of a continuous, even function f such that f(0) = 0, f'(x) = 2x if 0 < x < 1, f'(x) = -1 if 1 < x < 3, and f'(x) = 1 if x > 3.
  - 57. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst or not?
- **58.** Investigate the family of curves given by

$$f(x) = x^4 + x^3 + cx^3$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

- **59.** Show that the equation  $x^{101} + x^{51} + x 1 = 0$  has exactly one real root.
- **60.** Suppose that f(0) = 1 and  $2 \le f'(x) \le 5$  for all x in [0, 4]. Show that  $9 \le f(4) \le 21$ .
- **61.** A rectangular beam will be cut from a cylindrical log of radius 10 inches.
  - (a) Show that the beam of maximal cross-sectional area is a square.
  - (b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.

(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



**62.** If a projectile is fired with an initial velocity v at an angle of inclination  $\theta$  from the horizontal, then its trajectory, neglecting air resistance, is the parabola

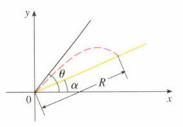
$$y = (\tan \theta)x - \frac{g}{2v^2 \cos^2 \theta}x^2 \qquad 0 \le \theta \le \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle  $\alpha$ ,  $\alpha > 0$ , from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

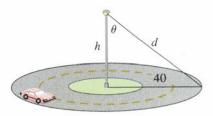
$$R(\theta) = \frac{2v^2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

- (b) Determine  $\theta$  so that R is a maximum.
- (c) Suppose the plane is at an angle  $\alpha$  below the horizontal. Determine the range *R* in this case, and

determine the angle at which the projectile should be fired to maximize R.



- 63. A light is to be placed atop a pole of height h feet to illuminate a busy traffic circle, which has a radius of 40 ft. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle  $\theta$  (see the figure) and inversely proportional to the square of the distance d from the source.
  - (a) How tall should the light pole be to maximize I?
  - (b) Suppose that the light pole is h feet tall and that a woman is walking away from the base of the pole at the rate of 4 ft/s. At what rate is the intensity of the light at the point on her back 4 ft above the ground decreasing when she reaches the outer edge of the traffic circle?





One of the most important principles of problem solving is *analogy* (see page 87). If you are having trouble getting started on a problem, it is sometimes helpful to start by solving a similar, but simpler, problem. The following example illustrates the principle.

**Example** If x, y, and z are positive numbers, prove that

Cover up the solution and try solving the problem yourself first.

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \ge 8$$

Solution It may be difficult to get started on this problem. (Some students have tackled it by multiplying out the numerator, but that just creates a mess.) Let's try to think of a similar, simpler problem. When several variables are involved, it's often helpful to think of an analogous problem with fewer variables. In the present case we can reduce the number of variables from three to one and prove the analogous inequality

$$\frac{x^2 + 1}{x} \ge 2 \qquad \text{for } x > 0$$

In fact, if we are able to prove (1), then the desired inequality follows because

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} = \left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)\left(\frac{z^2+1}{z}\right) \ge 2 \cdot 2 \cdot 2 = 8$$

The key to proving (1) is to recognize that it is a disguised version of a minimum problem. If we let

$$f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x} \qquad x > 0$$

then  $f'(x) = 1 - (1/x^2)$ , so f'(x) = 0 when x = 1. Also, f'(x) < 0 for 0 < x < 1 and f'(x) > 0 for x > 1. Therefore, the absolute minimum value of f is f(1) = 2. This means that

$$\frac{x^2 + 1}{x} \ge 2 \qquad \text{for all positive values of } x$$

and, as previously mentioned, the given inequality follows by multiplication.

The inequality in (1) could also be proved without calculus. In fact, if x > 0, we have

$$\frac{x^2+1}{x} \ge 2 \quad \Longleftrightarrow \quad x^2+1 \ge 2x \quad \Longleftrightarrow \quad x^2-2x+1 \ge 0$$
$$\iff (x-1)^2 \ge 0$$

Because the last inequality is obviously true, the first one is true too.

Look Back

What have we learned from the solution to this example?

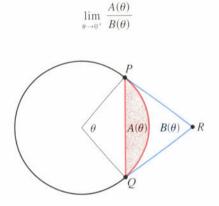
- To solve a problem involving several variables, it might help to solve a similar problem with just one variable.
- When trying to prove an inequality, it might help to think of it as a maximum or minimum problem.

## **Problems**

- 1. If a rectangle has its base on the x-axis and two vertices on the curve  $y = e^{-x^2}$ , show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.
- **2.** Show that  $|\sin x \cos x| \le \sqrt{2}$  for all x.
- 3. Show that, for all positive values of x and y,

$$\frac{e^{x+y}}{xy} \ge e^2$$

- **4.** Let a and b be positive numbers. Show that not both of the numbers a(1 b) and b(1 a) can be greater than  $\frac{1}{4}$ .
- 5. Find the highest and lowest points on the curve  $x^2 + xy + y^2 = 12$ .
- 6. An arc PQ of a circle subtends a central angle  $\theta$  as in the figure. Let  $A(\theta)$  be the area between the chord PQ and the arc PQ. Let  $B(\theta)$  be the area between the tangent lines PR, QR, and the arc. Find



7. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2|}$$

- 8. Find a function f such that  $f'(-1) = \frac{1}{2}$ , f'(0) = 0, and f''(x) > 0 for all x, or prove that such a function cannot exist.
- **9.** Show that, for x > 0,

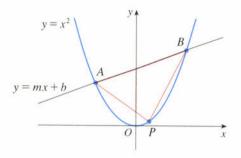
$$\frac{x}{1+x^2} < \tan^{-1}x < x$$

10. Sketch the region in the plane consisting of all points (x, y) such that

$$|2xy \le |x - y| \le x^2 + y^2$$

345

11. The line y = mx + b intersects the parabola  $y = x^2$  in points A and B (see the figure). Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.



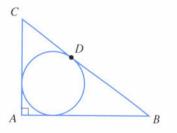
**12.** For what value of a is the following equation true?

$$\lim_{x \to \infty} \left( \frac{x+a}{x-a} \right)^x = e$$

- **13.** A triangle with sides a, b, and c varies with time t, but its area never changes. Let  $\theta$  be the angle opposite the side of length a and suppose  $\theta$  always remains acute.
  - (a) Express  $d\theta/dt$  in terms of b, c,  $\theta$ , db/dt, and dc/dt.
  - (b) Express da/dt in terms of the quantities in part (a).
- 14. Sketch the set of all points (x, y) such that  $|x + y| \le e^x$ .
- **15.** Let ABC be a triangle with  $\angle BAC = 120^{\circ}$  and  $|AB| \cdot |AC| = 1$ .
  - (a) Express the length of the angle bisector AD in terms of x = |AB|.
  - (b) Find the largest possible value of |AD|.
- **16.** (a) Let *ABC* be a triangle with right angle A and hypotenuse a = |BC| (see the figure). If the inscribed circle touches the hypotenuse at D, show that

$$|CD| = \frac{1}{2}(|BC| + |AC| - |AB|)$$

- (b) If  $\theta = \frac{1}{2} \angle C$ , express the radius r of the inscribed circle in terms of a and  $\theta$ .
- (c) If a is fixed and  $\theta$  varies, find the maximum value of r.



17. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal.

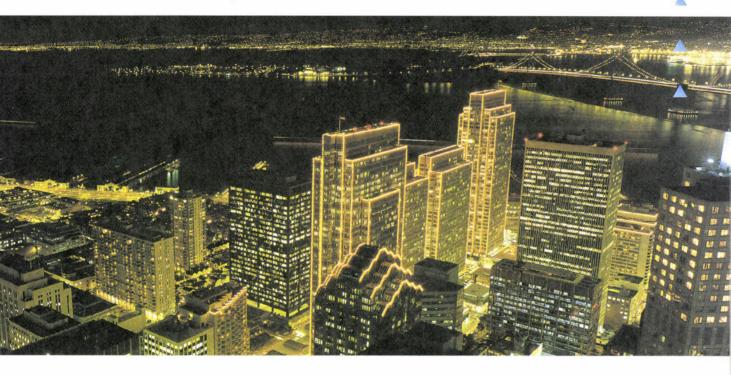
- **18.** *ABCD* is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from *B* to *D* with center *A*. The piece of paper is folded along *EF*, with *E* on *AB* and *F* on *AD*, so that *A* falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF could have.
- 19. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is  $\pi rl$ , where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm<sup>3</sup>/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?
- **20.** A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?





Integrals

• • • Integrals are involved in each of the situations pictured here: using the rate at which oil leaks from a tank to find the amount leaked over a certain time period; using velocity readings of the space shuttle Endeavour to calculuate the height it has reached at a given time; using the knowledge of power consumption to find the energy used on a given day in San Francisco. These problems are solved in Section 5.3.



In Chapter 2 we used the tangent and velocity problems to introduce the derivative, which is the central idea in differential calculus. In much the same way, this chapter starts with the area and distance problems and uses them to formulate the idea of a definite integral, which is the basic concept of integral calculus. We will see in Chapters 6 and 7 how to use the integral to solve problems concerning volumes, lengths of curves, population predictions, cardiac output, forces on a dam, work, consumer surplus, and baseball, among many others.

There is a connection between integral calculus and differential calculus. The Fundamental Theorem of Calculus relates the integral to the derivative, and we will see in this chapter that it greatly simplifies the solution of many problems.



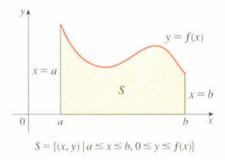
# Areas and Distances

In this section we discover that in attempting to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.



## The Area Problem

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve y = f(x) from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where  $f(x) \ge 0$ ], the vertical lines x = a and x = b, and the x-axis.





In trying to solve the area problem we have to ask ourselves: What is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

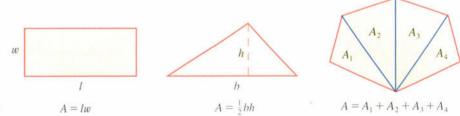


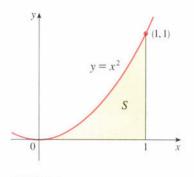
FIGURE 2

However, it is not so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles. The following example illustrates the procedure.

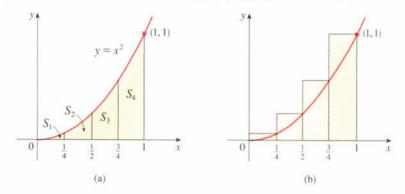
**EXAMPLE I** Use rectangles to estimate the area under the parabola  $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

Now is a good time to read (or reread) A Preview of Calculus (see page 2). It discusses the unifying ideas of calculus and helps put in perspective where we have been and where we are going.





**SOLUTION** We first notice that the area of *S* must be somewhere between 0 and 1 because *S* is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide *S* into four strips  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$  by drawing the vertical lines  $x = \frac{1}{4}$ ,  $x = \frac{1}{2}$ , and  $x = \frac{3}{4}$  as in Figure 4(a). We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)]. In other words, the heights of these rectangles are the values of the function  $f(x) = x^2$  at the right endpoints of the subintervals  $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], and [\frac{3}{4}, 1].$ 



Each rectangle has width  $\frac{1}{4}$  and the heights are  $(\frac{1}{4})^2$ ,  $(\frac{1}{2})^2$ ,  $(\frac{3}{4})^2$ , and  $1^2$ . If we let  $R_4$  be the sum of the areas of these approximating rectangles, we get

 $R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = \frac{15}{32} = 0.46875$ 

From Figure 4(b) we see that the area A of S is less than  $R_4$ , so

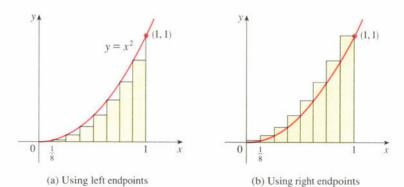
A < 0.46875

Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of f at the left-hand endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

We see that the area of S is larger than  $L_4$ , so we have lower and upper estimates for A:

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region *S* into eight strips of equal width.



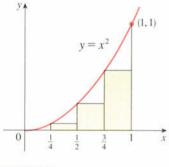


FIGURE 4

FIGURE 5

**FIGURE 6** Approximating *S* with eight rectangles

By computing the sum of the areas of the smaller rectangles  $(L_8)$  and the sum of the areas of the larger rectangles  $(R_8)$ , we obtain better lower and upper estimates for A:

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips. The table shows the results of similar calculations (with a computer) using *n* rectangles whose heights are found with left-hand endpoints ( $L_n$ ) or right-hand endpoints ( $R_n$ ). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.328335 and 0.3338335. A good estimate is obtained by averaging these numbers:  $A \approx 0.3333335$ .

From the values in the table it looks as if  $R_n$  is approaching  $\frac{1}{3}$  as *n* increases. We confirm this in the next example.

**EXAMPLE 2** For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches  $\frac{1}{3}$ , that is,

$$\lim_{n\to\infty}R_n=\frac{1}{3}$$

**SOLUTION**  $R_n$  is the sum of the areas of the *n* rectangles in Figure 7. Each rectangle has width 1/n and the heights are the values of the function  $f(x) = x^2$  at the points  $1/n, 2/n, 3/n, \ldots, n/n$ ; that is, the heights are  $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$ . Thus

$$R_n = \frac{1}{n} \left(\frac{1}{n}\right)^2 + \frac{1}{n} \left(\frac{2}{n}\right)^2 + \frac{1}{n} \left(\frac{3}{n}\right)^2 + \dots + \frac{1}{n} \left(\frac{n}{n}\right)^2$$
$$= \frac{1}{n} \cdot \frac{1}{n^2} (1^2 + 2^2 + 3^2 + \dots + n^2)$$
$$= \frac{1}{n^3} (1^2 + 2^2 + 3^2 + \dots + n^2)$$

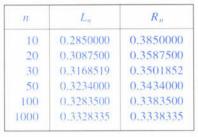
Here we need the formula for the sum of the squares of the first *n* positive integers:

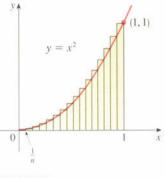
 $1^{2} + 2^{2} + 3^{2} + \dots + n^{2} = \frac{n(n+1)(2n+1)}{6}$ 

Perhaps you have seen this formula before. One method of proving it is by mathematical induction (see Exercise 12).

Putting Formula 1 into our expression for  $R_n$ , we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$







Thus, we have

Here we are computing the limit of the sequence  $\{R_n\}$ . Sequences were discussed in A Preview of Calculus and will be studied in detail in Chapter 8. Their limits are calculated in the same way as limits at infinity (Section 2.5). In particular, we know that

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2}$$
  
=  $\lim_{n \to \infty} \frac{1}{6} \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right)$   
=  $\lim_{n \to \infty} \frac{1}{6} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right)$   
=  $\frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}$ 

It can be shown that the lower approximating sums also approach  $\frac{1}{3}$ , that is,

$$\lim_{n\to\infty}L_n=\frac{1}{3}$$

From Figures 8 and 9 it appears that, as *n* increases, both  $L_n$  and  $R_n$  become better and better approximations to the area of *S*. Therefore, we *define* the area *A* to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

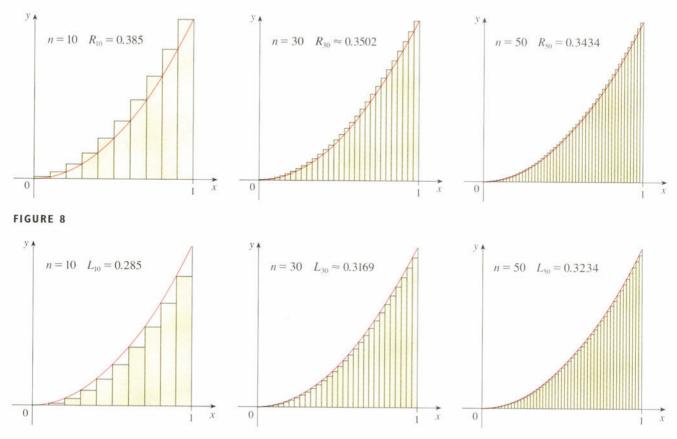


FIGURE 9

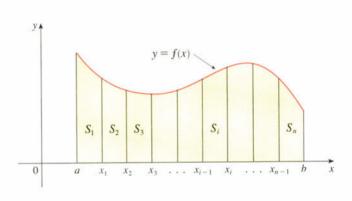
Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips  $S_1, S_2, \ldots, S_n$  of equal width as in Figure 10. The width of the interval [a, b] is b - a, so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n}$$

These strips divide the interval [a, b] into n subintervals

 $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ 

where  $x_0 = a$  and  $x_n = b$ . The right-hand endpoints of the subintervals are

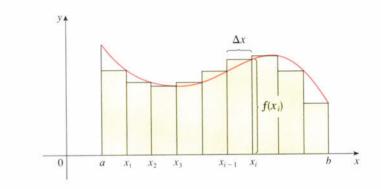


 $x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \quad \dots$ 

FIGURE 10

Let's approximate the *i*th strip  $S_i$  by a rectangle with width  $\Delta x$  and height  $f(x_i)$ , which is the value of f at the right-hand endpoint (see Figure 11). Then the area of the *i*th rectangle is  $f(x_i) \Delta x$ . What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$





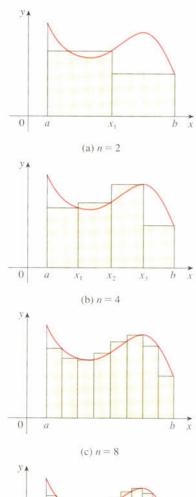


Figure 12 shows this approximation for n = 2, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as  $n \to \infty$ . Therefore, we define the area A of the region S in the following way.

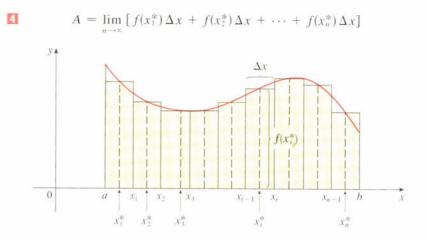
**2** Definition The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of areas of approximating rectangles:

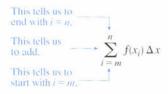
$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[ f(x_1) \,\Delta x + f(x_2) \,\Delta x + \dots + f(x_n) \,\Delta x \right]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

$$A = \lim L_n = [f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the *i*th rectangle to be the value of *f* at *any* number  $x_i^*$  in the *i*th subinterval  $[x_{i-1}, x_i]$ . We call the numbers  $x_1^*, x_2^*, \ldots, x_n^*$  the **sample points**. Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of *S* is





(d) n = 12

b

FIGURE 13

0 a

FIGURE 12

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^n f(x_i) \Delta x = f(x_1) \Delta x + f(x_2) \Delta x + \dots + f(x_n) \Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1}) \Delta x$$
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

We could also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**EXAMPLE 3** Let A be the area of the region that lies under the graph of  $f(x) = e^{-x}$  between x = 0 and x = 2.

(a) Using right endpoints, find an expression for *A* as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

#### SOLUTION

(a) Since a = 0 and b = 2, the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

So  $x_1 = 2/n$ ,  $x_2 = 4/n$ ,  $x_3 = 6/n$ ,  $x_i = 2i/n$ , and  $x_n = 2n/n$ . The sum of the areas of the approximating rectangles is

$$R_{n} = f(x_{1}) \Delta x + f(x_{2}) \Delta x + \dots + f(x_{n}) \Delta x$$
  
=  $e^{-x_{1}} \Delta x + e^{-x_{2}} \Delta x + \dots + e^{-x_{n}} \Delta x$   
=  $e^{-2/n} \left(\frac{2}{n}\right) + e^{-4/n} \left(\frac{2}{n}\right) + \dots + e^{-2n/n} \left(\frac{2}{n}\right)$ 

According to Definition 2, the area is

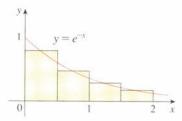
$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{2}{n} \left( e^{-2/n} + e^{-4/n} + e^{-6/n} + \dots + e^{-2n/n} \right)$$

Using sigma notation we could write

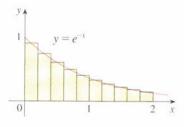
$$A = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

It is difficult to evaluate this limit directly by hand, but with the aid of a computer algebra system it is not hard (see Exercise 16). In Section 5.3 we will be able to find A more easily using a different method.

(b) With n = 4 the subintervals of equal width  $\Delta x = 0.5$  are [0, 0.5], [0.5, 1], [1, 1.5], and [1.5, 2]. The midpoints of these subintervals are  $x_1^* = 0.25$ ,









$$M_4 = \sum_{i=1}^{4} f(x_i^*) \Delta x$$
  
=  $f(0.25) \Delta x + f(0.75) \Delta x + f(1.25) \Delta x + f(1.75) \Delta x$   
=  $e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$   
=  $\frac{1}{2}(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) \approx 0.8557$ 

So an estimate for the area is

$$A \approx 0.8557$$

With n = 10 the subintervals are [0, 0.2], [0.2, 0.4], ..., [1.8, 2] and the midpoints are  $x_1^* = 0.1$ ,  $x_2^* = 0.3$ ,  $x_3^* = 0.5$ , ...,  $x_{10}^* = 1.9$ . Thus

$$A \approx M_{10} = f(0.1)\Delta x + f(0.3)\Delta x + f(0.5)\Delta x + \dots + f(1.9)\Delta x$$
$$= 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5} + \dots + e^{-1.9}) \approx 0.8632$$

From Figure 15 it appears that this estimate is better than the estimate with n = 4.

## The Distance Problem

Now let's consider the *distance problem:* Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. (In a sense this is the inverse problem of the velocity problem that we discussed in Section 2.1.) If the velocity remains constant, then the distance problem is easy to solve by means of the formula

distance = velocity  $\times$  time

But if the velocity varies, it is not so easy to find the distance traveled. We investigate the problem in the following example.

**EXAMPLE 4** Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	1.5	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when t = 5 s. So our estimate for the distance traveled from t = 5 s to t = 10 s is

31 ft/s 
$$\times$$
 5 s = 155 ft

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

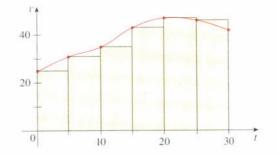
 $25 \times 5 + 31 \times 5 + 35 \times 5 + 43 \times 5 + 47 \times 5 + 46 \times 5 = 1135$  ft

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

 $31 \times 5 + 35 \times 5 + 43 \times 5 + 47 \times 5 + 46 \times 5 + 41 \times 5 = 1215$  ft

If we had wanted a more accurate estimate, we could have taken velocity readings every two seconds, or even every second.

Perhaps the calculations in Example 4 remind you of the sums we used earlier to estimate areas. The similarity is explained when we sketch a graph of the velocity function of the car in Figure 16. The area of the first rectangle is  $25 \times 5 = 125$ , which is also our estimate for the distanced traveled in the first five seconds. In fact, the area of each rectangle can be interpreted as a distance because the height represents velocity and the width represents time. The sum of the areas of the rectangles in Figure 16 is  $L_6 = 1135$ , which is our initial estimate for the total distance traveled.



#### FIGURE 16

In general, suppose an object moves with velocity v = f(t), where  $a \le t \le b$ and  $f(t) \ge 0$  (so the object always moves in the positive direction). We take velocity readings at times  $t_0 (= a), t_1, t_2, ..., t_n (= b)$  so that the velocity is approximately constant on each subinterval. If these times are equally spaced, then the time between consecutive readings is  $\Delta t = (b - a)/n$ . During the first time inter-

val the velocity is approximately  $f(t_0)$  and so the distance traveled is approximately  $f(t_0) \Delta t$ . Similarly, the distance traveled during the second time interval is about  $f(t_1) \Delta t$  and the total distance traveled during the time interval [a, b] is approximately

$$f(t_0) \Delta t + f(t_1) \Delta t + \dots + f(t_{n-1}) \Delta t = \sum_{i=1}^n f(t_{i-1}) \Delta t$$

If we use the velocity at right-hand endpoints instead of left-hand endpoints, our estimate for the total distance becomes

$$f(t_1)\Delta t + f(t_2)\Delta t + \dots + f(t_n)\Delta t = \sum_{i=1}^n f(t_i)\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance *d* traveled is the *limit* of such expressions:

$$d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1}) \Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \Delta t$$

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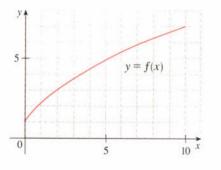
We will see in Section 5.3 that this is indeed true.

Because Equation 5 has the same form as our expressions for area in Equations 2 and 3, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapter 6 we will see that other quantities of interest in the natural and social sciences—such as the work done by a variable force or the cardiac output of the heart—can also be interpreted as the area under a curve. So when we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.



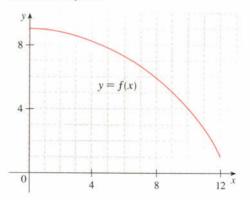
Exercises

- (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from x = 0 to x = 10. In each case sketch the rectangles that you use.
  - (b) Find new estimates using 10 rectangles in each case.



2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from x = 0 to x = 12.

- (i)  $L_6$  (sample points are left endpoints)
- (ii)  $R_6$  (sample points are right endpoints)
- (iii)  $M_6$  (sample points are midpoints)
- (b) Is L<sub>6</sub> an underestimate or overestimate of the true area?
- (c) Is R<sub>6</sub> an underestimate or overestimate of the true area?
- (d) Which of the numbers  $L_6$ ,  $R_6$ , or  $M_6$  gives the best estimate? Explain.



- 3. (a) Estimate the area under the graph of f(x) = x<sup>3</sup> + 2 from x = -1 to x = 2 using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
  - (b) Repeat part (a) using left endpoints.
  - (c) Repeat part (a) using midpoints.
  - (d) From your sketches in parts (a), (b), and (c), which appears to be the best estimate?
- **4.** (a) Graph the function  $f(x) = e^{-x^2}$ ,  $-2 \le x \le 2$ .
  - (b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be
    - (i) right endpoints(ii) midpointsIn each case sketch the curve and the rectangles.
  - (c) Improve your estimates in part (b) by using eight rectangles.

**5–6** With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for n = 10, 30, and 50. Then guess the value of the exact area.

- 5. The region under  $y = \sin x$  from 0 to  $\pi$
- 6. The region under  $y = 1/x^2$  from 1 to 2
- 7. Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x<sub>i</sub><sup>\*</sup> is a left or right endpoint. (For instance, in Maple use leftbox, rightbox, leftsum, and rightsum.)
  - (a) If  $f(x) = \sqrt{x}$ ,  $1 \le x \le 4$ , find the left and right sums for n = 10, 30, and 50.
  - (b) Illustrate by graphing the rectangles in part (a).
  - (c) Show that the exact area under *f* lies between 4.6 and 4.7.
- **6.** (a) If  $f(x) = \sin(\sin x)$ ,  $0 \le x \le \pi/2$ , use the commands discussed in Exercise 7 to find the left and right sums for n = 10, 30, and 50.
  - (b) Illustrate by graphing the rectangles in part (a).
  - (c) Show that the exact area under *f* lies between 0.87 and 0.91.
  - **9.** The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

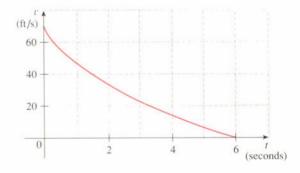
$T(\mathbf{s})$	0	0.5	1.0	1.5	2.0	2.5	3.0
v (f1/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

10. When we estimate distances from velocity data it is sometimes necessary to use times  $t_0, t_1, t_2, t_3, ...$  that are not equally spaced. We can still estimate distances using the time periods  $\Delta t_i = t_i - t_{i-1}$ . For example, on May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (ft/s)		
Launch	()	0		
Begin roll maneuver	10	185		
End roll maneuver	1.5	319		
Throttle to 89%	20	447		
Throttle to 67%	32	742		
Throttle to 104%	59	1325		
Maximum dynamic pressure	62	1445		
Solid rocket booster separation	125	4151		

Use these data to estimate the height above Earth's surface of the space shuttle *Endeavour*, 62 seconds after liftoff.

**II.** The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.



12. Prove Formula 1, that is,

$$1^{2} + 2^{2} + 3^{2} + \dots + n^{2} = \frac{n(n+1)(2n+1)}{6}$$

using mathematical induction (see the *Principles of Problem Solving*, page 88).

**13.** Use mathematical induction to prove the formula for the sum of the cubes of the first *n* positive integers:

$$1^{3} + 2^{3} + 3^{3} + \dots + n^{3} = \left[\frac{n(n+1)}{2}\right]^{2}$$

- 14. (a) Use Definition 2 to find an expression for the area under the curve  $y = x^3$  from 0 to 1 as a limit.
  - (b) Evaluate the limit in part (a) with the help of the formula in Exercise 13.
- **IS.** (a) Express the area under the curve  $y = x^5$  from 0 to 2 as a limit.
  - (b) Use a computer algebra system to find the sum in your expression from part (a).
  - (c) Evaluate the limit in part (a).
- **CAS** 16. Find the exact area of the region under the graph of  $y = e^{-x}$  from 0 to 2 by using a computer algebra system to evaluate the sum and then the limit in Example 3(a).

Compare your answer with the estimate obtained in Example 3(b).

- **17.** Find the exact area under the cosine curve  $y = \cos x$ from x = 0 to x = b, where  $0 \le b \le \pi/2$ . (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if  $b = \pi/2$ ?
  - **18.** (a) Let  $A_n$  be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle  $2\pi/n$ , show that  $A_n = \frac{1}{2}nr^2\sin(2\pi/n)$ .
    - (b) Show that  $\lim_{n\to\infty} A_n = \pi r^2$ . [*Hint*: Use Equation 2 in Section 3.4.]



## The Definite Integral

We saw in Section 5.1 that a limit of the form

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} \left[ f(x_1^*) \Delta x + f(x_2^*) \Delta x + \dots + f(x_n^*) \Delta x \right]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapter 6 we will see that limits of the form (1) also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities. We therefore give this type of limit a special name and notation.

**2** Definition of a Definite Integral If f is a continuous function defined for  $a \le x \le b$ , we divide the interval [a, b] into n subintervals of equal width  $\Delta x = (b - a)/n$ . We let  $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$  be the endpoints of these subintervals and we choose sample points  $x_1^*, x_2^*, \ldots, x_n^*$  in these subintervals, so  $x_i^*$  lies in the *i*th subinterval  $[x_{i-1}, x_i]$ . Then the definite integral of f from a to b is

$$\int_{a}^{b} f(x) dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i}^{*}) \Delta x$$

**NOTE 1** • The symbol  $\int$  was introduced by Leibniz and is called an **integral** sign. It is an elongated S and was chosen because an integral is a limit of sums. In the notation  $\int_{a}^{b} f(x) dx$ , f(x) is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx has no meaning by itself;  $\int_{a}^{b} f(x) dx$  is all one symbol. The procedure of calculating an integral is called **integration**.

**NOTE 2** • The definite integral  $\int_a^b f(x) dx$  is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(r) \, dr$$

**NOTE 3** • Because we have assumed that f is continuous, it can be proved that the limit in Definition 2 always exists and gives the same value no matter how we choose the sample points  $x_i^*$ . If we take the sample points to be right-hand endpoints, then  $x_i^* = x_i$  and the definition of an integral becomes

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \, \Delta x$$

If we choose the sample points to be left-hand endpoints, then  $x_i^* = x_{i-1}$  and the definition becomes

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_{i-1}) \, \Delta x$$

Alternatively, we could choose  $x_i^*$  to be the midpoint of the subinterval or any other number between  $x_{i-1}$  and  $x_i$ .

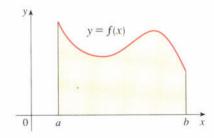
Although most of the functions that we encounter are continuous, the limit in Definition 2 also exists if f has a finite number of removable or jump discontinuities (but not infinite discontinuities). (See Section 2.4.) So we can also define the definite integral for such functions.

NOTE 4 • The sum

3

$$\sum_{i=1}^n f(x_i^*) \, \Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 5.1, we see that the definite integral  $\int_a^b f(x) dx$  can be interpreted as the area under the curve y = f(x) from a to b (see Figure 2).



**FIGURE 2** If  $f(x) \ge 0$ , the integral  $\int_{a}^{b} f(x) dx$  is the area under the curve y = f(x) from *a* to *b*.

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

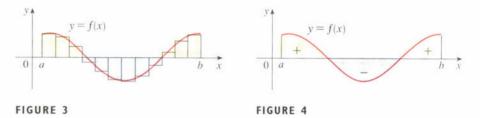


#### SECTION 5.2 THE DEFINITE INTEGRAL 363

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the *x*-axis and the *negatives* of the areas of the rectangles that lie below the *x*-axis (the areas of the gold rectangles *minus* the areas of the blue rectangles). When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x) \, dx = A_1 - A_2$$

where  $A_1$  is the area of the region above the x-axis and below the graph of f and  $A_2$  is the area of the region below the x-axis and above the graph of f.



**EXAMPLE I** Express

 $\lim_{n \to \infty} \sum_{i=1}^{n} \left[ x_i^3 + x_i \sin x_i \right] \Delta x$ 

as an integral on the interval  $[0,\pi]$ .

**SOLUTION** Comparing the given limit with the limit in Definition 2, we see that they will be identical if we choose

$$f(x) = x^3 + x \sin x$$
 and  $x_i^* = x_i$ 

(So the sample points are right endpoints and the given limit is of the form of Equation 3.) We are given that a = 0 and  $b = \pi$ . Therefore, by Definition 2 or Equation 3, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left[ x_i^3 + x_i \sin x_i \right] \Delta x = \int_0^{\pi} (x^3 + x \sin x) \, dx$$

Later, when we apply the definite integral to physical situations, it will be important to recognize limits of sums as integrals, as we did in Example 1. When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process. In general, when we write

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \int_a^b f(x) \, dx$$

we replace  $\lim \Sigma$  by  $\int$ ,  $x_i^*$  by x, and  $\Delta x$  by dx.

### Evaluating Integrals

When we use the definition to evaluate a definite integral, we need to know how to work with sums. The following three equations give formulas for sums of powers of positive integers. Equation 4 may be familiar to you from a course in algebra. Equations 5 and 6 were discussed in Section 5.1.

4  $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ 5  $\sum_{i=1}^{n} i^{2} = \frac{n(n+1)(2n+1)}{6}$ 6  $\sum_{i=1}^{n} i^{3} = \left[\frac{n(n+1)}{2}\right]^{2}$ 

The remaining formulas are simple rules for working with sigma notation:

### EXAMPLE 2

(a) Evaluate the Riemann sum for  $f(x) = x^3 - 6x$  taking the sample points to be right-hand endpoints and a = 0, b = 3, and n = 6.

(b) Evaluate  $\int_{0}^{3} (x^{3} - 6x) dx$ .

#### SOLUTION

(a) With n = 6 the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

and the right endpoints are  $x_1 = 0.5$ ,  $x_2 = 1.0$ ,  $x_3 = 1.5$ ,  $x_4 = 2.0$ ,  $x_5 = 2.5$ , and  $x_6 = 3.0$ . So the Riemann sum is

$$R_{6} = \sum_{i=1}^{6} f(x_{i}) \Delta x$$
  
=  $f(0.5) \Delta x + f(1.0) \Delta x + f(1.5) \Delta x + f(2.0) \Delta x + f(2.5) \Delta x + f(3.0) \Delta x$   
=  $\frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$   
=  $-3.9375$ 

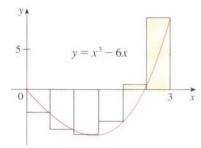
Formulas 7–10 are proved by writing out each side in expanded form. The left side of Equation 8 is

 $ca_1 + ca_2 + \cdots + ca_n$ 

The right side is

 $c(a_1 + a_2 + \cdots + a_n)$ 

These are equal by the distributive property.



Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the gold rectangles (above the x-axis) minus the sum of the areas of the blue rectangles (below the x-axis) in Figure 5.

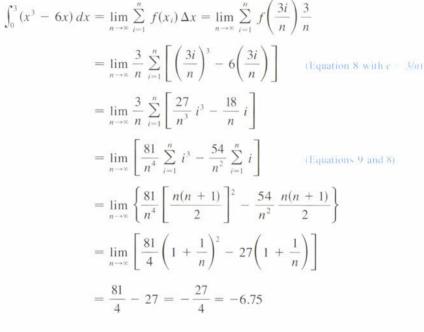
(b) With *n* subintervals we have

are using right endpoints, we can use Equation 3:

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

Thus  $x_0 = 0$ ,  $x_1 = 3/n$ ,  $x_2 = 6/n$ ,  $x_3 = 9/n$ , and, in general,  $x_i = 3i/n$ . Since we

FIGURE 5



 $y = x^{3} - 6x$   $A_{1}$   $A_{2}$   $A_{2}$ 

**FIGURE 7**  $R_{40} \approx -6.3998$  This integral cannot be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas  $A_1 - A_2$ , where  $A_1$  and  $A_2$  are shown in Figure 6.

Figure 7 illustrates the calculation by showing the positive and negative terms in the right Riemann sum  $R_n$  for n = 40. The values in the table show the Riemann sums approaching the exact value of the integral, -6.75, as  $n \to \infty$ .

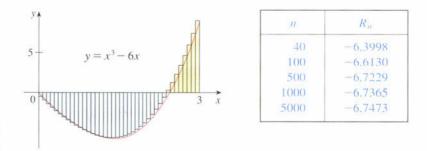


FIGURE 6

A much simpler method for evaluating the integral in Example 2 will be given in Section 5.3 after we have proved the Evaluation Theorem.

### CAS EXAMPLE 3

- (a) Set up an expression for  $\int_{1}^{3} e^{x} dx$  as a limit of sums.
- (b) Use a computer algebra system to evaluate the expression.

#### SOLUTION

(a) Here we have  $f(x) = e^x$ , a = 1, b = 3, and

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

So 
$$x_0 = 1$$
,  $x_1 = 1 + 2/n$ ,  $x_2 = 1 + 4/n$ ,  $x_3 = 1 + 6/n$ , and

$$x_i = 1 + \frac{2i}{n}$$

From Equation 3, we get

$$\int_{1}^{3} e^{x} dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i}) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \frac{2}{n}$$
$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} e^{1 + 2i/n}$$

(b) If we ask a computer algebra system to evaluate the sum and simplify, we obtain

$$\sum_{i=1}^{n} e^{1+2i/n} = \frac{e^{(3n+2)/n} - e^{(n+2)/n}}{e^{2/n} - 1}$$

Now we ask the computer algebra system to evaluate the limit:

$$\int_{1}^{3} e^{x} dx = \lim_{n \to \infty} \frac{2}{n} \cdot \frac{e^{(3n+2)/n} - e^{(n+2)/n}}{e^{2/n} - 1} = e^{3} - e$$

**EXAMPLE 4** ■ Evaluate the following integrals by interpreting each in terms of areas.

(a) 
$$\int_0^1 \sqrt{1 - x^2} \, dx$$
 (b)  $\int_0^3 (x - 1) \, dx$ 

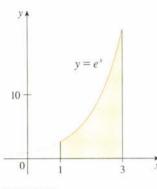
#### SOLUTION

(a) Since  $f(x) = \sqrt{1 - x^2} \ge 0$ , we can interpret this integral as the area under the curve  $y = \sqrt{1 - x^2}$  from 0 to 1. But, since  $y^2 = 1 - x^2$ , we get  $x^2 + y^2 = 1$ , which shows that the graph of f is the quarter-circle with radius 1 in Figure 9. Therefore

$$\int_0^1 \sqrt{1 - x^2} \, dx = \frac{1}{4} \pi (1)^2 = \frac{\pi}{4}$$

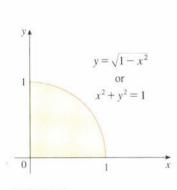
(In Section 5.5 we will be able to *prove* that the area of a circle of radius r is  $\pi r^2$ .)

Because  $f(x) = e^x$  is positive, the integral in Example 3 represents the area shown in Figure 8.





A computer algebra system is able to find an explicit expression for this sum because it is a geometric series. The limit could be found using l'Hospital's Rule.





(b) The graph of y = x - 1 is the line with slope 1 shown in Figure 10. We compute the integral as the difference of the areas of the two triangles:

$$\int_{0}^{3} (x - 1) dx = A_{1} - A_{2} = \frac{1}{2}(2 \cdot 2) - \frac{1}{2}(1 \cdot 1) = 1.5$$

$$y = x - 1$$

$$(3, 2)$$

$$y = x - 1$$

$$A_{1}$$

$$A_{1}$$

$$A_{2}$$

$$A_{2}$$

$$A_{3}$$

$$A_{3}$$

$$A_{3}$$



## The Midpoint Rule

We often choose the sample point  $x_i^*$  to be the right endpoint of the *i*th subinterval because it is convenient for computing the limit. But if the purpose is to find an *approximation* to an integral, it is usually better to choose  $x_i^*$  to be the midpoint of the interval, which we denote by  $\overline{x}_i$ . Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule  $\int_{a}^{b} f(x) dx \approx \sum_{i=1}^{n} f(\bar{x}_{i}) \Delta x = \Delta x [f(\bar{x}_{1}) + \dots + f(\bar{x}_{n})]$ where  $\Delta x = \frac{b-a}{n}$ and  $\bar{x}_{i} = \frac{1}{2} (x_{i-1} + x_{i}) = \text{midpoint of } [x_{i-1}, x_{i}]$ 

**EXAMPLE 5** Use the Midpoint Rule with n = 5 to approximate  $\int_{1}^{2} \frac{1}{x} dx$ .

**SOLUTION** The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. The width of the subintervals is  $\Delta x = (2 - 1)/5 = \frac{1}{5}$ , so the Midpoint Rule gives

$$\int_{1}^{2} \frac{1}{x} dx \approx \Delta x \left[ f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]$$
$$= \frac{1}{5} \left( \frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)$$
$$\approx 0.691908$$

Since f(x) = 1/x > 0 for  $1 \le x \le 2$ , the integral represents an area and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 11.

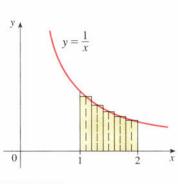


FIGURE II

At the moment we don't know how accurate the approximation in Example 5 is, but in Section 5.8 we will learn a method for estimating the error involved in using the Midpoint Rule. At that time we will discuss other methods for approximating definite integrals.

If we apply the Midpoint Rule to the integral in Example 2, we get the picture in Figure 12. The approximation  $M_{40} \approx -6.7563$  is much closer to the true value -6.75 than the right endpoint approximation,  $R_{40} \approx -6.3998$ , shown in Figure 7.

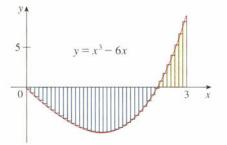


FIGURE 12  $M_{40} \approx -6.7563$ 



We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral 1.  $\int_{a}^{b} c \, dx = c(b - a)$ , where *c* is any constant 2.  $\int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$ 3.  $\int_{a}^{b} c f(x) \, dx = c \int_{a}^{b} f(x) \, dx$ , where *c* is any constant 4.  $\int_{a}^{b} [f(x) - g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx$ 

Property 1 says that the integral of a constant function f(x) = c is the constant times the length of the interval. If c > 0 and a < b, this is to be expected because c(b - a) is the area of the shaded rectangle in Figure 13.

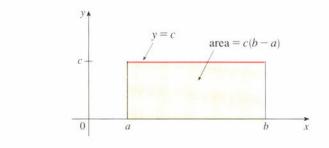
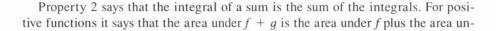
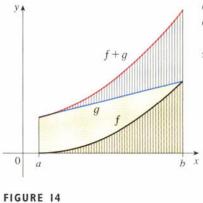
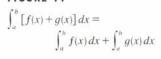


FIGURE 13  $\int_{a}^{b} c \, dx = c(b-a)$ 







Property 3 seems intuitively reasonable because we know that multiplying a function by a positive number cstretches or shrinks its graph vertically by a factor of c. So it stretches or shrinks each approximating rectangle by a factor c and therefore it has the effect of multiplying the area by c. der g. Figure 14 helps us understand why this is true: In view of how graphical addition works, the corresponding vertical line segments have equal height.

In general, Property 2 follows from Equation 3 and the fact that the limit of a sum is the sum of the limits:

$$\int_{a}^{b} [f(x) + g(x)] dx = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)] \Delta x$$
$$= \lim_{n \to \infty} \left[ \sum_{i=1}^{n} f(x_i) \Delta x + \sum_{i=1}^{n} g(x_i) \Delta x \right]$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x + \lim_{n \to \infty} \sum_{i=1}^{n} g(x_i) \Delta x$$
$$= \int_{a}^{b} f(x) dx + \int_{a}^{b} g(x) dx$$

Property 3 can be proved in a similar manner and says that the integral of a constant times a function is the constant times the integral of the function. In other words, a constant (but *only* a constant) can be taken in front of an integral sign. Property 4 is proved by writing f - g = f + (-g) and using Properties 2 and 3 with c = -1.

**EXAMPLE 6** Use the properties of integrals to evaluate  $\int_0^1 (4 + 3x^2) dx$ .

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4+3x^2) \, dx = \int_0^1 4 \, dx + \int_0^1 3x^2 \, dx = \int_0^1 4 \, dx + 3 \, \int_0^1 x^2 \, dx$$

We know from Property 1 that

$$\int_0^1 4 \, dx = 4(1 - 0) = 4$$

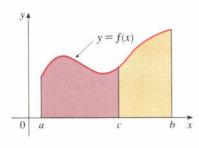
and we found in Example 2 in Section 5.1 that  $\int_0^1 x^2 dx = \frac{1}{3}$ . So

$$\int_0^1 (4 + 3x^2) \, dx = \int_0^1 4 \, dx + 3 \int_0^1 x^2 \, dx$$
$$= 4 + 3 \cdot \frac{1}{3} = 5$$

The final property tells us how to combine integrals of the same function over adjacent intervals:

5.  $\int_{a}^{c} f(x) dx + \int_{c}^{b} f(x) dx = \int_{a}^{b} f(x) dx$ 

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This is not easy to prove in general, but for the case where  $f(x) \ge 0$  and a < c < b Property 5 can be seen from the geometric interpretation in Figure 15: The area under y = f(x) from a to c plus the area from c to b is equal to the total area from a to b.

**EXAMPLE 7** If it is known that  $\int_0^{10} f(x) dx = 17$  and  $\int_0^8 f(x) dx = 12$ , find  $\int_8^{10} f(x) dx$ .

SOLUTION By Property 5, we have

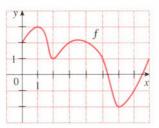
$$\int_0^8 f(x) \, dx \, + \, \int_8^{10} f(x) \, dx = \int_0^{10} f(x) \, dx$$
$$\int_8^{10} f(x) \, dx = \int_0^{10} f(x) \, dx - \int_0^8 f(x) \, dx = 17 - 12 = 5$$

FIGURE 15

 Evaluate the Riemann sum for f(x) = 2 - x<sup>2</sup>, 0 ≤ x ≤ 2, with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

SO

- If f(x) = ln x 1, 1 ≤ x ≤ 4, evaluate the Riemann sum with n = 6, taking the sample points to be left endpoints. (Give your answer correct to six decimal places.) What does the Riemann sum represent? Illustrate with a diagram.
- **3.** The graph of a function f is given. Estimate  $\int_0^{\infty} f(x) dx$  using four subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



4. The table gives the values of a function obtained from an experiment. Use them to estimate  $\int_0^6 f(x) dx$  using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be a decreasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

.X	0	Ţ	2	3	4	5	6
f(x)	9.3	9.()	8.3	6.5	2.3	-7.6	-10.5

5-8 Use the Midpoint Rule with the given value of n to approximate each integral. Round each answer to four decimal places.

5. 
$$\int_0^5 x^3 dx$$
,  $n = 5$   
6.  $\int_1^3 \frac{1}{2x - 7} dx$ ,  $n = 4$   
7.  $\int_1^2 \sqrt{1 + x^2} dx$ ,  $n = 10$   
8.  $\int_0^{\pi/4} \tan x \, dx$ ,  $n = 4$ 

- **9.** If you have a CAS that evaluates midpoint approximations and graphs the corresponding rectangles (use middlesum and middlebox commands in Maple), check the answer to Exercise 7 and illustrate with a graph. Then repeat with n = 20 and n = 30.
  - 10. With a programmable calculator or computer (see the instructions for Exercise 5 in Section 5.1), compute the left and right Riemann sums for the function  $f(x) = \sqrt{1 + x^2}$  on the interval [1, 2] with n = 100. Explain why these estimates show that

$$1.805 < \int_{1}^{2} \sqrt{1 + x^{2}} \, dx < 1.815$$

Deduce that the approximation using the Midpoint Rule with n = 10 in Exercise 7 is accurate to two decimal places.

**11–14** Express the limit as a definite integral on the given interval.

11. 
$$\lim_{n \to \infty} \sum_{i=1}^{n} \cos x_i \Delta x, \quad [0,\pi]$$
  
12. 
$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\tan x_i}{x_i} \Delta x, \quad [2,4]$$

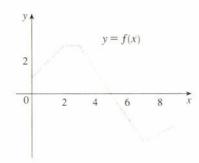
**13.**  $\lim_{n \to \infty} \sum_{i=1}^{n} [2(x_i^*)^2 - 5x_i^*] \Delta x, \quad [0, 1]$ **14.**  $\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i^*} \Delta x, \quad [1, 4]$ 

15-17 = Use the form of the definition of the integral given in Equation 3 to evaluate the integral.

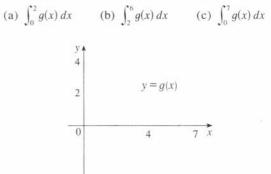
- **15.**  $\int_0^2 (2 x^2) dx$
- **16.**  $\int_0^5 (1 + 2x^3) dx$

**17.** 
$$\int_{1}^{2} x^{3} dx$$

- 18. (a) Find an approximation to the integral  $\int_0^4 (x^2 3x) dx$  using a Riemann sum with right endpoints and n = 8.
  - (b) Draw a diagram like Figure 3 to illustrate the approximation in part (a).
  - (c) Use Equation 3 to evaluate  $\int_0^4 (x^2 3x) dx$ .
  - (d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 4.
- **CNS** 19-20 Express the integral as a limit of sums. Then evaluate, using a computer algebra system to find both the sum and the limit.
  - **19.**  $\int_0^{\pi} \sin 5x \, dx$  **20.**  $\int_2^{10} x^6 \, dx$
  - **21.** The graph of *f* is shown. Evaluate each integral by interpreting it in terms of areas.
    - (a)  $\int_{0}^{2} f(x) dx$  (b)  $\int_{0}^{5} f(x) dx$ (c)  $\int_{5}^{7} f(x) dx$  (d)  $\int_{0}^{9} f(x) dx$



**22.** The graph of *g* consists of two straight lines and a semicircle. Use it to evaluate each integral.



**23–28** Evaluate each integral by interpreting it in terms of areas.

**23.**  $\int_{1}^{3} (1 + 2x) dx$  **24.**  $\int_{-2}^{2} \sqrt{4 - x^{2}} dx$  **25.**  $\int_{-3}^{0} (1 + \sqrt{9 - x^{2}}) dx$  **26.**  $\int_{-1}^{3} (2 - x) dx$  **27.**  $\int_{-2}^{2} (1 - |x|) dx$ **28.**  $\int_{0}^{3} |3x - 5| dx$ 

**29–30** Write the given sum or difference as a single integral in the form  $\int_{a}^{b} f(x) dx$ .

- **29.**  $\int_{1}^{3} f(x) dx + \int_{3}^{6} f(x) dx + \int_{6}^{12} f(x) dx$ **30.**  $\int_{0}^{10} f(x) dx - \int_{0}^{7} f(x) dx$
- **31.** If  $\int_{2}^{8} f(x) dx = 1.7$  and  $\int_{5}^{8} f(x) dx = 2.5$ , find  $\int_{2}^{5} f(x) dx$ .
- **32.** If  $\int_0^1 f(t) dt = 2$ ,  $\int_0^4 f(t) dt = -6$ , and  $\int_3^4 f(t) dt = 1$ , find  $\int_1^3 f(t) dt$ .
- **33.** In Example 2 in Section 5.1 we showed that  $\int_0^1 x^2 dx = \frac{1}{3}$ . Use this fact and the properties of integrals to evaluate  $\int_0^1 (5 6x^2) dx$ .
- **34.** Use the properties of integrals and the result of Example 3 to evaluate  $\int_{1}^{3} (2e^x 1) dx$ .
- **35.** Use the result of Example 3 to evaluate  $\int_{1}^{3} e^{x+2} dx$ .
- 36. Express the following limit as a definite integral:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$$



# Evaluating Definite Integrals

In Section 5.2 we computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. Sir Isaac Newton discovered a much simpler method for evaluating integrals and a few years later Leibniz made the same discovery. They realized that they could calculate  $\int_{a}^{b} f(x) dx$  if they happened to know an antiderivative *F* of *f*. Their discovery is part of the Fundamental Theorem of Calculus, which is discussed in the next section.

**Evaluation Theorem** If f is continuous on the interval [a, b], then

$$\int_{a}^{b} f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, F' = f.

The theorem states that if we know an antiderivative *F* of *f*, then we can evaluate  $\int_a^b f(x) dx$  simply by subtracting the values of *F* at the endpoints of the interval [a, b]. It is very surprising that  $\int_a^b f(x) dx$ , which was defined by a complicated procedure involving all of the values of f(x) for  $a \le x \le b$ , can be found by knowing the values of F(x) at only two points, *a* and *b*.

For instance, we know from Section 4.9 that an antiderivative of  $f(x) = x^2$  is  $F(x) = \frac{1}{3}x^3$ , so the Evaluation Theorem tells us that

$$\int_0^1 x^2 \, dx = F(1) - F(0) = \frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}$$

Comparing this method with the calculation in Example 2 in Section 5.1, where we found the area under the parabola  $y = x^2$  from 0 to 1 by computing a limit of sums, we see that the Evaluation Theorem provides us with a simple and powerful method.

Although the Evaluation Theorem may be surprising at first glance, it becomes plausible if we interpret it in physical terms. If v(t) is the velocity of an object and s(t) is its position at time t, then v(t) = s'(t), so s is an antiderivative of v. In Section 5.1 we considered an object that always moves in the positive direction and made the guess that the area under the velocity curve is equal to the distance traveled. In symbols:

$$\int_a^b v(t) \, dt = s(b) - s(a)$$

That is exactly what the Evaluation Theorem says in this context.

**Proof of the Evaluation Theorem** We divide the interval [a, b] into *n* subintervals with endpoints  $x_0 (= a), x_1, x_2, ..., x_n (= b)$  and with length  $\Delta x = (b - a)/n$ . Let *F* be any antiderivative of *f*. By subtracting and adding like terms, we can

express the total difference in the F values as the sum of the differences over the subintervals:

$$F(b) - F(a) = F(x_n) - F(x_0)$$
  
=  $F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \dots + F(x_2)$   
 $- F(x_1) + F(x_1) - F(x_0)$   
=  $\sum_{i=1}^n [F(x_i) - F(x_{i-1})]$ 

Now *F* is continuous (because it's differentiable) and so we can apply the Mean Value Theorem to *F* on each subinterval  $[x_{i-1}, x_i]$ . Thus, there exists a number  $x_i^*$  between  $x_{i-1}$  and  $x_i$  such that

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*)\Delta x$$
$$F(b) - F(a) = \sum_{i=1}^n f(x_i^*)\Delta x$$

Now we take the limit of each side of this equation as  $n \to \infty$ . The left side is a constant and the right side is a Riemann sum for the function *f*, so

$$F(b) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \int_a^b f(x) \, dx$$

When applying the Evaluation Theorem we use the notation

$$F(x)\Big|_a^b = F(b) - F(a)$$

and so we can write

Therefore

$$\int_{a}^{b} f(x) dx = F(x) \Big]_{a}^{b} \quad \text{where} \quad F' = f$$

Other common notations are  $F(x)|_a^b$  and  $[F(x)]_a^b$ .

**EXAMPLE 1** Evaluate  $\int_{1}^{3} e^{x} dx$ .

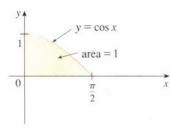
**SOLUTION** An antiderivative of  $f(x) = e^x$  is  $F(x) = e^x$ , so we use the Evaluation Theorem as follows:

$$\int_{1}^{3} e^{x} dx = e^{x} \Big]_{1}^{3} = e^{3} - e^{3}$$

If you compare the calculation in Example 1 with the one in Example 3 in Section 5.2, you will see that the Evaluation Theorem provides a much shorter method.

The Mean Value Theorem was discussed in Section 4.3.

In applying the Evaluation Theorem we use a particular antiderivative F of f. It is not necessary to use the most general antiderivative.





**EXAMPLE 2** Find the area under the cosine curve from 0 to b, where  $0 \le b \le \pi/2$ .

**SOLUTION** Since an antiderivative of  $f(x) = \cos x$  is  $F(x) = \sin x$ , we have

$$A = \int_0^b \cos x \, dx = \sin x \Big]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking  $b = \pi/2$ , we have proved that the area under the cosine curve from 0 to  $\pi/2$  is  $\sin(\pi/2) = 1$  (see Figure 1).

When the French mathematician Gilles de Roberval first found the area under the sine and cosine curves in 1635, this was a very challenging problem that required a great deal of ingenuity. If we didn't have the benefit of the Evaluation Theorem, we would have to compute a difficult limit of sums using obscure trigonometric identities (or a computer algebra system as in Exercise 17 in Section 5.1). It was even more difficult for Roberval because the apparatus of limits had not been invented in 1635. But in the 1660s and 1670s, when the Evaluation Theorem was discovered by Newton and Leibniz, such problems became very easy, as you can see from Example 2.

## Indefinite Integrals

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation given by the Evaluation Theorem between antiderivatives and integrals, the notation  $\int f(x) dx$  is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

 $\int f(x) dx = F(x)$  means F'(x) = f(x)

The connection between them is given by the Evaluation Theorem. If f is continuous on [a, b], then

$$\int_a^b f(x) \, dx = \int f(x) \, dx \Big]_a^b$$

Recall from Section 4.9 that if F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is F(x) + C, where C is an arbitrary constant. For instance, the formula

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

is valid (on any interval that doesn't contain 0) because  $(d/dx) \ln |x| = 1/x$ . So an indefinite integral  $\int f(x) dx$  can represent either a particular antiderivative of f or an entire *family* of antiderivatives (one for each value of the constant C).

The effectiveness of the Evaluation Theorem depends on having a supply of antiderivatives of functions. We therefore restate the Table of Antidifferentiation Formulas from Section 4.9, together with a few others, in the notation of indefinite integrals. Any formula can be verified by differentiating the function on the right side and obtaining the integrand. For instance,

$$\int \sec^2 x \, dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

**Table of Indefinite Integrals** 

$$\int cf(x) dx = c \int f(x) dx$$
  

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$
  

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x} dx = \ln|x| + C$$
  

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$
  

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$
  

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C$$
  

$$\int \sec x \tan x dx = \sec x + C \qquad \int \csc x \cot x dx = -\csc x + C$$
  

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1}x + C$$

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.

**EXAMPLE 3** Find the general indefinite integral The indefinite integral in Example 3 is graphed in Figure 2 for several values of C. The value of C is the y-intercept.

 $\int (10x^4 - 2\sec^2 x) \, dx$ 



$$\int (10x^4 - 2\sec^2 x) \, dx = 10 \frac{x^5}{5} - 2\tan x + C$$
$$= 2x^5 - 2\tan x + C$$

4 1.5 -15

You should check this answer by differentiating it.

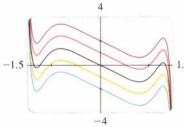


FIGURE 2

**EXAMPLE 4** Evaluate  $\int_0^3 (x^3 - 6x) dx$ .

SOLUTION Using the Evaluation Theorem and Table 1, we have

$$\int_0^3 (x^3 - 6x) \, dx = \frac{x^4}{4} - 6\frac{x^2}{2} \bigg]_0^3$$
$$= \left(\frac{1}{4} \cdot 3^4 - 3 \cdot 3^2\right) - \left(\frac{1}{4} \cdot 0^4 - 3 \cdot 0^2\right)$$
$$= \frac{81}{4} - 27 - 0 + 0 = -6.75$$

Compare this calculation with Example 2(b) in Section 5.2.

**EXAMPLE 5** Find  $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2 + 1}\right) dx$  and interpret the result in terms of areas.

**SOLUTION** The Evaluation Theorem gives

$$\int_{0}^{2} \left( 2x^{3} - 6x + \frac{3}{x^{2} + 1} \right) dx = 2\frac{x^{4}}{4} - 6\frac{x^{2}}{2} + 3\tan^{-1}x \Big]_{0}^{2}$$
$$= \frac{1}{2}x^{4} - 3x^{2} + 3\tan^{-1}x \Big]_{0}^{2}$$
$$= \frac{1}{2}(2^{4}) - 3(2^{2}) + 3\tan^{-1}2 - 0$$
$$= -4 + 3\tan^{-1}2$$

This is the exact value of the integral. If a decimal approximation is desired, we can use a calculator to approximate  $\tan^{-1} 2$ . Doing so, we get

$$\int_{0}^{2} \left( 2x^{3} - 6x + \frac{3}{x^{2} + 1} \right) dx \approx -0.67855$$

Figure 3 shows the graph of the integrand. We know from Section 5.2 that the value of the integral can be interpreted as the sum of the areas labeled with a plus sign minus the area labeled with a minus sign.

**EXAMPLE 6** Evaluate 
$$\int_{1}^{9} \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dt$$

**SOLUTION** First we need to write the integrand in a simpler form by carrying out the division:

$$\int_{1}^{9} \frac{2t^{2} + t^{2}\sqrt{t} - 1}{t^{2}} dt = \int_{1}^{9} (2 + t^{1/2} - t^{-2}) dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1} \Big]_{1}^{9}$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t} \Big]_{1}^{9}$$

$$= \left[2 \cdot 9 + \frac{2}{3}(9)^{3/2} + \frac{1}{9}\right] - \left(2 \cdot 1 + \frac{2}{3} \cdot 1^{3/2} + \frac{1}{1}\right)$$

$$= 18 + 18 + \frac{1}{9} - 2 - \frac{2}{3} - 1 = 32\frac{4}{9}$$

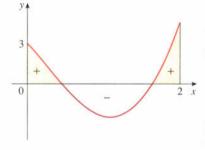


FIGURE 3

## Applications

The Evaluation Theorem says that if f is continuous on [a, b], then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

where F is any antiderivative of f. This means that F' = f, so the equation can be rewritten as

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

We know that F'(x) represents the rate of change of y = F(x) with respect to x and F(b) - F(a) is the change in y when x changes from a to b. So we can reformulate the Evaluation Theorem in words as follows.

**Total Change Theorem** The integral of a rate of change is the total change:

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences that we discussed in Section 3.3. Here are a few instances of this idea:

If V(t) is the volume of water in a reservoir at time *t*, then its derivative V'(t) is the rate at which water flows into the reservoir at time *t*. So

$$\int_{t_1}^{t_2} V'(t) \, dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time  $t_1$  and time  $t_2$ .

If [C] (t) is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative d[C]/dt. So

$$\int_{t_1}^{t_2} \frac{d[\mathbf{C}]}{dt} dt = [\mathbf{C}](t_2) - [\mathbf{C}](t_1)$$

is the change in the concentration of C from time  $t_1$  to time  $t_2$ .

• If the mass of a rod measured from the left end to a point x is m(x), then the linear density is  $\rho(x) = m'(x)$ . So

$$\int_a^b \rho(x) \, dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between x = a and x = b. If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt} \, dt = n(t_2) - n(t_1)$$

is the increase in population during the time period from  $t_1$  to  $t_2$ .

If C(x) is the cost of producing x units of a commodity, then the marginal cost is the derivative C'(x). So

$$\int_{x_1}^{x_2} C'(x) \, dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from  $x_1$  units to  $x_2$  units.

If an object moves along a straight line with position function s(t), then its velocity is v(t) = s'(t), so

2 
$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1)$$

is the change of position, or *displacement*, of the particle during the time period from  $t_1$  to  $t_2$ . In Section 5.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it always true.

• If we want to calculate the distance traveled during the time interval, we have to consider the intervals when  $v(t) \ge 0$  (the particle moves to the right) and also the intervals when  $v(t) \le 0$  (the particle moves to the left). In both cases the distance is computed by integrating |v(t)|, the speed. Therefore

$$\int_{t_1}^{t_2} |v(t)| dt = \text{total distance traveled}$$

Figure 4 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

• The acceleration of the object is a(t) = v'(t), so

$$\int_{t_1}^{t_2} a(t) \, dt = v(t_2) - v(t_1)$$

is the change in velocity from time  $t_1$  to time  $t_2$ .

**EXAMPLE 7** A particle moves along a line so that its velocity at time t is  $v(t) = t^2 - t - 6$  (measured in meters per second).

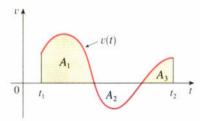
- (a) Find the displacement of the particle during the time period  $1 \le t \le 4$ .
- (b) Find the distance traveled during this time period.

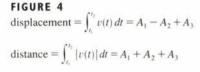
#### SOLUTION

(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_{1}^{4} v(t) dt = \int_{1}^{4} (t^{2} - t - 6) dt$$
$$= \left[\frac{t^{3}}{3} - \frac{t^{2}}{2} - 6t\right]_{1}^{4} = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.





t

(b) Note that  $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$  and so  $v(t) \le 0$  on the interval [1, 3] and  $v(t) \ge 0$  on [3, 4]. Thus, from Equation 3, the distance traveled is

$$\int_{1}^{4} |v(t)| dt = \int_{1}^{3} [-v(t)] dt + \int_{3}^{4} v(t) dt$$
  
=  $\int_{1}^{3} (-t^{2} + t + 6) dt + \int_{3}^{4} (t^{2} - t - 6) dt$   
=  $\left[ -\frac{t^{3}}{3} + \frac{t^{2}}{2} + 6t \right]_{1}^{3} + \left[ \frac{t^{3}}{3} - \frac{t^{2}}{2} - 6t \right]_{3}^{4}$   
=  $\frac{61}{6}$  m  $\approx 10.17$  m

**EXAMPLE 8** Figure 5 shows the power consumption in the city of San Francisco for September 19, 1996 (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.

P 800

600

400

200

0

3

6

**SOLUTION** Power is the rate of change of energy: P(t) = E'(t). So, by the Total Change Theorem,

9

$$\int_0^{24} P(t) \, dt = \int_0^{24} E'(t) \, dt = E(24) - E(0)$$

12

15

18

21

Pacific Gas & Electric

is the total amount of energy used on September 19, 1996. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and  $\Delta t = 2$ .

$$\int_{0}^{24} P(t) dt \approx [P(1) + P(3) + P(5) + \dots + P(21) + P(23)] \Delta t$$
$$\approx (440 + 400 + 420 + 620 + 790 + 840 + 850 + 840 + 810 + 690 + 670 + 550) (2)$$
$$= 15.840$$

The energy used was approximately 15,840 megawatt-hours.

How did we know what units to use for energy in Example 8? The integral  $\int_0^{24} P(t) dt$  is defined as the limit of sums of terms of the form  $P(t_i^*) \Delta t$ . Now  $P(t_i^*)$  is measured in megawatts and  $\Delta t$  is measured in hours, so their product is measured.

we use Property 5 of integrals from  
Section 5.2 to split the integral into  
two parts, one where 
$$v(t) \le 0$$
 and  
one where  $v(t) \ge 0$ .

To integrate the absolute value of v(t),

FIGURE 5

A note on units

sured in megawatt-hours. The same is true of the limit. In general, the unit of measurement for  $\int_{a}^{b} f(x) dx$  is the product of the unit for f(x) and the unit for x.



Exercises

- 1. If w'(t) is the rate of growth of a child in pounds per year, what does  $\int_{5}^{10} w'(t) dt$  represent?
- 2. The current in a wire is defined as the derivative of the charge: I(t) = Q'(t). (See Example 3 in Section 3.3.) What does  $\int_{a}^{b} I(t) dt$  represent?
- **3.** If oil leaks from a tank at a rate of r(t) gallons per minute at time t, what does  $\int_{0}^{120} r(t) dt$  represent?
- **4.** A honeybee population starts with 100 bees and increases at a rate of n'(t) bees per week. What does  $100 + \int_0^{15} n'(t) dt$  represent?
- 5. In Section 4.7 we defined the marginal revenue function R'(x) as the derivative of the revenue function R(x), where x is the number of units sold. What does  $\int_{1000}^{5000} R'(x) dx$  represent?
- **6.** If f(x) is the slope of a trail at a distance of x miles from the start of the trail, what does  $\int_{3}^{5} f(x) dx$  represent?

**7–30** ■ Evaluate the integral.

**7.**  $\int_{-\infty}^{4} (3x - 5) dx$ 8.  $\int_{1}^{2} (5x^2 - 4x + 3) dx$ **10.**  $\int_0^1 (y^9 - 2y^5 + 3y) \, dy$ **9.**  $\int_{-\infty}^{0} (2x - e^x) dx$ 12.  $\int_{1}^{2} \frac{t^{6} - t^{2}}{t^{4}} dt$ 11.  $\int_{0}^{4} \sqrt{x} dx$ 13.  $\int_{1}^{2} \frac{x^{2} + 1}{\sqrt{x}} dx$ 14.  $\int_{0}^{2} (x^{3} - 1)^{2} dx$ 16.  $\int_{1}^{8} \frac{x-1}{\sqrt[3]{x^2}} dx$ **15.**  $\int_{-\infty}^{1} u(\sqrt{u} + \sqrt[3]{u}) du$ **17.**  $\int_{-\pi/3}^{\pi/3} \sin t \, dt$ **18.**  $\int_{0}^{\pi/2} (\cos\theta + 2\sin\theta) d\theta$ 19.  $\int_{\pi/6}^{\pi/3} \csc^2\theta \, d\theta$ **20.**  $\int_{-\pi/2}^{\pi/2} \csc x \cot x \, dx$ **21.**  $\int_{4}^{8} \frac{1}{x} dx$ **22.**  $\int_{10^3}^{10^6} 8e^x dx$ **24.**  $\int_{-e^2}^{-e} \frac{3}{x} dx$ **23.**  $\int_{0}^{9} 2^{t} dt$ **25.**  $\int_{1}^{\sqrt{3}} \frac{6}{1+x^2} dx$ **26.**  $\int_{0}^{0.5} \frac{dx}{\sqrt{1-x^2}}$ 

**27.** 
$$\int_{1}^{e} \frac{x^{2} + x + 1}{x} dx$$
**28.** 
$$\int_{4}^{9} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{2} dx$$
**29.** 
$$\int_{-2}^{3} |x^{2} - 1| dx$$
**30.** 
$$\int_{-1}^{2} |x - x^{2}| dx$$

**31–34** Use a graph to give a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

- **31.**  $y = \sqrt[3]{x}, \ 0 \le x \le 27$  **32.**  $y = x^{-4}, \ 1 \le x \le 6$  **33.**  $y = \sin x, \ 0 \le x \le \pi$ **34.**  $y = \sec^2 x, \ 0 \le x \le \pi/3$
- 35. Use a graph to estimate the x-intercepts of the curve y = x + x<sup>2</sup> x<sup>4</sup>. Then use this information to estimate the area of the region that lies under the curve and above the x-axis.
- **36.** Repeat Exercise 35 for the curve  $y = 2x + 3x^4 2x^6$ .
- **37–38** Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch like Figure 3.

**37.** 
$$\int_{-1}^{2} x^{3} dx$$
 **38.**  $\int_{\pi/4}^{5\pi/2} \sin x \, dx$ 

**39–40** Verify by differentiation that the formula is correct.

**39.** 
$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$
  
**40.** 
$$\int \frac{1}{x^2 \sqrt{x^2 + a^2}} \, dx = -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C$$

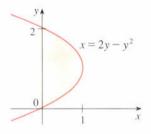
**41–42** Find the general indefinite integral. Illustrate by graphing several members of the family on the same screen.

**41.** 
$$\int x \sqrt{x} \, dx$$
 **42.**  $\int (\cos x - 2 \sin x) \, dx$ 

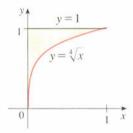
**43–46** ■ Find the general indefinite integral.

**43.** 
$$\int (2 - \sqrt{x})^2 dx$$
 **44.**  $\int \sqrt{x} (x^2 - 1/x) dx$ 

**45.**  $\int (2x + \sec x \tan x) dx$ **46.**  $\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) dx$  **47.** The area of the region that lies to the right of the y-axis and to the left of the parabola  $x = 2y - y^2$  (the shaded region in the figure) is given by the integral  $\int_0^2 (2y - y^2) dy$ . (Turn your head clockwise and think of the region as lying below the curve  $x = 2y - y^2$  from y = 0 to y = 2.) Find the area of the region.



**48.** The boundaries of the shaded region are the y-axis, the line y = 1, and the curve  $y = \sqrt[4]{x}$ . Find the area of this region by writing x as a function of y and integrating with respect to y (as in Exercise 47).



**49–50** ■ The velocity function (in meters per second) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

**49.** 
$$v(t) = 3t - 5, \quad 0 \le t \le 3$$

**50.**  $v(t) = t^2 - 2t - 8$ ,  $1 \le t \le 6$ 

**51–52** The acceleration function (in  $m/s^2$ ) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time *t* and (b) the distance traveled during the given time interval.

- **51.** a(t) = t + 4, v(0) = 5,  $0 \le t \le 10$
- **52.** a(t) = 2t + 3, v(0) = -4,  $0 \le t \le 3$
- 53. The linear density of a rod of length 4 m is given by  $\rho(x) = 9 + 2\sqrt{x}$  measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.
- 54. An animal population is increasing at a rate of 200 + 50t per year (where *t* is measured in years). By how much does the animal population increase between the fourth and tenth years?
- **55.** The velocity of a car was read from its speedometer at ten-second intervals and recorded in the table. Use

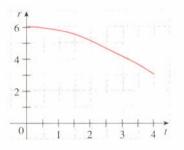
the Midpoint Rule to estimate the distance traveled by the car.

1 (8)	0 10		20	30	40	50	60				100	
<i>v</i> (mi/h)	0	38	52	58	55	51	56	53	50	17	45	

**56.** The inflation rate is often defined as the derivative of the Consumer Price Index (CPI), which is published by the U.S. Bureau of Labor Statistics and measures prices of items in a "representative market basket" of typical urban consumers. The table gives the inflation rate in the United States from 1980 to 1994. Write the total percentage increase in the CPI from 1980 to 1994 as a definite integral. Then use the Midpoint Rule to estimate it.

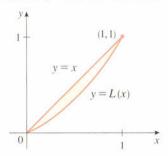
T	12(1)	1	
1980	13.5	1988	4.1
981	10.3	1989	1.8
982	6.2	1990	
983	3.2	1001	
984	4.3	1992	3,11
985	3.6	1002	
986	1.9	1004	
1987	3.6		

- 57. The marginal cost of manufacturing x yards of a certain fabric is  $C'(x) = 3 0.01x + 0.000006x^2$  (in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.
- **58.** Water leaked from a tank at a rate of r(t) liters per hour, where the graph of r is as shown. Express the total amount of water that leaked out during the first four hours as a definite integral. Then use the Midpoint Rule to estimate that amount.



59. Economists use a cumulative distribution called a *Lorenz curve* to describe the distribution of income between households in a given country. Typically, a Lorenz curve is defined on [0, 1], passes through (0, 0) and (1, 1), and is continuous, increasing, and concave upward. The points on this curve are determined by ranking all households by income and then computing the percentage of households whose income is less than or equal to a given percentage of the total income of

the country. For example, the point (a/100, b/100) is on the Lorenz curve if the bottom a% of the households receive less than or equal to b% of the total income. *Absolute equality* of income distribution would occur if the bottom a% of the households receive a% of the income, in which case the Lorenz curve would be the line y = x. The area between the Lorenz curve and the line y = x measures how much the income distribution differs from absolute equality. The *coefficient of inequality* is the ratio of the area between the Lorenz curve and the line y = x to the area under y = x.



(a) Show that the coefficient of inequality is twice the area between the Lorenz curve and the line y = x, that is, show that

coefficient of inequality =  $2 \int_0^1 [x - L(x)] dx$ 

(b) The income distribution for a certain country is represented by the Lorenz curve defined by the equation

$$L(x) = \frac{5}{12}x^2 + \frac{7}{12}x$$

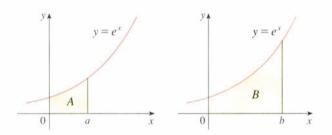
What is the percentage of total income received by the bottom 50% of the households? Find the coefficient of inequality.

**60.** On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was

to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure Solid rocket booster	62	1445
separation	125	4151

- (a) Use the methods of Section 1.7 to model these data by a third-degree polynomial.
- (b) Use the model in part (a) to estimate the height reached by the *Endeavour*, 125 seconds after liftoff.
- **61.** Suppose h is a function such that h(1) = -2, h'(1) = 2, h''(1) = 3, h(2) = 6, h'(2) = 5, h''(2) = 13, and h'' is continuous everywhere. Evaluate  $\int_{1}^{2} h''(u) du$ .
- **62.** The area labeled *B* is three times the area labeled *A*. Express *b* in terms of *a*.



Discovery	
Project	Area Functions

- 1. (a) Draw the line y = 2t + 1 and use geometry to find the area under this line, above the *t*-axis, and between the vertical lines t = 1 and t = 3.
  - (b) If x > 1, let A(x) be the area of the region that lies under the line y = 2t + 1 between t = 1 and t = x. Sketch this region and use geometry to find an expression for A(x).
  - (c) Differentiate the area function A(x). What do you notice?
- **2.** (a) If  $0 \le x \le \pi$ , let

$$A(x) = \int_0^x \sin t \, dt$$

A(x) represents the area of a region. Sketch that region.

- (b) Use the Evaluation Theorem to find an expression for A(x).
- (c) Find A'(x). What do you notice?
- (d) If x is any number between 0 and  $\pi$  and h is a small positive number, then A(x + h) A(x) represents the area of a region. Describe and sketch the region.
- (e) Draw a rectangle that approximates the region in part (d). By comparing the areas of these two regions, show that

$$\frac{A(x+h) - A(x)}{h} \approx \sin x$$

- (f) Use part (e) to give an intuitive explanation for the result of part (c).
- 3. (a) Draw the graph of the function  $f(x) = \cos(x^2)$  in the viewing rectangle [0, 2] by [-1.25, 1.25].
  - (b) If we define a new function g by

$$g(x) = \int_0^x \cos(t^2) \, dt$$

then g(x) is the area under the graph of f from 0 to x [until f(x) becomes negative, at which point g(x) becomes a difference of areas]. Use part (a) to determine the value of x at which g(x) starts to decrease. [Unlike the integral in Problem 2, it is impossible to evaluate the integral defining g to obtain an explicit expression for g(x).]

- (c) Use the integration command on your calculator or computer to estimate  $g(0.2), g(0.4), g(0.6), \ldots, g(1.8), g(2)$ . Then use these values to sketch a graph of g.
- (d) Use your graph of g from part (c) to sketch the graph of g' using the interpretation of g'(x) as the slope of a tangent line. How does the graph of g' compare with the graph of f?
- **4.** Suppose *f* is a continuous function on the interval [*a*, *b*] and we define a new function *g* by the equation

$$g(x) = \int_{a}^{x} f(t) dt$$

Based on your results in Problems 1–3, conjecture an expression for g'(x).



## The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method.

In preparation for the Fundamental Theorem we first investigate some additional properties of the definite integral.

### **Properties of Integrals**

When we defined the definite integral  $\int_a^b f(x) dx$ , we implicitly assumed that a < b. But the definition as a limit of Riemann sums makes sense even if a > b. Notice that if we reverse a and b, then  $\Delta x$  changes from (b - a)/n to (a - b)/n. Therefore

$$\int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx$$

If a = b, then  $\Delta x = 0$  and so

$$\int_{a}^{a} f(x) \, dx = 0$$

In the following properties we compare sizes of functions and sizes of integrals. These properties are true only if  $a \leq b$ .

**Comparison Properties of the Integral** 

1. If 
$$f(x) \ge 0$$
 for  $a \le x \le b$ , then  $\int_a^b f(x) dx \ge 0$ .  
2. If  $f(x) \ge g(x)$  for  $a \le x \le b$ , then  $\int_a^b f(x) dx \ge \int_a^b g(x) dx$   
3. If  $m \le f(x) \le M$  for  $a \le x \le b$ , then  
 $m(b - a) \le \int_a^b f(x) dx \le M(b - a)$ 

If  $f(x) \ge 0$ , then  $\int_a^b f(x) dx$  represents the area under the graph of f, so the geometric interpretation of Property 1 is simply that areas are positive. (It also follows directly from the definition because all the quantities involved are positive.) Property 2 says that a bigger function has a bigger integral. It follows from Property 1 because  $f - g \ge 0$ .

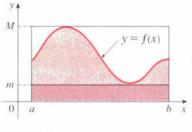
Property 3 is illustrated by Figure 1 for the case where  $f(x) \ge 0$ . If f is continuous we could take m and M to be the absolute minimum and maximum values of fon the interval [a, b]. In this case Property 3 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

In general, since  $m \leq f(x) \leq M$ , Property 2 gives

$$\int_{a}^{b} m \, dx \leq \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} M \, dx$$

Evaluating the integrals on the left- and right-hand sides, we obtain

$$m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)$$



Property 3 is useful when all we want is a rough estimate of the size of an integral without going to the bother of using the Midpoint Rule.

**EXAMPLE I** Use Property 3 to estimate 
$$\int_0^1 e^{-x^2} dx$$

**SOLUTION** Because  $f(x) = e^{-x^2}$  is a decreasing function on [0, 1], its absolute maximum value is M = f(0) = 1 and its absolute minimum value is  $m = f(1) = e^{-1}$ . Thus, by Property 3,

$$e^{-1}(1-0) \le \int_0^1 e^{-x^2} dx \le 1(1-0)$$

 $e^{-1} \le \int_0^1 e^{-x^2} dx \le 1$ 

or

Since  $e^{-1} \approx 0.3679$ , we can write

$$0.367 \leq \int_0^1 e^{-x^2} dx \leq 1$$

The result of Example 1 is illustrated in Figure 2. The integral is greater than the area of the lower rectangle and less than the area of the square.

### The Fundamental Theorem

П

In order to motivate the Fundamental Theorem, let f be a continuous function on [a, b] and define a new function g by

$$g(x) = \int_{-\infty}^{x} f(t) dt$$

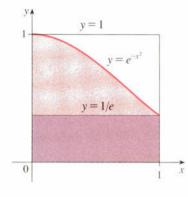
where  $a \le x \le b$ . Observe that *g* depends only on *x*, which appears as the variable upper limit in the integral. If *x* is a fixed number, then the integral  $\int_a^x f(t) dt$  is a definite number. If we then let *x* vary, the number  $\int_a^x f(t) dt$  also varies and defines a function of *x* denoted by g(x). For instance, if we take  $f(t) = t^2$  and a = 1, then, by the Evaluation Theorem, we have

$$g(x) = \int_{1}^{x} t^{2} dt = \frac{x^{3} - 1}{3}$$

Notice that  $g'(x) = x^2$ , that is, g' = f. In other words, if g is defined as the integral of f by Equation 1, then g turns out to be an antiderivative of f, at least in this case.

To see why this might be generally true we consider any continuous function f with  $f(x) \ge 0$ . Then  $g(x) = \int_a^x f(t) dt$  can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 3.)

In order to compute g'(x) from the definition of derivative we first observe that, for h > 0, g(x + h) - g(x) is obtained by subtracting areas, so it is the area under





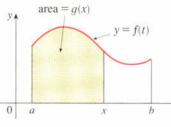


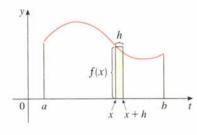
FIGURE 3



SO

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FIGURE 4



the graph of f from x to x + h (the gold area in Figure 4). For small h you can see from the figure that this area is approximately equal to the area of the rectangle with height f(x) and width h:

$$g(x + h) - g(x) \approx hf(x)$$
$$\frac{g(x + h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

We abbreviate the name of this theorem as FTCI. In words, it says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit. **The Fundamental Theorem of Calculus, Part 1** If f is continuous on [a, b], then the function g defined by

$$g(x) = \int_{a}^{x} f(t) dt \qquad a \le x \le b$$

is an antiderivative of f, that is, g'(x) = f(x) for a < x < b.

Using Leibniz notation for derivatives, we can write this theorem as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

when f is continuous. Roughly speaking, this equation says that if we first integrate f and then differentiate the result, we get back to the original function f.

It is easy to prove the Fundamental Theorem if we make the assumption that f possesses an antiderivative F. (This is certainly plausible. After all, we sketched graphs of antiderivatives in Sections 2.10 and 4.9.) Then, by the Evaluation Theorem,

$$\int_{a}^{x} f(t) dt = F(x) - F(a)$$

for any x between a and b. Therefore

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) = f(x)$$

as required. At the end of this section we present a proof without the assumption that an antiderivative exists.

**EXAMPLE 2** Find the derivative of the function  $g(x) = \int_0^x \sqrt{1+t^2} dt$ .

**SOLUTION** Since  $f(t) = \sqrt{1 + t^2}$  is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

**EXAMPLE 3** Although a formula of the form  $g(x) = \int_a^x f(t) dt$  may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function** 

$$S(x) = \int_0^x \sin(\pi t^2/2) dt$$

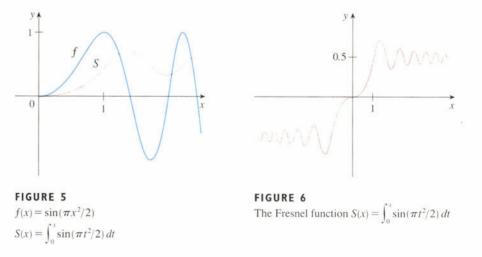
is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics. This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

Part 1 of the Fundamental Theorem tells us how to differentiate the Fresnel function:

$$S'(x) = \sin(\pi x^2/2)$$

This means that we can apply all the methods of differential calculus to analyze S (see Exercise 25).

Figure 5 shows the graphs of  $f(x) = \sin(\pi x^2/2)$  and the Fresnel function  $S(x) = \int_0^x f(t) dt$ . A computer was used to graph S by computing the value of this integral for many values of x. It does indeed look as if S(x) is the area under the graph of f from 0 to x [until  $x \approx 1.4$ , when S(x) becomes a difference of areas]. Figure 6 shows a larger part of the graph of S.



If we now start with the graph of *S* in Figure 5 and think about what its derivative should look like, it seems reasonable that S'(x) = f(x). [For instance, *S* is increasing when f(x) > 0 and decreasing when f(x) < 0.] So this gives a visual confirmation of Part 1 of the Fundamental Theorem of Calculus.

**EXAMPLE 4** Find  $\frac{d}{dx} \int_{1}^{x^4} \sec t \, dt$ .

**SOLUTION** Here we have to be careful to use the Chain Rule in conjunction with Part 1 of the Fundamental Theorem. Let  $u = x^4$ . Then

$$\frac{d}{dx} \int_{1}^{x^{4}} \sec t \, dt = \frac{d}{dx} \int_{1}^{u} \sec t \, dt$$
$$= \frac{d}{du} \left[ \int_{1}^{u} \sec t \, dt \right] \frac{du}{dx} \qquad \text{(by the Chain Rule)}$$
$$= \sec u \, \frac{du}{dx} \qquad \text{(by FTC1)}$$
$$= \sec(x^{4}) \cdot 4x^{3}$$

#### Differentiation and Integration as Inverse Processes

We now bring together the two parts of the Fundamental Theorem. We regard Part 1 as fundamental because it relates integration and differentiation. But the Evaluation Theorem from Section 5.3 also relates integrals and derivatives, so we rename it Part 2 of the Fundamental Theorem.

The Fundamental Theorem of Calculus Suppose f is continuous on [a, b]. 1. If  $g(x) = \int_a^x f(t) dt$ , then g'(x) = f(x). 2.  $\int_a^b f(x) dx = F(b) - F(a)$ , where F is any antiderivative of f, that is, F' = f.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt\,=\,f(x)$$

which says that if f is integrated and the result is then differentiated, we arrive back at the original function f. In Section 5.3 we reformulated Part 2 as the Total Change Theorem:

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

This version says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function F, but in the form F(b) - F(a). Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are inverse processes. Each undoes what the other does.

The Fundamental Theorem of Calculus is unquestionably the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were so difficult that only a genius could meet the challenge. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

### Proof of FTCI

Here we give a proof of Part 1 of the Fundamental Theorem of Calculus without assuming the existence of an antiderivative of f. Let  $g(x) = \int_a^x f(t) dt$ . If x and x + h are in the open interval (a, b), then

$$g(x + h) - g(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt$$
$$= \left(\int_a^x f(t) dt + \int_x^{x+h} f(t) dt\right) - \int_a^x f(t) dt$$
$$= \int_x^{x+h} f(t) dt$$

and so, for  $h \neq 0$ ,

2

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_{x}^{x+h} f(t) \, dt$$

For now let's assume that h > 0. Since f is continuous on [x, x + h], the Extreme Value Theorem says that there are numbers u and v in [x, x + h] such that f(u) = m and f(v) = M, where m and M are the absolute minimum and maximum values of f on [x, x + h] (see Figure 7).

By Property 3, we have

$$mh \leq \int_{x}^{x+h} f(t) dt \leq Mh$$
$$f(u)h \leq \int_{x}^{x+h} f(t) dt \leq f(v)h$$

Since h > 0, we can divide this inequality by h:

$$f(u) \leq \frac{1}{h} \int_{x}^{x+h} f(t) \, dt \leq f(v)$$

Now we use Equation 2 to replace the middle part of this inequality:

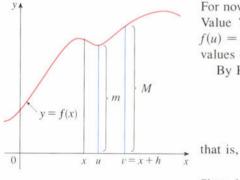
$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v)$$

Inequality 3 can be proved in a similar manner for the case where h < 0. Now we let  $h \rightarrow 0$ . Then  $u \rightarrow x$  and  $v \rightarrow x$ , since u and v lie between x and x + h. Thus

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \qquad \lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from (3) and the Squeeze Theorem, that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$





5.4

Exercises

- 1. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."
- 2. Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must  $\int_{0}^{2} f(x) dx$  lie? Which property of integrals allows you to make your conclusion?
- 3. Use the properties of integrals to verify that

$$0 \le \int_1^3 \ln x \, dx \le 2 \ln 3$$

4–5 ■ Use Property 3 to estimate the value of the integral.

**4.** 
$$\int_0^2 \sqrt{x^3 + 1} \, dx$$
 **5.**  $\int_{-1}^1 e^{x^2} dx$ 

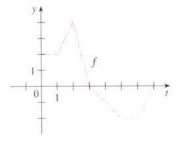
6. (a) Use Property 3 to show that

$$\int_1^4 \sqrt{1 + x^2} \, dx \ge 3\sqrt{2}$$

- (b) Show that  $\sqrt{1 + x^2} \ge x$ .
- (c) Use part (b) and Property 2 to show that

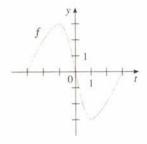
$$\int_1^4 \sqrt{1+x^2} \ dx \ge 7.5$$

- 7. Let  $g(x) = \int_0^x f(t) dt$ , where f is the function whose graph is shown.
  - (a) Evaluate g(0), g(1), g(2), g(3), and g(6).
  - (b) On what intervals is *g* increasing?
  - (c) Where does *g* have a maximum value?
  - (d) Sketch a rough graph of g.



- **8.** Let  $g(x) = \int_{-3}^{x} f(t) dt$ , where f is the function whose graph is shown.
  - (a) Evaluate g(-3) and g(3).
  - (b) Estimate g(-2), g(-1), and g(0).
  - (c) On what interval is g increasing?
  - (d) Where does q have a maximum value?
  - (e) Sketch a rough graph of g.

(f) Use the graph in part (e) to sketch the graph of g'(x). Compare with the graph of f.



9-10 Sketch the area represented by g(x). Then find g'(x) in two ways: (a) by using Part 1 of the Fundamental Theorem and (b) by evaluating the integral using Part 2 and then differentiating.

**9.** 
$$g(x) = \int_0^x (1 + t^2) dt$$
 **10.**  $g(x) = \int_\pi^x (2 + \cos t) dt$ 

**11–20** Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the given function.

11. 
$$g(x) = \int_{1}^{x} (t^{2} - 1)^{20} dt$$
  
12.  $g(x) = \int_{-1}^{x} \sqrt{t^{3} + 1} dt$   
13.  $g(u) = \int_{\pi}^{u} \frac{1}{1 + t^{4}} dt$   
14.  $F(x) = \int_{x}^{2} \cos(t^{2}) dt$   
 $\begin{bmatrix} Hint: \int_{x}^{2} \cos(t^{2}) dt = -\int_{2}^{x} \cos(t^{2}) dt \end{bmatrix}$   
15.  $h(x) = \int_{2}^{1/x} \sin^{4}t dt$   
16.  $h(x) = \int_{1}^{\sqrt{x}} \frac{s^{2}}{s^{2} + 1} ds$   
17.  $y = \int_{\tan x}^{17} \sin(t^{4}) dt$   
18.  $y = \int_{-5}^{\sin x} t \cos(t^{3}) dt$   
19.  $g(x) = \int_{2x}^{3x} \frac{u - 1}{u + 1} du$   
 $\begin{bmatrix} Hint: \int_{2x}^{3x} f(u) du = \int_{2x}^{0} f(u) du + \int_{0}^{3x} f(u) du \end{bmatrix}$   
20.  $g(x) = \int_{\tan x}^{x^{2}} \frac{1}{\sqrt{2 + t^{4}}} dt$   
21. If  $F(x) = \int_{1}^{x} f(t) dt$ , where  $f(t) = \int_{1}^{t^{2}} \frac{\sqrt{1 + u^{4}}}{u} du$ , find  $F''(2)$ .

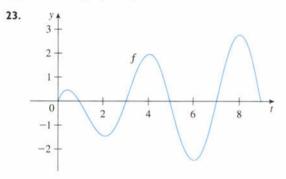
22. Find the interval on which the curve

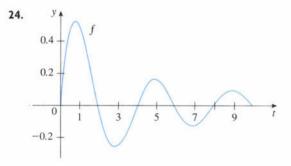
$$y = \int_0^x \frac{1}{1 + t + t^2} \, dt$$

is concave upward.

**23–24** Let  $g(x) = \int_0^x f(t) dt$ , where f is the function whose graph is shown.

- (a) At what values of x do the local maximum and minimum values of g occur?
- (b) Where does g attain its absolute maximum value?
- (c) On what intervals is g concave downward?
- (d) Sketch the graph of g.





- **25.** The Fresnel function S was defined in Example 3 and graphed in Figures 5 and 6.
  - (a) At what values of x does this function have local maximum values?
  - (b) On what intervals is the function concave upward?
  - (c) Use a graph to solve the following equation correct to one decimal place:

$$\int_0^x \sin(\pi t^2/2) \, dt = 0.2$$

**CAS** 26. The sine integral function

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt$$

is important in electrical engineering. [The integrand  $f(t) = (\sin t)/t$  is not defined when t = 0 but we know

that its limit is 1 when  $t \to 0$ . So we define f(0) = 1 and this makes f a continuous function everywhere.]

- (a) Draw the graph of Si.
- (b) At what values of x does this function have local maximum values?
- (c) Find the coordinates of the first inflection point to the right of the origin.
- (d) Does this function have horizontal asymptotes?
- (e) Solve the following equation correct to one decimal place:

$$\int_0^x \frac{\sin t}{t} \, dt = 1$$

**27.** Find a function f such that f(1) = 0 and  $f'(x) = 2^{x/x}$ .

28. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \\ 0 & \text{if } x > 2 \end{cases}$$

and  $g(x) = \int_0^x f(t) dt$ 

- (a) Find an expression for g(x) similar to the one for f(x).
- (b) Sketch the graphs of f and g.
- (c) Where is f differentiable? Where is g differentiable?
- **29.** Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$$

- 30. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate f = f(t) and will accumulate maintenance costs at the rate g = g(t), where t is the time measured in months. The company wants to determine the optimal time to replace the system.
  (a) Let
  - $C(t) = \frac{1}{t} \int_0^t \left[ f(s) + g(s) \right] ds$

Show that the critical numbers of C occur at the numbers t where C(t) = f(t) + g(t).

(b) Suppose that

and

$$f(t) = \begin{cases} \frac{V}{15} - \frac{V}{450} t & \text{if } 0 < t \le 30\\ 0 & \text{if } t > 30 \end{cases}$$
  
$$d \qquad g(t) = \frac{Vt^2}{12,900} \quad t > 0 \end{cases}$$

Determine the length of time *T* for the total depreciation  $D(t) = \int_{0}^{t} f(s) ds$  to equal the initial value *V*.

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- (c) Determine the absolute minimum of C on (0, T].
- (d) Sketch the graphs of C and f + g in the same coordinate system, and verify the result in part (a) in this case.
- **31.** A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate f = f(t), where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.
- (a) Show that  $\int_0^t f(s) ds$  represents the loss in value of the machine over the period of time t since the last overhaul.
- (b) Let C = C(t) be given by

$$C(t) = \frac{1}{t} \left[ A + \int_0^t f(s) \, ds \right]$$

What does *C* represent and why would the company want to minimize *C*?

(c) Show that C has a minimum value at the numbers t = T where C(T) = f(T).

## Writing Project

## Newton, Leibniz, and the Invention of Calculus

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks such as Eudoxus and Archimedes, and methods for finding tangents were pioneered by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow, Newton's teacher at Cambridge, was the first to understand the inverse relationship between differentiation and integration. What Newton and Leibniz did was to use this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. It is in this sense that Newton and Leibniz are credited with the invention of calculus.

Read about the contributions of these men in one or more of the given references and write a report on one of the following three topics. You can include biographical details, but the main thrust of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the sourcebooks, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

#### References

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- 4. Howard Eves, An Introduction to the History of Mathematics, 6th ed. (New York: Saunders, 1990), Chapter 11.
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- 6. Victor Katz, A History of Mathematics: An Introduction (New York: Harper-Collins, 1993), Chapter 12.
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- 1. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987), Chapters 12 and 13.
- 2. D. E. Smith, ed., A Sourcebook in Mathematics (New York: Dover, 1959), Chapter V.
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. . . . .



### The Substitution Rule •••••

Because of the Evaluation Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

1

3

 $\int 2x \sqrt{1 + x^2} \, dx$ 

To find this integral we use the problem-solving strategy of *introducing something extra*. Here the "something extra" is a new variable; we change from the variable x to a new variable u. Suppose that we let u be the quantity under the root sign in (1),  $u = 1 + x^2$ . Then the differential of u is du = 2x dx. Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential 2x dx would occur in (1) and so, formally, without justifying our calculation, we could write

2  

$$\int 2x \sqrt{1 + x^2} \, dx = \int \sqrt{1 + x^2} \, 2x \, dx$$

$$= \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (x^2 + 1)^{3/2} + C$$

But now we could check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx} \left[ \frac{2}{3} (x^2 + 1)^{3/2} + C \right] = \frac{2}{3} \cdot \frac{3}{2} (x^2 + 1)^{1/2} \cdot 2x = 2x \sqrt{x^2 + 1}$$

In general, this method works whenever we have an integral of the form  $\int f(g(x))g'(x) dx$ . Observe that if F' = f, then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

Differentials were defined in Section 3.8. If u = f(x), then

 $du = f'(x) \, dx$ 

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

If we make the "change of variable" or "substitution" u = g(x), then from Equation 3 we have

$$\int F'(g(x))g'(x) \, dx = F(g(x)) + C = F(u) + C = \int F'(u) \, du$$

or, writing F' = f, we get

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Thus, we have proved the following rule.

**4** The Substitution Rule If u = g(x) is a differentiable function whose range is an interval *I* and *f* is continuous on *I*, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation. Notice also that if u = g(x), then du = g'(x) dx, so a way to remember the Substitution Rule is to think of dx and du in (4) as differentials.

Thus, the Substitution Rule says: It is permissible to operate with dx and du after integral signs as if they were differentials.

**EXAMPLE I** Find  $\int x^3 \cos(x^4 + 2) dx$ .

**SOLUTION** We will make the substitution  $u = x^4 + 2$  because its differential is  $du = 4x^3 dx$ , which, apart from the constant factor 4, occurs in the integral. Thus, using  $x^3 dx = du/4$  and the Substitution Rule, we have

$$\int x^{3} \cos(x^{4} + 2) dx = \int \cos u \cdot \frac{1}{4} du = \frac{1}{4} \int \cos u du$$
$$= \frac{1}{4} \sin u + C$$
$$= \frac{1}{4} \sin(x^{4} + 2) + C$$

Check the answer by differentiating it.

Notice that at the final stage we had to return to the original variable x.

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral. This is accomplished by changing from the original variable x to a new variable u that is a function of x. Thus, in Example 1 we replaced the integral  $\int x^3 \cos(x^4 + 2) dx$  by the simpler integral  $\frac{1}{4} \int \cos u du$ . The main challenge in using the Substitution Rule is to think of an appropriate substitution. You should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor). This was the case in Example 1. If that is not possible, try choosing u to be some complicated part of the integrand. Finding the right substitution is a bit of an art. It's not unusual to guess wrong; if your first guess doesn't work, try another substitution.

**EXAMPLE 2** Evaluate 
$$\int \sqrt{2x+1} dx$$
.

**SOLUTION I** Let u = 2x + 1. Then du = 2 dx, so dx = du/2. Thus, the Substitution Rule gives

$$\int \sqrt{2x+1} \, dx = \int \sqrt{u} \, \frac{du}{2} = \frac{1}{2} \int u^{1/2} \, du$$
$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \frac{1}{3} u^{3/2} + C$$
$$= \frac{1}{3} (2x+1)^{3/2} + C$$

**SOLUTION 2** Another possible substitution is  $u = \sqrt{2x + 1}$ . Then

$$du = \frac{dx}{\sqrt{2x+1}}$$
 so  $dx = \sqrt{2x+1} du = u du$ 

(Or observe that  $u^2 = 2x + 1$ , so  $2u \, du = 2 \, dx$ .) Therefore

$$\int \sqrt{2x+1} \, dx = \int u \cdot u \, du = \int u^2 \, du$$
$$= \frac{u^3}{3} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

**EXAMPLE 3** Find  $\int \frac{x}{\sqrt{1-4x^2}} dx$ .

**SOLUTION** Let  $u = 1 - 4x^2$ . Then du = -8x dx, so  $x dx = -\frac{1}{8} du$  and

$$\int \frac{x}{\sqrt{1 - 4x^2}} \, dx = -\frac{1}{8} \int \frac{du}{\sqrt{u}} = -\frac{1}{8} \int u^{-1/2} \, du$$
$$= -\frac{1}{8} (2\sqrt{u}) + C = -\frac{1}{4} \sqrt{1 - 4x^2} + C$$

The answer to Example 3 could be checked by differentiation, but instead let's check it visually. In Figure 1 we have used a computer to graph both the integrand  $f(x) = x/\sqrt{1 - 4x^2}$  and its indefinite integral  $g(x) = -\frac{1}{4}\sqrt{1 - 4x^2}$  (we take the case C = 0). Notice that g(x) decreases when f(x) is negative, increases when f(x) is positive, and has its minimum value when f(x) = 0. So it seems reasonable, from the graphical evidence, that g is an antiderivative of f.

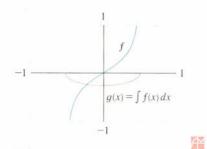


FIGURE 1  $f(x) = \frac{x}{\sqrt{1 - 4x^2}}$   $g(x) = \int f(x) \, dx = -\frac{1}{4}\sqrt{1 - 4x^2}$  **EXAMPLE 4** Calculate  $\int \cos 5x \, dx$ .

**SOLUTION** If we let u = 5x, then du = 5 dx, so  $dx = \frac{1}{5} du$ . Therefore

$$\int \cos 5x \, dx = \frac{1}{5} \int \cos u \, du = \frac{1}{5} \sin u + C = \frac{1}{5} \sin 5x + C$$

**EXAMPLE 5** Calculate  $\int \tan x \, dx$ .

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

This suggests that we should substitute  $u = \cos x$ , since then  $du = -\sin x \, dx$ and so  $\sin x \, dx = -du$ :

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u}$$
$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since  $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$ , the result of Example 5 can also be written as

$$\int \tan x \, dx = \ln |\sec x| + C$$

**EXAMPLE 6** • Evaluate  $\int \cos^3 x \, dx$ .

**SOLUTION** In order to use the Substitution Rule, we first use the trigonometric identity  $\cos^2 x = 1 - \sin^2 x$  and write

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$$

It is useful to have the extra factor of  $\cos x$  because if we make the substitution  $u = \sin x$ , then we have  $du = \cos x dx$ . Thus

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$
$$= \int (1 - u^2) \, du = u - \frac{1}{3}u^3 + C$$
$$= \sin x - \frac{1}{3}\sin^3 x + C$$

**EXAMPLE 7** Find  $\int \frac{1}{x(x+1)} dx$ .

**SOLUTION** We rewrite the integrand in terms of simpler fractions (called partial fractions) using the identity

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

#### SECTION 5.5 THE SUBSTITUTION RULE 397

which you can verify by combining the fractions on the right-hand side. Then

Computer algebra systems are also able to obtain partial fraction decompositions. See Exercise 59.

$$\int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$
$$= \int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$

In the last integral we make the substitution u = x + 1. Thus

$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$
$$= \ln|x| - \ln|x+1| + C$$

### Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Evaluation Theorem. For instance, using the result of Example 2, we have

$$\int_0^4 \sqrt{2x+1} \, dx = \int \sqrt{2x+1} \, dx \Big]_0^4 = \frac{1}{3}(2x+1)^{3/2} \Big]_0^4$$
$$= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{1}{3}(27-1) = \frac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

**I** The Substitution Rule for Definite Integrals If g' is continuous on [a, b] and f is continuous on the range of u = g(x), then

$$\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

**Proof** Let *F* be an antiderivative of *f*. Then, by (3), F(g(x)) is an antiderivative of f(g(x))g'(x), so by the Evaluation Theorem, we have

$$\int_{a}^{b} f(g(x))g'(x) \, dx = F(g(x))]_{a}^{b} = F(g(b)) - F(g(a))$$

But, applying the Evaluation Theorem a second time, we also have

$$\int_{g(a)}^{g(b)} f(u) \, du = F(u) \Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

**EXAMPLE 8** Evaluate  $\int_0^4 \sqrt{2x+1} \, dx$  using (5).

**SOLUTION** Using the substitution from Solution 1 of Example 2, we have u = 2x + 1 and dx = du/2. To find the new limits of integration we note that

when 
$$x = 0$$
,  $u = 1$  and when  $x = 4$ ,  $u = 9$ 

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u, not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to x = a and x = b.

Therefore

The geometric interpretation of Example 8 is shown in Figure 2. The substitution u = 2x + 1 stretches the interval [0, 4] by a factor of 2 and translates it to the right by 1 unit. The Substitution Rule shows that the two areas are equal.

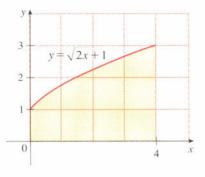
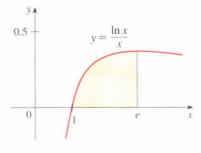
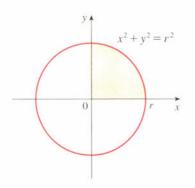


FIGURE 2

Since the function  $f(x) = (\ln x)/x$  in Example 9 is positive for x > 1, the integral represents the area of the shaded region in Figure 3.

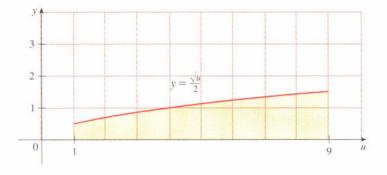






 $\int_0^4 \sqrt{2x+1} \, dx = \int_1^9 \frac{1}{2} \sqrt{u} \, du$  $= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big]_1^9$  $= \frac{1}{3} (9^{3/2} - 1^{3/2}) = \frac{26}{3}$ 

Observe that when using (5) we do not return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u.



**EXAMPLE 9** Calculate  $\int_{1}^{e} \frac{\ln x}{x} dx$ .

.

**SOLUTION** We let  $u = \ln x$  because its differential du = dx/x occurs in the integral. When x = 1,  $u = \ln 1 = 0$ ; when x = e,  $u = \ln e = 1$ . Thus

$$\int_{1}^{e} \frac{\ln x}{x} \, dx = \int_{0}^{1} u \, du = \frac{u^{2}}{2} \bigg]_{0}^{1} = \frac{1}{2}$$

**EXAMPLE 10** Prove that the area of a circle with radius r is  $\pi r^2$ .

**SOLUTION** This is, of course, a well-known formula. You were *told* that it's true a long time ago; but the only way to actually *prove* it is by integration.

For simplicity, let's place the circle with its center at the origin, so its equation is  $x^2 + y^2 = r^2$ . Solving this equation for y, we get

$$y = \pm \sqrt{r^2 - x^2}$$

Because the circle is symmetric with respect to both axes, the total area *A* is four times the area in the first quadrant (see Figure 4). The part of the circle in the first quadrant is given by the function

$$y = \sqrt{r^2 - x^2} \qquad 0 \le x \le r$$
$$\frac{1}{4}A = \int_0^r \sqrt{r^2 - x^2} \, dx$$

and so

To simplify this integral, we would like to make a substitution that turns  $r^2 - x^2$  into the square of something. The trigonometric identity  $1 - \sin^2\theta = \cos^2\theta$  is useful here. In fact, because

$$r^2 - r^2 \sin^2 \theta = r^2 \cos^2 \theta$$

FIGURE 4

we make the substitution

This substitution is a bit different from  
our previous substitutions. Here the old  
variable x is a function of the new vari-  
able 
$$\theta$$
 instead of the other way around.  
But our substitution  $x = r \sin \theta$  is  
equivalent to saying that  $\theta = \sin^{-1}(x/r)$ .

This our

able

This is Formula 17a in Appendix C.

$$\frac{1}{4}A = r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{1}{2}r^2 \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta$$
$$= \frac{1}{2}r^2 \Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = \frac{1}{2}r^2 \bigg(\frac{\pi}{2} + 0 - 0\bigg)$$
$$= \frac{1}{4}\pi r^2$$

 $x = r \sin \theta$ 

Since  $0 \le x \le r$ , we restrict  $\theta$  so that  $0 \le \theta \le \pi/2$ . We have  $dx = r \cos \theta d\theta$  and

 $\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2 \sin^2 \theta} = \sqrt{r^2 \cos^2 \theta} = r \cos \theta$ 

 $\int_0^r \sqrt{r^2 - x^2} \, dx = \int_0^{\pi/2} (r\cos\theta) r\cos\theta \, d\theta = r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$ 

because  $\cos \theta \ge 0$  when  $0 \le \theta \le \pi/2$ . Therefore, the Substitution Rule gives

Here we make the mental substitution  $u = 2\theta$ .

We have therefore proved the famous formula  $A = \pi r^2$ .

### ٠y

The next theorem uses the Substitution Rule for Definite Integrals (5) to simplify the calculation of integrals of functions that possess symmetry properties.

**6** Integrals of Symmetric Functions Suppose f is continuous on [-a, a]. (a) If f is even [f(-x) = f(x)], then  $\int_{-a}^{a} f(x) dx = 2 \int_{0}^{a} f(x) dx$ .

(b) If f is odd [f(-x) = -f(x)], then  $\int_{-a}^{a} f(x) dx = 0$ .

**Proof** We split the integral in two:

$$\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = -\int_{0}^{-a} f(x) \, dx + \int_{0}^{a} f(x) \, dx$$

In the first integral on the far right side we make the substitution u = -x. Then du = -dx and when x = -a, u = a. Therefore

$$-\int_0^{-a} f(x) \, dx = -\int_0^a f(-u) \, (-du) = \int_0^a f(-u) \, du$$

and so Equation 7 becomes

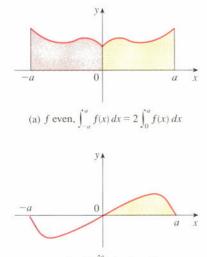
**B** 
$$\int_{-a}^{a} f(x) \, dx = \int_{0}^{a} f(-u) \, du \, + \, \int_{0}^{a} f(x) \, dx$$

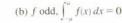
(a) If f is even, then f(-u) = f(u) so Equation 8 gives

$$\int_{-a}^{a} f(x) \, dx = \int_{0}^{a} f(u) \, du \, + \, \int_{0}^{a} f(x) \, dx = 2 \, \int_{0}^{a} f(x) \, dx$$

We integrate 
$$\cos^2 \theta$$
 by means of trigonometric identity  
 $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ 

Thus





(b) If f is odd, then f(-u) = -f(u) and so Equation 8 gives

$$\int_{-a}^{a} f(x) \, dx = -\int_{0}^{a} f(u) \, du \, + \, \int_{0}^{a} f(x) \, dx = 0$$

Theorem 6 is illustrated by Figure 5. For the case where f is positive and even, part (a) says that the area under y = f(x) from -a to a is twice the area from 0 to a because of symmetry. Recall that an integral  $\int_a^b f(x) dx$  can be expressed as the area above the x-axis and below y = f(x) minus the area below the axis and above the curve. Thus, part (b) says the integral is 0 because the areas cancel.

**EXAMPLE II** Since  $f(x) = x^6 + 1$  satisfies f(-x) = f(x), it is even and so

$$\int_{-2}^{2} (x^{6} + 1) dx = 2 \int_{0}^{2} (x^{6} + 1) dx$$
$$= 2 \left[ \frac{x^{7}}{7} + x \right]_{0}^{2} = 2 \left( \frac{128}{7} + 2 \right) = \frac{284}{7}$$

FIGURE 5

**EXAMPLE 12** Since  $f(x) = (\tan x)/(1 + x^2 + x^4)$  satisfies f(-x) = -f(x), it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4} \, dx = 0$$



**5.5** Exercises

substitution.

- 1.  $\int x(x^2 1)^{99} dx$ ,  $u = x^2 1$ 2.  $\int \frac{x^2}{\sqrt{2 + x^3}} dx$ ,  $u = 2 + x^3$
- **3.**  $\int e^{4x} dx$ , u = 4x **4.**  $\int \frac{dx}{(2x+1)^2}$ , u = 2x + 1
- 5.  $\int \frac{x+3}{(x^2+6x)^2} dx$ ,  $u = x^2 + 6x$
- **6.**  $\int \sec a\theta \tan a\theta \, d\theta, \quad u = a\theta$
- **7–32** Evaluate the indefinite integral.
- **7.**  $\int \frac{(\ln x)^2}{x} dx$  **8.**  $\int x e^{x^2} dx$  **9.**  $\int \sqrt{x 1} dx$  **10.**  $\int t^2 \cos(1 t^3) dt$  **11.**  $\int x^3 \sqrt{2 + x^4} dx$  **12.**  $\int x (x^2 + 1)^{3/2} dx$
- 13.  $\int \frac{2}{(t+1)^6} dt$ 14.  $\int \sqrt[5]{3-5y} \, dy$ **16.**  $\int \frac{\tan^{-1}x}{1+x^2} dx$ **15.**  $\int e^{x}(1 + e^{x})^{10} dx$ 18.  $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$ **17.**  $\int \sec^2 3\theta \, d\theta$  $19. \int \cos^4 x \sin x \, dx$ **20.**  $\int \cot x \, dx$ **21.**  $\int \sin^5 x \cos^2 x \, dx$ **22.**  $\int \cos x \cos(\sin x) dx$ 23.  $\int \frac{dx}{2x-1}$ **24.**  $\int \frac{x}{x^2 + 1} dx$ 25.  $\int \frac{dx}{x \ln x}$ **26.**  $\int e^x \sin(e^x) dx$ **27.**  $\int \frac{e^x + 1}{e^x} dx$ 28.  $\int \frac{e^x}{e^x + 1} dx$ **29.**  $\int \frac{x+1}{x^2+2x} dx$ **30.**  $\int \frac{\sin x}{1 + \cos^2 x} dx$
- **31.**  $\int \frac{1+x}{1+x^2} dx$  **32.**  $\int \frac{x}{1+x^4} dx$

**33–36** Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take C = 0).

- **33.**  $\int \frac{3x-1}{(3x^2-2x+1)^4} dx$  **34.**  $\int \frac{x}{\sqrt{x^2+1}} dx$ **35.**  $\int \sin^3 x \cos x \, dx$ **36.**  $\int \tan^2\theta \sec^2\theta \, d\theta$ **37–52** ■ Evaluate the definite integral. **37.**  $\int_{0}^{1} (2x-1)^{100} dx$  **38.**  $\int_{0}^{\pi/2} e^{\sin x} \cos x \, dx$ **40.**  $\int_{2}^{3} \frac{3x^2 - 1}{(x^3 - x)^2} dx$ **39.**  $\int_{0}^{1/2} \frac{1}{1+4x^2} dx$ **42.**  $\int_0^4 \frac{x}{\sqrt{1+2x}} dx$ **41.**  $\int_{1}^{2} x \sqrt{x-1} dx$ **43.**  $\int_{0}^{1} \cos \pi t \, dt$ **44.**  $\int_{0}^{\pi/4} \sin 4t \, dt$ **45.**  $\int_{1}^{4} \frac{1}{x^2} \sqrt{1 + \frac{1}{x}} dx$  **46.**  $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6} dx$ **48.**  $\int_{0}^{\pi/3} \frac{\sin\theta}{\cos^{2}\theta} d\theta$ **47.**  $\int_{a}^{a} x \sqrt{x^{2} + a^{2}} dx$ **49.**  $\int_0^3 \frac{dx}{2x+3}$ **50.**  $\int_{-1}^{1} t^2 2^{-t^3} dt$ **52.**  $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx$ 51.  $\int_{e}^{e^4} \frac{dx}{x \sqrt{\ln x}}$
- 53-54 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.
  - **53.**  $y = \sqrt{2x + 1}, \quad 0 \le x \le 1$
  - **54.**  $y = 2 \sin x \sin 2x$ ,  $0 \le x \le \pi$
  - **55.** Evaluate  $\int_{-2}^{2} (x + 3)\sqrt{4 x^2} dx$  by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.
  - **56.** Evaluate  $\int_0^1 x \sqrt{1 x^4} dx$  by making a substitution and interpreting the resulting integral in terms of an area.
  - 57. Verify that

$$\frac{1}{x^2 - 1} = \frac{1}{2} \left( \frac{1}{x - 1} - \frac{1}{x + 1} \right)$$

 $\int \frac{4x^2}{2x+1} dx$ 

and use this equation to evaluate  $\int \frac{1}{x^2 - 1} dx$ .

by first performing long division.

**CAS 59–60** Evaluate the integral by first using a computer algebra system to rewrite the integrand as a sum of simpler fractions. The Maple command is convert (f, parfrac, x); the Mathematica command is Apart[f]; the Derive command is Expand.

**59.** 
$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$
 **60.**  $\int \frac{1}{x(x+1)(2x+3)} dx$ 

**61.** Prove that the area enclosed by the given ellipse is  $\pi ab$ :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**62.** Use the trigonometric substitution  $x = \sin \theta$ , where  $-\pi/2 \le \theta \le \pi/2$ , to evaluate

$$\int \frac{1}{x^2 \sqrt{1-x^2}} \, dx$$

**63.** Use the trigonometric substitution  $x = 2 \tan \theta$ , where  $-\pi/2 < \theta < \pi/2$ , to evaluate

$$\int \frac{1}{x^2 \sqrt{x^2 + 4}} \, dx$$

- **64.** A bacteria population starts with 400 bacteria and grows at a rate of  $r(t) = (450.268)e^{1.12567t}$  bacteria per hour. How many bacteria will there be after three hours?
- **65.** Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function  $f(t) = \frac{1}{2}\sin(2\pi t/5)$  has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time *t*.
- **66.** Alabama Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after *t* weeks is modeled by the equation

$$\frac{dx}{dt} = 5000 \left( 1 - \frac{100}{(t+10)^2} \right) \text{ calculators/week}$$

(Notice that production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

**67.** If f is continuous and 
$$\int_0^4 f(x) dx = 10$$
, find  $\int_0^2 f(2x) dx$ .

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**68.** If f is continuous and 
$$\int_0^9 f(x) dx = 4$$
, find  $\int_0^3 x f(x^2) dx$ 

**69.** If f is continuous on  $\mathbb{R}$ , prove that

$$\int_{a}^{b} f(-x) \, dx = \int_{-b}^{-a} f(x) \, dx$$

For the case where  $f(x) \ge 0$ , draw a diagram to interpret this equation geometrically as an equality of areas.

**70.** If f is continuous on  $\mathbb{R}$ , prove that

$$\int_a^b f(x + c) \, dx = \int_{a+c}^{b+c} f(x) \, dx$$

For the case where  $f(x) \ge 0$ , draw a diagram to interpret this equation geometrically as an equality of areas.

**71.** If a and b are positive numbers, show that

$$\int_{0}^{1} x^{a} (1-x)^{b} dx = \int_{0}^{1} x^{b} (1-x)^{a} dx$$



# Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx} \left[ f(x)g(x) \right] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int \left[ f(x)g'(x) + g(x)f'(x) \right] dx = f(x)g(x)$$

or

 $\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$ 

We can rearrange this last equation as

 $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$ 

Formula 1 is called **the formula for integration by parts**. It is perhaps easier to remember in the following notation. Let u = f(x) and v = g(x). Then the differentials are du = f'(x) dx and dv = g'(x) dx, so, by the Substitution Rule, the formula for integration by parts becomes

$$\int u\,dv = uv - \int v\,du$$

**EXAMPLE I** Find  $\int x \sin x \, dx$ .

**SOLUTION USING FORMULA 1** Suppose we choose f(x) = x and  $g'(x) = \sin x$ . Then f'(x) = 1 and  $g(x) = -\cos x$ . (For g we can choose any antiderivative of g'.)

Thus, using Formula 1, we have

$$\int x \sin x \, dx = f(x)g(x) - \int g(x)f'(x) \, dx$$
$$= x(-\cos x) - \int (-\cos x) \, dx$$
$$= -x \cos x + \int \cos x \, dx$$
$$= -x \cos x + \sin x + C$$

It is wise to check the answer by differentiating it. If we do so, we get  $x \sin x$ , as expected.

u = x  $dv = \sin x \, dx$ 

du = dx  $v = -\cos x$ 

SOLUTION USING FORMULA 2 Let

It is helpful to use the pattern u = 1  $dv = \Box$  $du = \Box$  v = 1

Then

and so 
$$\int x \sin x \, dx = \int \frac{u}{x} \frac{dv}{\sin x \, dx} = \frac{u}{x} \frac{v}{(-\cos x)} - \int \frac{v}{(-\cos x)} \frac{du}{dx}$$
$$= -x \cos x + \int \cos x \, dx$$
$$= -x \cos x + \sin x + C$$

**NOTE** • Our objective in using integration by parts is to obtain a simpler integral than the one we started with. Thus, in Example 1 we started with  $\int x \sin x \, dx$  and expressed it in terms of the simpler integral  $\int \cos x \, dx$ . If we had chosen  $u = \sin x$  and  $dv = x \, dx$ , then  $du = \cos x \, dx$  and  $v = x^2/2$ , so integration by parts gives

$$\int x \sin x \, dx = (\sin x) \frac{x^2}{2} - \frac{1}{2} \int x^2 \cos x \, dx$$

But  $\int x^2 \cos x \, dx$  is a more difficult integral than the one we started with. In general, when deciding on a choice for u and dv, we usually try to choose u = f(x) to be a function that becomes simpler when differentiated (or at least not more complicated) as long as  $dv = g'(x) \, dx$  can be readily integrated to give v.

**EXAMPLE 2** Evaluate  $\int \ln x \, dx$ .

**SOLUTION** Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$
  
Then 
$$du = \frac{1}{x} dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x \, dx = x \ln x - \int x \, \frac{dx}{x}$$
$$= x \ln x - \int dx$$
$$= x \ln x - x + C$$

It's customary to write  $\int 1 dx \operatorname{as} \int dx$ .

Check the answer by differentiating it.

Integration by parts is effective in this example because the derivative of the function  $f(x) = \ln x$  is simpler than f.

**EXAMPLE 3** Find 
$$\int x^2 e^x dx$$
.  
SOLUTION Let  $u = x^2$   $dv = e^x dx$   
Then  $du = 2x dx$   $v = e^x$ 

Integration by parts gives

3

4

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

The integral that we obtained,  $\int xe^x dx$ , is simpler than the original integral but is still not obvious. Therefore, we use integration by parts a second time, this time with u = x and  $dv = e^x dx$ . Then du = dx,  $v = e^x$ , and

$$\int xe^{x} dx = xe^{x} - \int e^{x} dx$$
$$= xe^{x} - e^{x} + C$$

Putting this in Equation 3, we get

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$
  
=  $x^2 e^x - 2(x e^x - e^x + C)$   
=  $x^2 e^x - 2x e^x + 2e^x + C_1$  where  $C_1 = -2C$ 

**EXAMPLE 4** Evaluate  $\int e^x \sin x \, dx$ .

**SOLUTION** Let  $u = e^x$  and  $dv = \sin x \, dx$ . Then  $du = e^x \, dx$  and  $v = -\cos x$ , so integration by parts gives

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

The integral that we have obtained,  $\int e^x \cos x \, dx$ , is no simpler than the original one, but at least it is no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again.

An easier method, using complex numbers, is given in Exercise 50 in Appendix H.

#### SECTION 5.6 INTEGRATION BY PARTS 405

This time we use  $u = e^x$  and  $dv = \cos x \, dx$ . Then  $du = e^x \, dx$ ,  $v = \sin x$ , and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

At first glance, it appears as if we have accomplished nothing because we have arrived at  $\int e^x \sin x \, dx$ , which is where we started. However, if we put Equation 5 into Equation 4 we get

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

This can be regarded as an equation to be solved for the unknown integral. Solving, we obtain

$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

and, dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between a and b, assuming f' and g' are continuous, and using the Evaluation Theorem, we obtain

$$\int_{a}^{b} f(x)g'(x) \, dx = f(x)g(x) \Big]_{a}^{b} - \int_{a}^{b} g(x)f'(x) \, dx$$

 $u = \tan^{-1} x$  dv = dx

**EXAMPLE 5** Calculate  $\int_0^1 \tan^{-1} x \, dx$ .

SOLUTION Let

Then

Since 
$$\tan^{-1} x \ge 0$$
 for  $x \ge 0$ , the integral  
in Example 5 can be interpreted as the  
area of the region shown in Figure 2.

So Formula 6 gives

$$\int_0^1 \tan^{-1} x \, dx = x \tan^{-1} x \Big]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$
$$= 1 \cdot \tan^{-1} 1 - 0 \cdot \tan^{-1} 0 - \int_0^1 \frac{x}{1+x^2} \, dx$$
$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx$$

To evaluate this integral we use the substitution  $t = 1 + x^2$  (since u has another meaning in this example). Then dt = 2x dx, so x dx = dt/2. When x = 0, t = 1;

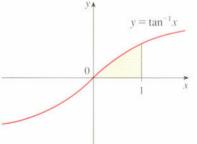


FIGURE I

Figure 1 illustrates Example 4 by showing the graphs of  $f(x) = e^x \sin x$  and  $F(x) = \frac{1}{2}e^{x}(\sin x - \cos x)$ . As a visual check on our work, notice that f(x) = 0when F has a maximum or minimum. 10

FIGURE 2

 $du = \frac{dx}{1 + x^2} \qquad v = x$ 

when x = 1, t = 2; so

$$\int_{0}^{1} \frac{x}{1+x^{2}} dx = \frac{1}{2} \int_{1}^{2} \frac{dt}{t} = \frac{1}{2} \ln|t| \Big]_{1}^{2}$$
$$= \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Therefore  $\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx = \frac{\pi}{4} - \frac{\ln 2}{2}$ 

**EXAMPLE 6** Prove the reduction formula

$$\int \sin^{n} x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

where  $n \ge 2$  is an integer.

**SOLUTION** Let 
$$u = \sin^{n-1}x$$
  $dv = \sin x \, dx$ 

Then  $du = (n-1)\sin^{n-2}x\cos x \, dx$   $v = -\cos x$ 

so integration by parts gives

$$\int \sin^{n} x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^{2} x \, dx$$

Since  $\cos^2 x = 1 - \sin^2 x$ , we have

$$\int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus, we have

$$n \int \sin^{n} x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$
  
$$\int \sin^{n} x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

or

The reduction formula (7) is useful because by using it repeatedly we could eventually express  $\int \sin^n x \, dx$  in terms of  $\int \sin x \, dx$  (if *n* is odd) or in terms of  $\int (\sin x)^0 \, dx = \int dx$  (if *n* is even). For instance, if we put n = 2 in Formula 7, we get

$$\int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

If we now put n = 4 in Formula 7, we obtain

$$\int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x \, dx$$
$$= -\frac{1}{4} \cos x \sin^3 x + \frac{3}{8} x - \frac{3}{16} \sin 2x + C$$

Exercises · · · ·

**1–24** ■ Evaluate the integral.

$\int x e^{2x} dx$	<b>2.</b> $\int x \cos x  dx$
<b>3.</b> $\int x \sin 4x  dx$	$4. \int x \ln x  dx$
$5. \int x^2 \cos 3x  dx$	$6. \ \int x^2 \sin 2x  dx$
$7. \int (\ln x)^2 dx$	$8. \int \sin^{-1}x  dx$
<b>9.</b> $\int \theta \sin \theta \cos \theta  d\theta$	$10. \int \theta \sec^2 \theta  d\theta$
$\mathbf{II.} \int t^2 \ln t  dt$	$12. \int t^3 e^t dt$
$13. \int e^{2\theta} \sin 3\theta  d\theta$	$14. \int e^{-\theta} \cos 3\theta  d\theta$
<b>15.</b> $\int_0^1 t e^{-t} dt$	<b>16.</b> $\int_{1}^{4} \sqrt{t} \ln t  dt$
<b>17.</b> $\int_0^{\pi/2} x \cos 2x  dx$	<b>18.</b> $\int_0^1 x^2 e^{-x} dx$
<b>19.</b> $\int_0^{1/2} \sin^{-1}x  dx$	<b>20.</b> $\int_{\pi/4}^{\pi/2} x \csc^2 x  dx$
$21. \ \int_{1}^{4} \ln \sqrt{x} \ dx$	<b>22.</b> $\int_0^1 x \tan^{-1} x  dx$
<b>23.</b> $\int_0^1 (x^2 - 1) e^x dx$	$24. \int_{1}^{e} \cos(\ln x)  dx$
a a a a a	6 10 10 10 10 10 10 10 10 10 10 10 10 10

**25–28** First make a substitution and then use integration by parts to evaluate the integral.

<b>25.</b> $\int \sin \sqrt{x}  dx$	$26. \int x^5 \cos(x^3)  dx$
$27. \int x^5 e^{x^2} dx$	$28. \ \int_{1}^{4} e^{\sqrt{x}} dx$

**29–32** Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take C = 0).

**29.**  $\int x \cos \pi x \, dx$  **30.**  $\int \sqrt{x} \ln x \, dx$  **31.**  $\int (2x + 3)e^x \, dx$ **32.**  $\int x^3 e^{x^2} \, dx$ 

**33.** Prove the reduction formula

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

34. (a) Use Exercise 33 to evaluate ∫ cos²x dx.
(b) Use part (a) and Exercise 33 to evaluate ∫ cos⁴x dx.

**35.** (a) Use the reduction formula in Example 6 to show that

$$\int_{0}^{\pi/2} \sin^{n} x \, dx = \frac{n-1}{n} \int_{0}^{\pi/2} \sin^{n-2} x \, dx$$

where  $n \ge 2$  is an integer.

- (b) Use part (a) to evaluate  $\int_0^{\pi/2} \sin^3 x \, dx$  and  $\int_0^{\pi/2} \sin^5 x \, dx$ .
- (c) Use part (a) to show that, for odd powers of sin,

$$\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1)}$$

36. Prove that, for even powers of sin,

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \frac{\pi}{2}$$

**37–38** ■ Use integration by parts to prove the reduction formula.

**37.**  $\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$ 

**38.** 
$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

- **39.** Use Exercise 37 to find  $\int (\ln x)^3 dx$ .
- **40.** Use Exercise 38 to find  $\int x^4 e^x dx$ .
- **41.** A particle that moves along a straight line has velocity  $v(t) = t^2 e^{-t}$  meters per second after t seconds. How far will it travel during the first t seconds?
- **42.** A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity  $v_e$  (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If  $g = 9.8 \text{ m/s}^2$ , m = 30,000 kg, r = 160 kg/s, and  $v_e = 3000 \text{ m/s}$ , find the height of the rocket one minute after liftoff.

43. Use integration by parts to show that

$$\int f(x) \, dx = x f(x) - \int x f'(x) \, dx$$

**44.** (a) If f is one-to-one and f' is continuous, prove that

$$\int_{a}^{b} f(x) \, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y) \, dy$$

[*Hint*: Use Exercise 43 and make the substitution y = f(x).]

(b) In the case where f is a positive function and b > a > 0, draw a diagram to give a geometric interpretation of part (a).

**45.** If f(0) = g(0) = 0, show that

$$\int_0^a f(x)g''(x) \, dx = f(a)g'(a) \, - \, f'(a)g(a) \, + \, \int_0^a f''(x)g(x) \, dx$$

- **46.** Let  $I_n = \int_0^{\pi/2} \sin^n x \, dx$ .
  - (a) Show that  $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ .
  - (b) Use Exercise 36 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

and deduce that  $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$ .

(d) Use part (c) and Exercises 35 and 36 to show that

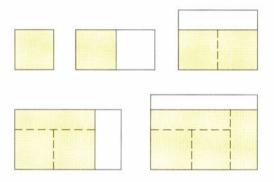
12	2	2	4	4	6	6	2n	2n		$\pi$
$n \rightarrow \infty$	1	3	3	5	5	$\frac{0}{7}$ · · · · ·	2n - 1	2n + 1	_	2

This formula is usually written as an infinite product:

 $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots$ 

and is called the Wallis product.

(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle. (See the figure.) Find the limit of the ratios of width to height of these rectangles.





### Integration Using Tables and Computer Algebra Systems

In this section we describe how to evaluate integrals using tables and computer algebra systems.

#### Tables of Integrals

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don't have access to a computer algebra system. A relatively brief table of 120 integrals is provided on the back endpapers. More extensive tables are available in the *CRC Mathematical Tables* (463 entries) or in Gradshteyn and Ryzhik's *Table of Integrals, Series and Products* (New York: Academic Press, 1979), which contains hundreds of pages of integrals. It should be remembered, however, that integrals do not often occur in exactly the form listed in a table. Usually we need to use the Substitution Rule or algebraic simplification to transform a given integral into one of the forms in the table.

**EXAMPLE I** Use the Table of Integrals to evaluate 
$$\int_0^2 \frac{x^2 + 12}{x^2 + 4} dx$$
.

The Table of Integrals appears on the back endpapers.

**SOLUTION** The only formula in the table that resembles our given integral is number 17:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

If we perform long division, we get

$$\frac{x^2 + 12}{x^2 + 4} = 1 + \frac{8}{x^2 + 4}$$

Now we can use Formula 17 with a = 2:

$$\int_{0}^{2} \frac{x^{2} + 12}{x^{2} + 4} dx = \int_{0}^{2} \left( 1 + \frac{8}{x^{2} + 4} \right) dx$$
$$= x + 8 \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} \Big]_{0}^{2}$$
$$= 2 + 4 \tan^{-1} 1 = 2 + \pi$$

**EXAMPLE 2** Use the Table of Integrals to find  $\int \frac{x^2}{\sqrt{5-4x^2}} dx$ .

**SOLUTION** If we look at the section of the table entitled *Forms involving*  $\sqrt{a^2 - u^2}$ , we see that the closest entry is number 34:

$$\int \frac{u^2}{\sqrt{a^2 - u^2}} \, du = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

This is not exactly what we have, so we make the substitution u = 2x:

$$\int \frac{x^2}{\sqrt{5-4x^2}} \, dx = \int \frac{(u/2)^2}{\sqrt{5-u^2}} \, \frac{du}{2} = \frac{1}{8} \int \frac{u^2}{\sqrt{5-u^2}} \, du$$

Then we use Formula 34 with  $a^2 = 5$ :

$$\int \frac{x^2}{\sqrt{5 - 4x^2}} \, dx = \frac{1}{8} \int \frac{u^2}{\sqrt{5 - u^2}} \, du$$
$$= \frac{1}{8} \left[ -\frac{u}{2} \sqrt{5 - u^2} + \frac{5}{2} \sin^{-1} \frac{u}{\sqrt{5}} \right] + C$$
$$= -\frac{x}{8} \sqrt{5 - 4x^2} + \frac{5}{16} \sin^{-1} \left(\frac{2x}{\sqrt{5}}\right) + C$$

**EXAMPLE 3** Use the Table of Integrals to find  $\int x^3 \sin x \, dx$ .

**SOLUTION** We look in the section called *Trigonometric Forms* and use the reduction formula in entry 84 with n = 3:

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3 \int x^2 \cos x \, dx$$

Then we use entries 85 and 82:

**85.** 
$$\int u^n \cos u \, du = u^n \sin u$$
$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx$$
$$= x^2 \sin x - 2(\sin x - x \cos x) + K$$

Combining these calculations, we get

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

where C = 3K.

**EXAMPLE 4** Use the Table of Integrals to find  $\int x \sqrt{x^2 + 2x + 4} \, dx$ .

**SOLUTION** Since the table gives forms involving  $\sqrt{a^2 + x^2}$ ,  $\sqrt{a^2 - x^2}$ , and  $\sqrt{x^2 - a^2}$ , but not  $\sqrt{ax^2 + bx + c}$ , we first complete the square:

$$x^{2} + 2x + 4 = (x + 1)^{2} + 3$$

Therefore, we make the substitution u = x + 1:

$$\int x \sqrt{x^2 + 2x + 4} \, dx = \int (u - 1) \sqrt{u^2 + 3} \, du$$
$$= \int u \sqrt{u^2 + 3} \, du - \int \sqrt{u^2 + 3} \, du$$

The first integral is evaluated using the substitution  $t = u^2 + 3$ :

$$\int u\sqrt{u^2+3} \, du = \frac{1}{2} \int \sqrt{t} \, dt = \frac{1}{2} \cdot \frac{2}{3}t^{3/2} = \frac{1}{3}(u^2+3)^{3/2}$$

For the second integral we use Formula 21 with  $a = \sqrt{3}$ :

21. 
$$\int \sqrt{a^2 + u^2} \, du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$$

$$\sqrt{u^2 + 3} \, du = \frac{u}{2} \sqrt{u^2 + 3} + \frac{3}{2} \ln(u + \sqrt{u^2 + 3})$$

Thus

$$\int x\sqrt{x^2 + 2x + 4} \, dx = \frac{1}{3}(x^2 + 2x + 4)^{3/2} - \frac{x+1}{2}\sqrt{x^2 + 2x + 4}$$
$$- \frac{3}{2}\ln(x + 1 + \sqrt{x^2 + 2x + 4}) + C$$

### **Computer Algebra Systems**

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We have seen that the use of tables involves matching the form of the given integrand with the forms of the integrands in the tables. Computers are particularly good at matching patterns. And just as we used substitutions in conjunction with tables, a CAS can perform substitutions that transform a given integral into one that occurs in its stored formulas. So it is not surprising that computer algebra systems excel at integration. That does not mean that integration by hand is an obsolete skill. We will see that a hand computation sometimes produces an indefinite integral in a form that is more convenient than a machine answer.

To begin, let's see what happens when we ask a machine to integrate the relatively simple function y = 1/(3x - 2). Using the substitution u = 3x - 2, an easy calculation by hand gives

$$\int \frac{1}{3x - 2} \, dx = \frac{1}{3} \ln|3x - 2| + C$$

whereas Derive, Mathematica, and Maple all return the answer

$$\frac{1}{3}\ln(3x-2)$$

The first thing to notice is that computer algebra systems omit the constant of integration. In other words, they produce a *particular* antiderivative, not the most general one. Therefore, when making use of a machine integration, we might have to add a constant. Second, the absolute value signs are omitted in the machine answer. That is fine if our problem is concerned only with values of x greater than  $\frac{2}{3}$ . But if we are interested in other values of x, then we need to insert the absolute value symbol.

In the next example we reconsider the integral of Example 4, but this time we ask a machine for the answer.

**EXAMPLE 5** Use a computer algebra system to find 
$$\int x\sqrt{x^2 + 2x + 4} \, dx$$
.

**SOLUTION** Maple responds with the answer

$$\frac{1}{3}(x^2 + 2x + 4)^{3/2} - \frac{1}{4}(2x + 2)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\ln(2\sqrt{x^2 + 2x + 4} + 2x + 2)$$

Notice that this is equivalent to the answer we got in Example 4 because the third term can be rewritten as

$$-\frac{3}{2}\ln[2(\sqrt{x^2+2x+4}+x+1)] = -\frac{3}{2}\ln 2 - \frac{3}{2}\ln(x+1+\sqrt{x^2+2x+4})$$

The extra term  $-\frac{3}{2} \ln 2$  can be absorbed into the constant of integration.

Mathematica gives the answer

$$\left(\frac{5}{6} + \frac{x}{6} + \frac{x^2}{3}\right)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\operatorname{arcsinh}\left(\frac{1+x}{\sqrt{3}}\right)$$

The first term corresponds to the first two terms in the answer in Example 4. The last terms are equivalent because of the identity

$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$$

Derive gives the answer

$$\frac{1}{6}\sqrt{x^2 + 2x + 4}(2x^2 + x + 5) - \frac{3}{2}\ln(\sqrt{x^2 + 2x + 4} + x + 1)$$

The first term is like the first term in the Mathematica answer. The second term is identical to the last term in Example 4.

**EXAMPLE 6** Use a CAS to evaluate 
$$\int x(x^2 + 5)^8 dx$$
.

**SOLUTION** Maple and Mathematica give the same answer:

$$\frac{1}{18}x^{18} + \frac{5}{2}x^{16} + 50x^{14} + \frac{1750}{3}x^{12} + 4375x^{10} + 21875x^8 + \frac{218750}{3}x^6 + 156250x^4 + \frac{390625}{2}x^2$$

This is the formula in Problem 9(c) in the Discovery Project on page 254.

It's clear that both systems must have expanded  $(x^2 + 5)^8$  by the Binomial Theorem and then integrated each term.

If we integrate by hand instead, using the substitution  $u = x^2 + 5$ , we get

$$\int x(x^2+5)^8 dx = \frac{1}{18}(x^2+5)^9 + C$$

For most purposes, this is a more convenient form of the answer.

**EXAMPLE 7** Use a CAS to find 
$$\int \sin^5 x \cos^2 x \, dx$$
.

**SOLUTION** Derive and Maple report the answer

$$-\frac{1}{7}\sin^4 x \cos^3 x - \frac{4}{35}\sin^2 x \cos^3 x - \frac{8}{105}\cos^3 x$$

whereas Mathematica produces

$$-\frac{5}{64}\cos x - \frac{1}{192}\cos 3x + \frac{3}{320}\cos 5x - \frac{1}{448}\cos 7x$$

We suspect that there are trigonometric identities which show these answers are equivalent. Indeed, if we ask Derive, Maple, and Mathematica to simplify their expressions using trigonometric identities, they ultimately produce the same form of the answer:

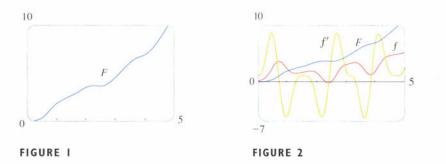
$$\int \sin^5 x \cos^2 x \, dx = -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x$$

**EXAMPLE 8** If  $f(x) = x + 60 \sin^4 x \cos^5 x$ , find the antiderivative *F* of *f* such that F(0) = 0. Graph *F* for  $0 \le x \le 5$ . Where does *F* have maximum and minimum values and inflection points?

**SOLUTION** The antiderivative of *f* produced by Maple is

$$F(x) = \frac{1}{2}x^2 - \frac{20}{3}\sin^3 x \cos^6 x - \frac{20}{7}\sin x \cos^6 x + \frac{4}{7}\cos^4 x \sin x + \frac{16}{21}\cos^2 x \sin x + \frac{32}{21}\sin x$$

and we note that F(0) = 0. This expression could probably be simplified, but there is no need to do so because a computer algebra system can graph this version of *F* as easily as any other version. A graph of *F* is shown in Figure 1. To locate the maximum and minimum values of *F* we graph its derivative F' = fin Figure 2 and observe that *F* has a local maximum when  $x \approx 2.3$  and a local minimum when  $x \approx 2.5$ . The graph of F'' = f' in Figure 2 shows that *F* has inflection points when  $x \approx 0.7$ , 1.3, 1.8, 2.4, 3.3, and 3.9.



Derive also gives this answer.

#### Can We Integrate All Continuous Functions?

The question arises: Will our basic integration formulas, together with the Substitution Rule, integration by parts, tables of integrals, and computer algebra systems, enable us to find the integral of every continuous function? In particular, can we use it to evaluate  $\int e^{x^2} dx$ ? The answer is No, at least not in terms of the functions that we are familiar with.

Most of the functions that we have been dealing with in this book are what are called **elementary functions**. These are the polynomials, rational functions, power functions  $(x^a)$ , exponential functions  $(a^x)$ , logarithmic functions, trigonometric and inverse trigonometric functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cos x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but  $\int f(x) dx$ need not be an elementary function. Consider  $f(x) = e^{x^2}$ . Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2} dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^x$$

Thus,  $f(x) = e^{x^2}$  has an antiderivative *F*, but it has been proved that *F* is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating  $\int e^{x^2} dx$  in terms of the functions we know. (In Chapter 8, however, we will see how to express  $\int e^{x^2} dx$  as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x} dx \qquad \int \sin(x^2) dx \qquad \int \cos(e^x) dx$$
$$\int \sqrt{x^3 + 1} dx \qquad \int \frac{1}{\ln x} dx \qquad \int \frac{\sin x}{x} dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives.

**5.**  $\int \frac{\sqrt{9x^2 - 1}}{x^2} dx$  **6.**  $\int \frac{\sin x \cos x}{\sqrt{1 + \sin x}} dx$ 

. . . . . . . . . . .

5.7

Exercises

I-22 Use the Table of Integrals on the back endpapers to evaluate the integral.

1. 
$$\int \frac{x^3 - x^2 + x - 1}{x^2 + 9} dx$$
 2.  $\int \csc^3(x/2) dx$  7.  $\int x \sin^{-1}(x^2) dx$  8.  $\int x^3 \sin^{-1}(x^2) dx$   
3.  $\int e^{-3x} \cos 4x \, dx$  4.  $\int \frac{\sqrt{4 - 3x^2}}{x} \, dx$  9.  $\int \sqrt{5 - 4x - x^2} \, dx$  10.  $\int x^2 \cos 3x \, dx$ 

11. 
$$\int \sec^5 x \, dx$$
 12.  $\int \sin^6 2x \, dx$ 

 13.  $\int \sin^2 x \cos x \ln(\sin x) \, dx$ 
 14.  $\int \frac{dx}{e^x (1 + 2e^x)}$ 

 15.  $\int_0^{\pi/2} \cos^5 x \, dx$ 
 16.  $\int \frac{x}{\sqrt{x^2 - 4x}} \, dx$ 

 17.  $\int \frac{x^4 \, dx}{\sqrt{x^{10} - 2}}$ 
 18.  $\int_0^1 x^4 e^{-x} \, dx$ 

 19.  $\int e^x \ln(1 + e^x) \, dx$ 
 20.  $\int x^2 \tan^{-1}x \, dx$ 

 21.  $\int \sqrt{e^{2x} - 1} \, dx$ 
 22.  $\int e^{\sin x} \sin 2x \, dx$ 

- 23. Verify Formula 53 in the Table of Integrals (a) by differentiation and (b) by using the substitution t = a + bu.
- **24.** Verify Formula 31 (a) by differentiation and (b) by substituting  $u = a \sin \theta$ .
- CAS 25-32 Use a computer algebra system to evaluate the integral. Compare the answer with the result of using tables. If the answers don't appear to be the same, show that they are equivalent.
  - **25.**  $\int x^2 \sqrt{5 x^2} \, dx$  **26.**  $\int x^2 (1 + x^3)^4 \, dx$  **27.**  $\int \sin^3 x \cos^2 x \, dx$  **28.**  $\int \tan^2 x \sec^4 x \, dx$  **29.**  $\int x \sqrt{1 + 2x} \, dx$  **30.**  $\int \sin^4 x \, dx$  **31.**  $\int \tan^3 x \, dx$ **32.**  $\int x^5 \sqrt{x^2 + 1} \, dx$

**CAS** 33. Computer algebra systems sometimes need a helping hand from human beings. Ask your CAS to evaluate

$$\int 2^x \sqrt{4^x - 1} \, dx$$

If it doesn't return an answer, ask it to try

$$\int 2^x \sqrt{2^{2x} - 1} \, dx$$

instead. Why do you think it was successful with this form of the integrand?

**CAS** 34. Try to evaluate

$$\int (1 + \ln x) \sqrt{1 + (x \ln x)^2} dx$$

with a computer algebra system. If it doesn't return an answer, make a substitution that changes the integral into one that the CAS *can* evaluate.

**CAS** 35-36 Use a CAS to find an antiderivative F of f such that F(0) = 0. Graph f and F and locate approximately the *x*-coordinates of the extreme points and inflection points of F.

**35.** 
$$f(x) = \frac{x^2 - 1}{x^4 + x^2 + 1}$$

**36.**  $f(x) = xe^{-x}\sin x, -5 \le x \le 5$ 

**CAS** 37-38 Use a graphing device to draw a graph of f and use this graph to make a rough sketch, by hand, of the graph of the antiderivative F such that F(0) = 0. Then use a CAS to find F explicitly and graph it. Compare the machine graph with your sketch.

**37.** 
$$f(x) = \sin^4 x \cos^6 x, \ 0 \le x \le \pi$$

**38.** 
$$f(x) = \frac{x^3 - x}{x^6 + 1}$$

Discovery Project

### Patterns in Integrals

In this project a computer algebra system is used to investigate indefinite integrals of families of functions. By observing the patterns that occur in the integrals of several members of the family, you will first guess, and then prove, a general formula for the integral of any member of the family.

I. (a) Use a computer algebra system to evaluate the following integrals.

(i) 
$$\int \frac{1}{(x+2)(x+3)} dx$$
  
(ii)  $\int \frac{1}{(x+1)(x+5)} dx$   
(iii)  $\int \frac{1}{(x+2)(x-5)} dx$   
(iv)  $\int \frac{1}{(x+2)^2} dx$ 

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \frac{1}{(x+a)(x+b)} \, dx$$

if  $a \neq b$ . What if a = b?

- (c) Check your guess by asking your CAS to evaluate the integral in part (b). Then prove it by differentiation.
- 2. (a) Use a computer algebra system to evaluate the following integrals.
  - (i)  $\int \sin x \cos 2x \, dx$  (ii)  $\int \sin 3x \cos 7x \, dx$
  - (iii)  $\int \sin 8x \cos 3x \, dx$
  - (b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\sin ax \cos bx \, dx$$

- (c) Check your guess with a CAS. Then prove it either by differentiation or by integration by parts. For what values of *a* and *b* is it valid?
- **3.** (a) Use a computer algebra system to evaluate the following integrals.
  - (i)  $\int \ln x \, dx$ (ii)  $\int x \ln x \, dx$ (iii)  $\int x^2 \ln x \, dx$ (iv)  $\int x^3 \ln x \, dx$ (v)  $\int x^7 \ln x \, dx$
  - (b) Based on the pattern of your responses in part (a), guess the value of

$$\int x^n \ln x \, dx$$

- (c) Use integration by parts to prove the conjecture that you made in part (b). For what values of *n* is it valid?
- 4. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\int x e^x dx$	(ii) $\int x^2 e^x dx$
(iii) $\int x^3 e^x dx$	(iv) $\int x^4 e^x dx$
(v) $\int x^5 e^x dx$	

- (b) Based on the pattern of your responses in part (a), guess the value of  $\int x^6 e^x dx$ . Then use your CAS to check your guess.
- (c) Based on the patterns in parts (a) and (b), make a conjecture as to the value of the integral

$$\int x^n e^x dx$$

when n is a positive integer.

(d) Use mathematical induction to prove the conjecture you made in part (c).



## Approximate Integration

There are two situations in which it is impossible to find the exact value of a definite integral.

The first situation arises from the fact that in order to evaluate  $\int_a^b f(x) dx$  using the Evaluation Theorem we need to know an antiderivative of *f*. Sometimes, however, it is difficult, or even impossible, to find an antiderivative (see Section 5.7). For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2} dx \qquad \int_{-1}^1 \sqrt{1 + x^3} \, dx$$

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data. There may be no formula for the function (see Example 5).

In both cases we need to find approximate values of definite integrals. We already know one such method. Recall that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as an approximation to the integral. In particular, let us divide [a, b] into n subintervals of equal length  $\Delta x = (b - a)/n$ . Then we have

$$\int_a^b f(x) \, dx \approx \sum_{i=1}^n f(x_i^*) \, \Delta x$$

where  $x_i^*$  is any point in the *i*th subinterval  $[x_{i-1}, x_i]$ . If  $x_i^*$  is chosen to be the left endpoint of the interval, then  $x_i^* = x_{i-1}$  and we have

 $\int_{a}^{b} f(x) dx \approx L_{n} = \sum_{i=1}^{n} f(x_{i-1}) \Delta x$ 

If  $f(x) \ge 0$ , then the integral represents an area and (1) represents an approximation of this area by the rectangles shown in Figure 1(a). If we choose  $x_i^*$  to be the right endpoint, then  $x_i^* = x_i$  and we have

2 
$$\int_{a}^{b} f(x) dx \approx R_{n} = \sum_{i=1}^{n} f(x_{i}) \Delta x$$

[See Figure 1(b).] The approximations  $L_n$  and  $R_n$  defined by Equations 1 and 2 are called the **left endpoint approximation** and **right endpoint approximation**, respectively.

In Section 5.2 we also considered the case where  $x_i^*$  is chosen to be the midpoint  $\overline{x}_i$  of the subinterval  $[x_{i-1}, x_i]$ . Figure 1(c) shows the midpoint approximation  $M_n$ , which appears to be better than either  $L_n$  or  $R_n$ .

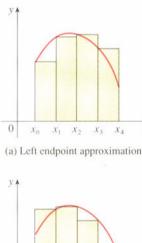
**Midpoint Rule** 

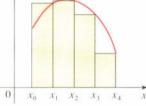
$$\int_{0}^{t_{n}} f(x) dx \approx M_{n} = \Delta x [f(\overline{x}_{1}) + f(\overline{x}_{2}) + \dots + f(\overline{x}_{n})]$$

$$\Delta x = \frac{b-a}{a}$$

where

and  $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$ 





(b) Right endpoint approximation

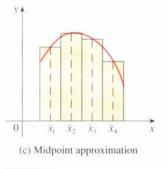


FIGURE I

Another approximation results from averaging the approximations in Equations 1 and 2:

$$\int_{a}^{b} f(x) dx \approx \frac{1}{2} \left[ \sum_{i=1}^{n} f(x_{i-1}) \Delta x + \sum_{i=1}^{n} f(x_{i}) \Delta x \right] = \frac{\Delta x}{2} \left[ \sum_{i=1}^{n} \left( f(x_{i-1}) + f(x_{i}) \right) \right]$$
$$= \frac{\Delta x}{2} \left[ \left( f(x_{0}) + f(x_{1}) \right) + \left( f(x_{1}) + f(x_{2}) \right) + \dots + \left( f(x_{n-1}) + f(x_{n}) \right) \right]$$
$$= \frac{\Delta x}{2} \left[ f(x_{0}) + 2f(x_{1}) + 2f(x_{2}) + \dots + 2f(x_{n-1}) + f(x_{n}) \right]$$

**Trapezoidal Rule** 

.

$$\int_{a}^{b} f(x) dx \approx T_{n} = \frac{\Delta x}{2} [f(x_{0}) + 2f(x_{1}) + 2f(x_{2}) + \dots + 2f(x_{n-1}) + f(x_{n})]$$
  
where  $\Delta x = (b - a)/n$  and  $x_{i} = a + i\Delta x$ .

The reason for the name Trapezoidal Rule can be seen from Figure 2, which illustrates the case  $f(x) \ge 0$ . The area of the trapezoid that lies above the *i*th sub-interval is

$$\Delta x \left( \frac{f(x_{i-1}) + f(x_i)}{2} \right) = \frac{\Delta x}{2} [f(x_{i-1}) + f(x_i)]$$

and if we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

**EXAMPLE 1** Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with n = 5 to approximate the integral  $\int_{1}^{2} (1/x) dx$ .

# SOLUTION

(a) With n = 5, a = 1, and b = 2, we have  $\Delta x = (2 - 1)/5 = 0.2$ , and so the Trapezoidal Rule gives

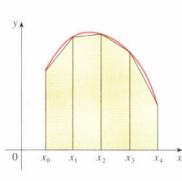
$$\int_{1}^{2} \frac{1}{x} dx \approx T_{5} = \frac{0.2}{2} [f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$
$$= 0.1 \left[ \frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2} \right]$$
$$\approx 0.695635$$

This approximation is illustrated in Figure 3.

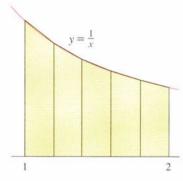
(b) The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9, so the Midpoint Rule gives

$$\int_{1}^{2} \frac{1}{x} dx \approx \Delta x [f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$
$$= \frac{1}{5} \left( \frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)$$
$$\approx 0.691908$$

This approximation is illustrated in Figure 4.









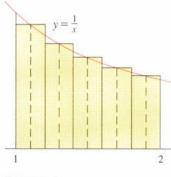


FIGURE 4

In Example 1 we deliberately chose an integral whose value can be computed explicitly so that we can see how accurate the Trapezoidal and Midpoint Rules are. By the Evaluation Theorem,

$$\int_{1}^{2} \frac{1}{x} dx = \ln x \Big]_{1}^{2} = \ln 2 = 0.693147...$$

 $\int_{a}^{b} f(x) \, dx = \text{approximation} + \text{error}$ 

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact. From the values in Example 1 we see that the errors in the Trapezoidal and Midpoint Rule approximations for n = 5 are

 $E_T \approx -0.002488$  and  $E_M \approx 0.001239$ 

In general, we have

$$E_T = \int_a^b f(x) \, dx - T_n$$
 and  $E_M = \int_a^b f(x) \, dx - M_n$ 

The following tables show the results of calculations similar to those in Example 1, but for n = 5, 10, and 20 and for the left and right endpoint approximations as well as the Trapezoidal and Midpoint Rules.

n	$L_n$	$R_n$	$T_{\mu}$	$M_n$	
5	0.745635	0.645635	0.695635	0.691908	
10	0.718771	0.668771	0.693771	0.692835	
20	0.705803	0.680803	0.693303	0.693069	

, H	$E_{t}$	$E_R$	$E_{I}$	$E_M$	
5	-0.052488	0.047512	-0.002488	0.001239	
10	-0.025624	0.024376	-0.000624	0.000312	
20	-0.012656	0.012344	-0.000156	0.000078	

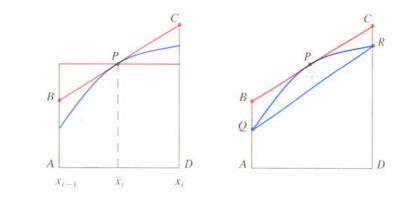
We can make several observations from these tables:

- 1. In all of the methods we get more accurate approximations when we increase the value of *n*. (But very large values of *n* result in so many arithmetic operations that we have to beware of accumulated round-off error.)
- 2. The errors in the left and right endpoint approximations are opposite in sign and appear to decrease by a factor of about 2 when we double the value of n.
- **3.** The Trapezoidal and Midpoint Rules are much more accurate than the endpoint approximations.
- **4.** The errors in the Trapezoidal and Midpoint Rules are opposite in sign and appear to decrease by a factor of about 4 when we double the value of *n*.
- **5.** The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

Approximations to  $\int_{1}^{2} \frac{1}{x} dx$ 

Corresponding errors

Figure 5 shows why we can expect the Midpoint Rule to be more accurate than the Trapezoidal Rule. The area of a typical rectangle in the Midpoint Rule is the same as the trapezoid *ABCD* whose upper side is tangent to the graph at *P*. The area of this trapezoid is closer to the area under the graph than is the area of the trapezoid *AQRD* used in the Trapezoidal Rule. [The midpoint error (shaded red) is smaller than the trapezoidal error (shaded blue).]





These observations are corroborated in the following error estimates, which are proved in books on numerical analysis. Notice that Observation 4 corresponds to the  $n^2$  in each denominator because  $(2n)^2 = 4n^2$ . The fact that the estimates depend on the size of the second derivative is not surprising if you look at Figure 5, because f''(x) measures how much the graph is curved. [Recall that f''(x) measures how fast the slope of y = f(x) changes.]

**3 Error Bounds** Suppose  $|f''(x)| \le K$  for  $a \le x \le b$ . If  $E_T$  and  $E_M$  are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$
 and  $|E_M| \leq \frac{K(b-a)^3}{24n^2}$ 

Let's apply this error estimate to the Trapezoidal Rule approximation in Example 1. If f(x) = 1/x, then  $f'(x) = -1/x^2$  and  $f''(x) = 2/x^3$ . Since  $1 \le x \le 2$ , we have  $1/x \le 1$ , so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2$$

Therefore, taking K = 2, a = 1, b = 2, and n = 5 in the error estimate (3), we see that

$$|E_T| \le \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Comparing this error estimate of 0.006667 with the actual error of about 0.002488, we see that it can happen that the actual error is substantially less than the upper bound for the error given by (3).

**EXAMPLE 2** How large should we take *n* in order to guarantee that the Trapezoidal and Midpoint Rule approximations for  $\int_{1}^{2} (1/x) dx$  are accurate to within 0.0001?

**SOLUTION** We saw in the preceding calculation that  $|f''(x)| \le 2$  for  $1 \le x \le 2$ , so we can take K = 2, a = 1, and b = 2 in (3). Accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore, we choose n so that

$$\frac{2(1)^3}{12n^2} < 0.0001$$

Solving the inequality for n, we get

$$n^2 > \frac{2}{12(0.0001)}$$

 $n > \frac{1}{\sqrt{0.0006}} \approx 40.8$ 

Oľ

Thus, n = 41 will ensure the desired accuracy.

For the same accuracy with the Midpoint Rule we choose n so that

$$\frac{2(1)^3}{24n^2} < 0.0001$$
$$n > \frac{1}{\sqrt{0.0012}} \approx 29$$

which gives

#### EXAMPLE 3

(a) Use the Midpoint Rule with n = 10 to approximate the integral  $\int_0^1 e^{x^2} dx$ .

(b) Give an upper bound for the error involved in this approximation.

#### SOLUTION

(a) Since a = 0, b = 1, and n = 10, the Midpoint Rule gives

$$\int_0^1 e^{x^2} dx \approx \Delta x [f(0.05) + f(0.15) + \dots + f(0.85) + f(0.95)]$$
  
= 0.1[ $e^{0.0025} + e^{0.0225} + e^{0.0625} + e^{0.1225} + e^{0.2025} + e^{0.3025}$   
+  $e^{0.4225} + e^{0.5625} + e^{0.7225} + e^{0.9025}]$   
 $\approx 1.460393$ 

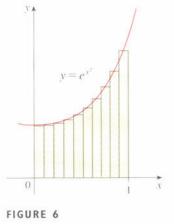
Figure 6 illustrates this approximation.

(b) Since  $f(x) = e^{x^2}$ , we have  $f'(x) = 2xe^{x^2}$  and  $f''(x) = (2 + 4x^2)e^{x^2}$ . Also, since  $0 \le x \le 1$ , we have  $x^2 \le 1$  and so

$$0 \le f''(x) = (2 + 4x^2)e^{x^2} \le 6e$$

Taking K = 6e, a = 0, b = 1, and n = 10 in the error estimate (3), we see that an upper bound for the error is

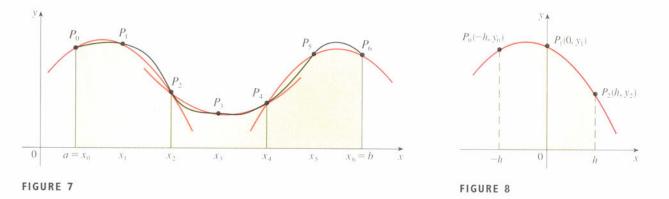
$$\frac{6e(1)^3}{24(10)^2} = \frac{e}{400} \approx 0.007$$



Error estimates are upper bounds for the error. They give theoretical, worstcase scenarios. The actual error in this case turns out to be about 0.0023.

# Simpson's Rule

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate a curve. As before, we divide [a, b] into n subintervals of equal length  $h = \Delta x = (b - a)/n$ , but this time we assume that n is an *even* number. Then on each consecutive pair of intervals we approximate the curve  $y = f(x) \ge 0$  by a parabola as shown in Figure 7. If  $y_i = f(x_i)$ , then  $P_i(x_i, y_i)$  is the point on the curve lying above  $x_i$ . A typical parabola passes through three consecutive points  $P_i$ ,  $P_{i+1}$ , and  $P_{i+2}$ .



In order to simplify our calculations, we first consider the case where  $x_0 = -h$ ,  $x_1 = 0$ , and  $x_2 = h$ . (See Figure 8.) We know that the equation of the parabola through  $P_0$ ,  $P_1$ , and  $P_2$  is of the form  $y = Ax^2 + Bx + C$  and so the area under the parabola from x = -h to x = h is

Here we have used Theorem 6 in Section 5.5. Notice that  $Ax^2 + C$  is even and Bx is odd.

$$\int_{-h}^{h} (Ax^{2} + Bx + C) dx = 2 \int_{0}^{h} (Ax^{2} + C) dx$$
$$= 2 \left[ A \frac{x^{3}}{3} + Cx \right]_{0}^{h}$$
$$= 2 \left( A \frac{h^{3}}{3} + Ch \right) = \frac{h}{3} (2Ah^{2} + 6C)$$

But, since the parabola passes through  $P_0(-h, y_0)$ ,  $P_1(0, y_1)$ , and  $P_2(h, y_2)$ , we have

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$
$$y_1 = C$$
$$y_2 = Ah^2 + Bh + C$$

and therefore

$$y_0 + 4y_1 + y_2 = 2Ah^2 + 60$$

Thus, we can rewrite the area under the parabola as

$$\frac{h}{3}(y_0+4y_1+y_2)$$

Now, by shifting this parabola horizontally we do not change the area under it. This means that the area under the parabola through  $P_0$ ,  $P_1$ , and  $P_2$  from  $x = x_0$  to  $x = x_2$  in Figure 7 is still

$$\frac{h}{3}(y_0+4y_1+y_2)$$

Similarly, the area under the parabola through  $P_2$ ,  $P_3$ , and  $P_4$  from  $x = x_2$  to  $x = x_4$  is

$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$

If we compute the areas under all the parabolas in this manner and add the results, we get

$$\int_{a}^{b} f(x) dx \approx \frac{h}{3} (y_{0} + 4y_{1} + y_{2}) + \frac{h}{3} (y_{2} + 4y_{3} + y_{4}) + \dots + \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_{n}) = \frac{h}{3} (y_{0} + 4y_{1} + 2y_{2} + 4y_{3} + 2y_{4} + \dots + 2y_{n-2} + 4y_{n-1} + y_{n})$$

Although we have derived this approximation for the case in which  $f(x) \ge 0$ , it is a reasonable approximation for any continuous function *f* and is called Simpson's Rule after the English mathematician Thomas Simpson (1710–1761). Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, ..., 4, 2, 4, 1.

Simpson's Rule

$$\int_{a}^{b} f(x) dx \approx S_{n} = \frac{\Delta x}{3} [f(x_{0}) + 4f(x_{1}) + 2f(x_{2}) + 4f(x_{3}) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_{n})]$$

where *n* is even and  $\Delta x = (b - a)/n$ .

**EXAMPLE 4** Use Simpson's Rule with n = 10 to approximate  $\int_{1}^{2} (1/x) dx$ .

**SOLUTION** Putting f(x) = 1/x, n = 10, and  $\Delta x = 0.1$  in Simpson's Rule, we obtain

$$\int_{1}^{2} \frac{1}{x} dx \approx S_{10} = \frac{\Delta x}{3} [f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \dots + 2f(1.8) + 4f(1.9) + f(2)]$$

$$= \frac{0.1}{3} \left[ \frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2} \right]$$

$$\approx 0.693150$$

Notice that, in Example 4, Simpson's Rule gives us a much better approximation  $(S_{10} \approx 0.693150)$  to the true value of the integral  $(\ln 2 \approx 0.693147...)$  than does

Thomas Simpson was a weaver who taught himself mathematics and went on to become one of the best English mathematicians of the 18th century. What we call Simpson's Rule was actually known to Cavalieri and Gregory in the 17th century, but Simpson popularized it in his best-selling calculus textbook, entitled A New Treatise of Fluxions. the Trapezoidal Rule ( $T_{10} \approx 0.693771$ ) or the Midpoint Rule ( $M_{10} \approx 0.692835$ ). It turns out (see Exercise 32) that the approximations in Simpson's Rule are weighted averages of those in the Trapezoidal and Midpoint Rules:

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$$

(Recall that  $E_T$  and  $E_M$  have opposite signs and  $|E_M|$  is about half the size of  $|E_T|$ .)

In many applications of calculus we need to evaluate an integral even if no explicit formula is known for y as a function of x. A function may be given graphically or as a table of values of collected data. If there is evidence that the values are not changing rapidly, then the Trapezoidal Rule or Simpson's Rule can still be used to find an approximate value for  $\int_a^b y \, dx$ , the integral of y with respect to x.

**EXAMPLE 5** The inflation rate r(t) is the derivative of the Consumer Price Index (CPI), which measures average prices of items in a "representative market basket" of urban consumers. The table gives the inflation rate (as a percentage) in the United States from 1984 to 1994 as published by the U.S. Bureau of Labor Statistics. Use Simpson's Rule to estimate the total percentage increase in the CPI from 1984 to 1994.

**SOLUTION** Because the derivative of the CPI is the inflation rate r(t), the Total Change Theorem (see Section 5.3) tells us that the increase in the CPI from 1984 to 1994 is

$$\int_{1984}^{1994} r(t) \, dt$$

To estimate the value of this integral we use Simpson's Rule with n = 10 intervals. The interval length is  $\Delta t = 1$ , so

$$\int_{1984}^{1994} r(t) dt$$

$$\approx \frac{\Delta t}{3} [r(1984) + 4r(1985) + 2r(1986) + \dots + 4r(1993) + r(1994)]$$

$$\approx \frac{1}{3} [4.3 + 4(3.6) + 2(1.9) + 4(3.6) + 2(4.1) + 4(4.8) + 2(5.4) + 4(4.2) + 2(3.0) + 4(3.0) + 2.6]$$

$$= 37.5$$

Thus, the CPI increased by about 37.5% from 1984 to 1994.

In Exercise 20 you are asked to demonstrate, in a particular case, that the error in Simpson's Rule decreases by a factor of about 16 when *n* is doubled. That is consistent with the appearance of  $n^4$  in the denominator of the following error estimate for Simpson's Rule. It is analogous to the estimates given in (3) for the Trapezoidal and Midpoint Rules, but it uses the fourth derivative of *f*.

**4 Error Bound for Simpson's Rule** Suppose that  $|f^{(4)}(x)| \le K$  for  $a \le x \le b$ . If  $E_s$  is the error involved in using Simpson's Rule, then

$$|E_s| \leq \frac{K(b-a)^5}{180n^4}$$

1	r(t)
1984	4.3
1985	3.6
1986	1.9
1987	3.6
1988	4.1
1989	4.8
1990	5.4
1991	4.2
1992	3.0
1993	3.0
1994	2.6

**EXAMPLE 6** How large should we take *n* in order to guarantee that the Simpson's Rule approximation for  $\int_{1}^{2} (1/x) dx$  is accurate to within 0.0001?

**SOLUTION** If f(x) = 1/x, then  $f^{(4)}(x) = 24/x^5$ . Since  $x \ge 1$ , we have  $1/x \le 1$  and so

 $\left| f^{(4)}(x) \right| = \left| \frac{24}{x^5} \right| \le 24$ 

Therefore, we can take K = 24 in (4). Thus, for an error less than 0.0001 we should choose *n* so that

 $\frac{24(1)^5}{180n^4} < 0.0001$ 

This gives

$$n^{4} > \frac{24}{180(0.0001)}$$
$$n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04$$

or

Therefore, n = 8 (*n* must be even) gives the desired accuracy. (Compare this with Example 2, where we obtained n = 41 for the Trapezoidal Rule and n = 29 for the Midpoint Rule.)

### EXAMPLE 7

- (a) Use Simpson's Rule with n = 10 to approximate the integral  $\int_0^1 e^{x^2} dx$ .
- (b) Estimate the error involved in this approximation.

### SOLUTION

(a) If n = 10, then  $\Delta x = 0.1$  and Simpson's Rule gives

$$\int_{0}^{1} e^{x^{2}} dx \approx \frac{\Delta x}{3} [f(0) + 4f(0.1) + 2f(0.2) + \dots + 2f(0.8) + 4f(0.9) + f(1)]$$
  
=  $\frac{0.1}{3} [e^{0} + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36} + 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^{1}]$ 

≈ 1.462681

(b) The fourth derivative of  $f(x) = e^{x^2}$  is

$$f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$$

and so, since  $0 \le x \le 1$ , we have

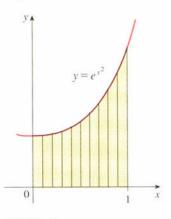
$$0 \le f^{(4)}(x) \le (12 + 48 + 16)e^1 = 76e$$

Therefore, putting K = 76e, a = 0, b = 1, and n = 10 in (4), we see that the error is at most

$$\frac{76e(1)^5}{180(10)^4} \approx 0.000115$$

Many calculators and computer algebra systems have a built-in algorithm that computes an approximation of a definite integral. Some of these machines use Simpson's Rule; others use more sophisticated techniques such as *adaptive* numerical integration. This means that if a function fluctuates much more on a certain part of the interval than it does elsewhere, then that part gets divided into more subintervals. This strategy reduces the number of calculations required to achieve a prescribed accuracy.

Figure 9 illustrates the calculation in Example 7. Notice that the parabolic arcs are so close to the graph of  $y = e^{x^2}$  that they are practically indistinguishable from it.





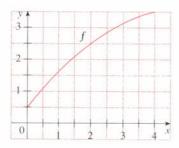
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(Compare this with Example 3.) Thus, correct to three decimal places, we have

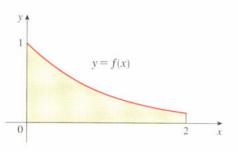
$$\int_0^1 e^{x^2} dx \approx 1.463$$

8 Exercises

- **1.** Let  $I = \int_0^4 f(x) dx$ , where f is the function whose graph is shown.
  - (a) Use the graph to find  $L_2$ ,  $R_2$ , and  $M_2$ .
  - (b) Are these underestimates or overestimates of I?
  - (c) Use the graph to find *T*<sub>2</sub>. How does it compare with *I*?
  - (d) For any value of *n*, list the numbers  $L_n$ ,  $R_n$ ,  $M_n$ ,  $T_n$ , and *I* in increasing order.



**2.** The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate  $\int_0^2 f(x) dx$ , where f is the function whose graph is shown. The estimates were 0.9540, 0.7811, 0.8675, and 0.8632, and the same number of subintervals were used in each case.



- (a) Which rule produced which estimate?
- (b) Between which two approximations does the true value of  $\int_0^2 f(x) dx$  lie?
- A Estimate ∫<sub>0</sub><sup>1</sup> cos(x<sup>2</sup>) dx using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with n = 4. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of  $f(x) = \sin(x^2/2)$  in the viewing rectangle [0, 1] by [0, 0.5] and let  $I = \int_0^1 f(x) dx$ .

- (a) Use the graph to decide whether  $L_2$ ,  $R_2$ ,  $M_2$ , and  $T_2$  underestimate or overestimate *I*.
- (b) For any value of n, list the numbers L<sub>n</sub>, R<sub>n</sub>, M<sub>n</sub>, T<sub>n</sub>, and I in increasing order.
- (c) Compute  $L_5$ ,  $R_5$ ,  $M_5$ , and  $T_5$ . From the graph, which do you think gives the best estimate of I?

5–12 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.)

5.  $\int_0^1 e^{-x^2} dx$ , n = 106.  $\int_0^2 \frac{1}{\sqrt{1 + x^3}} dx$ , n = 107.  $\int_0^{1/2} \cos(e^x) dx$ , n = 88.  $\int_2^3 \frac{1}{\ln x} dx$ , n = 10

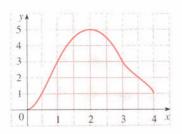
**9.** 
$$\int_{a}^{1} x^{5} e^{x} dx, n = 10$$
 **10.**  $\int_{a}^{4} \sqrt{x} \sin x dx, n = 8$ 

- **11.**  $\int_0^3 \frac{1}{1+x^4} dx$ , n = 6 **12.**  $\int_2^4 \frac{e^x}{x} dx$ , n = 10
- **13.** (a) Find the approximations  $T_{10}$  and  $M_{10}$  for the integral  $\int_0^2 e^{-x^2} dx$ .
  - (b) Estimate the errors in the approximations of part (a).
  - (c) How large do we have to choose n so that the approximations T<sub>n</sub> and M<sub>n</sub> to the integral in part (a) are accurate to within 0.00001?
- **14.** (a) Find the approximations  $T_4$ ,  $T_8$ ,  $M_4$ , and  $M_8$  for  $\int_0^1 \cos(x^2) dx$ .
  - (b) Estimate the errors involved in the approximations of part (a).
- **15.** (a) Find the approximations  $T_{10}$  and  $S_{10}$  for  $\int_0^1 e^x dx$  and the corresponding errors  $E_T$  and  $E_S$ .
  - (b) Compare the actual errors in part (a) with the error estimates given by (3) and (4).
  - (c) How large do we have to choose n so that the approximations  $T_n$ ,  $M_n$ , and  $S_n$  to the integral in part (a) are accurate to within 0.00001?

- 16. How large should *n* be to guarantee that the Simpson's Rule approximation to  $\int_0^1 e^{x^2} dx$  is accurate to within 0.00001?
- **CAS** 17. The trouble with the error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound K for  $|f^{(4)}(x)|$  by hand. But computer algebra systems have no problem computing  $f^{(4)}$  and graphing it, so we can easily find a value for K from a machine graph. This exercise deals with approximations to the integral  $I = \int_{0}^{2\pi} f(x) dx$ , where  $f(x) = e^{\cos x}$ .
  - (a) Use a graph to get a good upper bound for | f"(x)|.
    (b) Use M<sub>10</sub> to approximate I.
  - (c) Use part (a) to estimate the error in part (b).
  - (d) Use the built-in numerical integration capability of your CAS to approximate *I*.
  - (e) How does the actual error compare with the error estimate in part (c)?
  - (f) Use a graph to get a good upper bound for  $|f^{(4)}(x)|$ .
  - (g) Use  $S_{10}$  to approximate *I*.
  - (h) Use part (f) to estimate the error in part (g).
  - (i) How does the actual error compare with the error estimate in part (h)?
  - (j) How large should *n* be to guarantee that the size of the error in using  $S_n$  is less than 0.0001?

**CAS** 18. Repeat Exercise 17 for the integral  $\int_{-1}^{1} \sqrt{4 - x^3} dx$ .

- **19.** Find the approximations  $L_n$ ,  $R_n$ ,  $T_n$ , and  $M_n$  to the integral  $\int_0^1 x^3 dx$  for n = 4, 8, and 16. Then compute the corresponding errors  $E_L$ ,  $E_R$ ,  $E_T$ , and  $E_M$ . (Round your answers to six decimal places.) What observations can you make? In particular, what happens to the errors when *n* is doubled?
- **20.** Find the approximations  $T_n$ ,  $M_n$ , and  $S_n$  to the integral  $\int_{-1}^{2} xe^x dx$  for n = 6 and 12. Then compute the corresponding errors  $E_7$ ,  $E_M$ , and  $E_5$ . (Round your answers to six decimal places.) What observations can you make? In particular, what happens to the errors when n is doubled?
- 21. Estimate the area under the graph in the figure by using(a) the Trapezoidal Rule, (b) the Midpoint Rule, and(c) Simpson's Rule, each with n = 4.



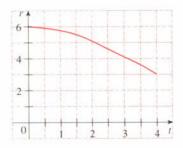
**22.** (a) Use Simpson's Rule and the given data to estimate the value of the integral  $\int_{2}^{6} f(x) dx$ .

X	f(x)
2.0	9.22
2.5	9.01
3.0	8.76
3.5	8.30
4.0	7.52
4.5	6.83
5.0	7.32
5.5	7.69
6.0	7.91

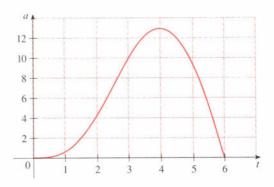
- (b) If it is known that  $-2 \le f^{(4)}x \le 5$  for all x, estimate the error involved in the approximation in part (a).
- **23.** The speedometer reading (*v*) on a car was observed at 1-minute intervals and recorded in the chart. Use Simpson's Rule to estimate the distance traveled by the car.

t (min)											
<i>v</i> (mi/h)	40	42	45	49	52	54	56	57	57	55	56

24. Water leaked from a tank at a rate of r(t) liters per hour, where the graph of r is as shown. Use Simpson's Rule to estimate the total amount of water that leaked out during the first four hours.



**25.** The graph of the acceleration a(t) of a car measured in  $ft/s^2$  is shown. Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.



**26.** The table (supplied by Pacific Gas and Electric) gives the power consumption in megawatts in the San Francisco Bay Area from midnight to noon on September 19, 1996. Use Simpson's Rule to estimate the energy used during that time period. (Use the fact that power is the derivative of energy.)

1	12	1	2	3	4	5	6
P	4182	3856	3640	3558	3547	3679	4112

1	7	8	9	10	11	12
P	4699	5151	5514	5751	6044	6206

27. In addition to the general inflation rate (Example 5), the U.S. Bureau of Labor Statistics also publishes the rates of increase of prices of more specialized goods. The table gives the rate of change in the price of food in the United States (as a percentage). Use Simpson's Rule to estimate the total percentage increase in the cost of food from 1986 to 1994.

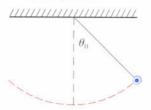
1	1986	1987	1988	1989	1990	1991	1992	1993	1994
r	3.2	4.1	4.1	5.8	5.8	2.9	1.2	2.2	2.4

Improper Integrals ·

**CAS** 28. The figure shows a pendulum with length L that makes a maximum angle  $\theta_0$  with the vertical. Using Newton's Second Law it can be shown that the period T (the time for one complete swing) is given by

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where  $k = \sin(\frac{1}{2}\theta_0)$  and g is the acceleration due to gravity. If L = 1 m and  $\theta_0 = 42^\circ$ , use Simpson's Rule with n = 10 to find the period.



**29.** If f is a positive function and f''(x) < 0 for  $a \le x \le b$ , show that

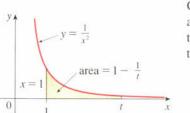
$$T_n < \int_a^b f(x) \, dx < M_n$$

- **30.** Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of  $\int_a^b f(x) dx$ .
- **31.** Show that  $\frac{1}{2}(T_n + M_n) = T_{2n}$ .
- **32.** Show that  $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$ .



In defining a definite integral  $\int_{a}^{b} f(x) dx$  we dealt with a function f defined on a finite interval [a, b] and we assumed that f does not have an infinite discontinuity (see Section 5.2). In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in [a, b]. In either case the integral is called an *improper* integral. One of the most important applications of this idea, probability distributions, will be studied in Section 6.7.

### Type I: Infinite Intervals

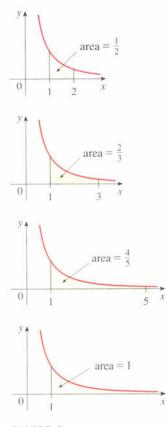


Consider the infinite region S that lies under the curve  $y = 1/x^2$ , above the x-axis, and to the right of the line x = 1. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line x = t (shaded in Figure 1) is

$$A(t) = \int_{1}^{t} \frac{1}{x^{2}} dx = -\frac{1}{x} \bigg]_{1}^{t} = 1 - \frac{1}{t}$$

Notice that A(t) < 1 no matter how large t is chosen.

FIGURE I





We also observe that

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left( 1 - \frac{1}{t} \right) = 1$$

The area of the shaded region approaches 1 as  $t \to \infty$  (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^2} dx = 1$$

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

Definition of an Improper Integral of Type 1

(a) If  $\int_{a}^{t} f(x) dx$  exists for every number  $t \ge a$ , then

$$\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

provided this limit exists (as a finite number).

(b) If  $\int_{t}^{b} f(x) dx$  exists for every number  $t \le b$ , then

$$\int_{-\infty}^{b} f(x) \, dx = \lim_{t \to -\infty} \int_{t}^{b} f(x) \, dx$$

provided this limit exists (as a finite number).

The improper integrals  $\int_{a}^{\infty} f(x) dx$  and  $\int_{-\infty}^{b} f(x) dx$  are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both  $\int_a^{\infty} f(x) dx$  and  $\int_{-\infty}^a f(x) dx$  are convergent, then we define

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx$$

In part (c) any real number a can be used (see Exercise 52).

Any of the improper integrals in Definition 1 can be interpreted as an area provided that f is a positive function. For instance, in case (a) if  $f(x) \ge 0$  and the integral  $\int_a^{\infty} f(x) dx$  is convergent, then we define the area of the region  $S = \{(x, y) | x \ge a, 0 \le y \le f(x)\}$  in Figure 3 to be

$$A(S) = \int_{a}^{\infty} f(x) \, dx$$

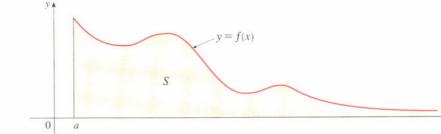


FIGURE 3

This is appropriate because  $\int_a^{\infty} f(x) dx$  is the limit as  $t \to \infty$  of the area under the graph of f from a to t.

**EXAMPLE I** Determine whether the integral  $\int_{1}^{\infty} (1/x) dx$  is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x} dx = \lim_{t \to \infty} \ln |x|]_{1}^{t}$$
$$= \lim_{t \to \infty} (\ln t - \ln 1) = \lim_{t \to \infty} \ln t = \infty$$

The limit does not exist as a finite number and so the improper integral  $\int_{1}^{\infty} (1/x) dx$  is divergent.

Let's compare the result of Example 1 with the example at the beginning of this section:

$$\int_{1}^{\infty} \frac{1}{x^2} dx \text{ converges} \qquad \int_{1}^{\infty} \frac{1}{x} dx \text{ diverges}$$

Geometrically, this says that although the curves  $y = 1/x^2$  and y = 1/x look very similar for x > 0, the region under  $y = 1/x^2$  to the right of x = 1 (the shaded region in Figure 4) has finite area whereas the corresponding region under y = 1/x (in Figure 5) has infinite area. Note that both  $1/x^2$  and 1/x approach 0 as  $x \to \infty$  but  $1/x^2$  approaches 0 faster than 1/x. The values of 1/x don't decrease fast enough for its integral to have a finite value.

**EXAMPLE 2** Evaluate  $\int_{-\infty}^{0} x e^x dx$ .

SOLUTION Using part (b) of Definition 1, we have

$$\int_{-\infty}^{0} x e^{x} dx = \lim_{t \to -\infty} \int_{t}^{0} x e^{x} dx$$

We integrate by parts with u = x,  $dv = e^{x} dx$  so that du = dx,  $v = e^{x}$ :

$$\int_{t}^{0} x e^{x} dx = x e^{x} \Big]_{t}^{0} - \int_{t}^{0} e^{x} dx$$
$$= -t e^{t} - 1 + e^{t}$$

We know that  $e' \rightarrow 0$  as  $t \rightarrow -\infty$ , and by l'Hospital's Rule we have

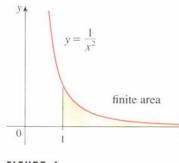
Therefore  

$$\lim_{t \to -\infty} te^{t} = \lim_{t \to -\infty} \frac{t}{e^{-t}} = \lim_{t \to -\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t \to -\infty} (-e^{t}) = 0$$

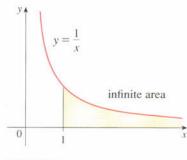
$$\int_{-\infty}^{0} xe^{x} dx = \lim_{t \to -\infty} (-te^{t} - 1 + e^{t})$$

$$= -0 - 1 + 0 = -1$$



x







**EXAMPLE 3** Evaluate 
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$
.

**SOLUTION** It is convenient to choose a = 0 in Definition 1(c):

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{\infty} \frac{1}{1+x^2} \, dx$$

We must now evaluate the integrals on the right side separately:

$$\int_{0}^{\infty} \frac{1}{1+x^{2}} dx = \lim_{t \to \infty} \int_{0}^{t} \frac{dx}{1+x^{2}} = \lim_{t \to \infty} \tan^{-1}x \Big]_{0}^{t}$$
$$= \lim_{t \to \infty} (\tan^{-1}t - \tan^{-1}0) = \lim_{t \to \infty} \tan^{-1}t = \frac{\pi}{2}$$
$$\int_{-\infty}^{0} \frac{1}{1+x^{2}} dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{1+x^{2}} = \lim_{t \to -\infty} \tan^{-1}x \Big]_{t}^{0}$$
$$= \lim_{t \to -\infty} (\tan^{-1}0 - \tan^{-1}t)$$
$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since  $1/(1 + x^2) > 0$ , the given improper integral can be interpreted as the area of the infinite region that lies under the curve  $y = 1/(1 + x^2)$  and above the *x*-axis (see Figure 6).

**EXAMPLE 4** For what values of p is the integral

$$\int_{1}^{\infty} \frac{1}{x^{p}} dx$$

convergent?

**SOLUTION** We know from Example 1 that if p = 1, then the integral is divergent, so let's assume that  $p \neq 1$ . Then

$$\int_{1}^{\infty} \frac{1}{x^{p}} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^{p}} dx$$
$$= \lim_{t \to \infty} \frac{x^{-p+1}}{-p+1} \Big]_{x=1}^{x=t}$$
$$= \lim_{t \to \infty} \frac{1}{1-p} \left[ \frac{1}{t^{p-1}} - 1 \right]_{x=1}^{x=t}$$

If p > 1, then p - 1 > 0, so as  $t \to \infty$ ,  $t^{p-1} \to \infty$  and  $1/t^{p-1} \to 0$ . Therefore

$$\int_{1}^{\infty} \frac{1}{x^{p}} \, dx = \frac{1}{p-1} \qquad \text{if } p > 1$$

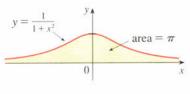


FIGURE 6

and so the integral converges. But if p < 1, then p - 1 < 0 and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \qquad \text{as } t \to \infty$$

and the integral diverges.

We summarize the result of Example 4 for future reference:

2. 
$$\int_{1}^{\infty} \frac{1}{x^{p}} dx$$
 is convergent if  $p > 1$  and divergent if  $p \le 1$ .

### Type 2: Discontinuous Integrands

Suppose that f is a positive continuous function defined on a finite interval [a, b) but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between a and b. (For Type 1 integrals, the regions extended indefinitely in a horizontal direction.) Here the region is infinite in a vertical direction.) The area of the part of S between a and t (the shaded region in Figure 7) is

$$A(t) = \int_a^t f(x) \, dx$$

If it happens that A(t) approaches a definite number A as  $t \to b^-$ , then we say that the area of the region S is A and we write

$$\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

### 3 Definition of an Improper Integral of Type 2

(a) If f is continuous on [a, b) and is discontinuous at b, then

$$\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx$$

if this limit exists (as a finite number).

(b) If f is continuous on (a, b] and is discontinuous at a, then

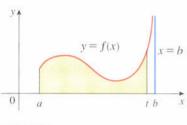
$$\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx$$

if this limit exists (as a finite number).

The improper integral  $\int_a^b f(x) dx$  is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

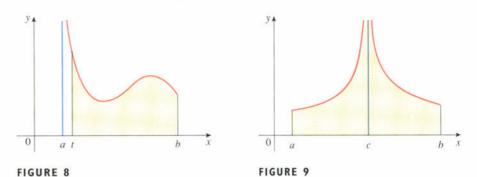
(c) If f has a discontinuity at c, where a < c < b, and both  $\int_a^c f(x) dx$  and  $\int_c^b f(x) dx$  are convergent, then we define

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx \, + \, \int_c^b f(x) \, dx$$





Parts (b) and (c) of Definition 3 are illustrated in Figures 8 and 9 for the case where  $f(x) \ge 0$  and f has vertical asymptotes at a and c, respectively.



**EXAMPLE 5** Find  $\int_{2}^{5} \frac{1}{\sqrt{x-2}} dx$ .

**SOLUTION** We note first of all that the given integral is improper because  $f(x) = 1/\sqrt{x-2}$  has the vertical asymptote x = 2. Since the infinite discontinuity occurs at the left endpoint of [2, 5], we use part (b) of Definition 3:

$$\int_{2}^{5} \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^{+}} \int_{t}^{5} \frac{dx}{\sqrt{x-2}}$$
$$= \lim_{t \to 2^{+}} 2\sqrt{x-2} \Big]_{t}^{5}$$
$$= \lim_{t \to 2^{+}} 2(\sqrt{3} - \sqrt{t-2})$$
$$= 2\sqrt{3}$$

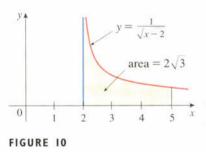
Thus, the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure 10.

**EXAMPLE 6** Determine whether  $\int_0^{\pi/2} \sec x \, dx$  converges or diverges.

**SOLUTION** Note that the given integral is improper because  $\lim_{x\to(\pi/2)^-} \sec x = \infty$ . Using part (a) of Definition 3 and Formula 14 from the Table of Integrals, we have

$$\int_0^{\pi/2} \sec x \, dx = \lim_{t \to (\pi/2)^-} \int_0^t \sec x \, dx$$
$$= \lim_{t \to (\pi/2)^-} \ln|\sec x + \tan x|\Big]_0^t$$
$$= \lim_{t \to (\pi/2)^-} \left[\ln(\sec t + \tan t) - \ln 1\right]$$
$$= \infty$$

because sec  $t \to \infty$  and  $\tan t \to \infty$  as  $t \to (\pi/2)^-$ . Thus, the given improper integral is divergent.



**EXAMPLE 7** Evaluate  $\int_0^3 \frac{dx}{x-1}$  if possible.

W

**SOLUTION** Observe that the line x = 1 is a vertical asymptote of the integrand. Since it occurs in the middle of the interval [0, 3], we must use part (c) of Definition 3 with c = 1:

here 
$$\int_{0}^{3} \frac{dx}{x-1} = \int_{0}^{1} \frac{dx}{x-1} + \int_{1}^{3} \frac{dx}{x-1}$$
$$= \lim_{t \to 1^{-}} \int_{0}^{t} \frac{dx}{x-1} = \lim_{t \to 1^{-}} \ln|x-1|]_{0}^{t}$$
$$= \lim_{t \to 1^{-}} (\ln|t-1| - \ln|-1|)$$
$$= \lim_{t \to 1^{-}} \ln(1-t) = -\infty$$

because  $1 - t \to 0^+$  as  $t \to 1^-$ . Thus,  $\int_0^1 dx/(x - 1)$  is divergent. This implies that  $\int_0^3 dx/(x - 1)$  is divergent. [We do not need to evaluate  $\int_1^3 dx/(x - 1)$ .]

*Warning:* If we had not noticed the asymptote x = 1 in Example 7 and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:

$$\int_{0}^{3} \frac{dx}{x-1} = \ln|x-1|]_{0}^{3} = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.

From now on, whenever you meet the symbol  $\int_a^b f(x) dx$  you must decide, by looking at the function f on [a, b], whether it is an ordinary definite integral or an improper integral.

**EXAMPLE 8** Evaluate  $\int_0^1 \ln x \, dx$ .

**SOLUTION** We know that the function  $f(x) = \ln x$  has a vertical asymptote at 0 since  $\lim_{x\to 0^+} \ln x = -\infty$ . Thus, the given integral is improper and we have

$$\int_{0}^{1} \ln x \, dx = \lim_{t \to 0^{+}} \int_{t}^{1} \ln x \, dx$$

Now we integrate by parts with  $u = \ln x$ , dv = dx, du = dx/x, and v = x:

$$\int_{t}^{1} \ln x \, dx = x \ln x \Big]_{t}^{1} - \int_{t}^{1} dx$$
$$= 1 \ln 1 - t \ln t - (1 - t)$$
$$= -t \ln t - 1 + t$$

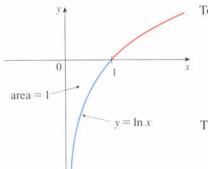


FIGURE II

# To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t} = \lim_{t \to 0^+} \frac{1/t}{-1/t^2}$$
$$= \lim_{t \to 0^+} (-t) = 0$$

Therefore

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \left( -t \ln t - 1 + t \right)$$
$$= -0 - 1 + 0 = -1$$

Figure 11 shows the geometric interpretation of this result. The area of the shaded region above  $y = \ln x$  and below the x-axis is 1.

### A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

**Comparison Theorem** Suppose that *f* and *g* are continuous functions with  $f(x) \ge g(x) \ge 0$  for  $x \ge a$ . (a) If  $\int_a^{\infty} f(x) dx$  is convergent, then  $\int_a^{\infty} g(x) dx$  is convergent.

(b) If  $\int_a^{\infty} g(x) dx$  is divergent, then  $\int_a^{\infty} f(x) dx$  is divergent.

We omit the proof of the Comparison Theorem, but Figure 12 makes it seem plausible. If the area under the top curve y = f(x) is finite, then so is the area under the bottom curve y = g(x). And if the area under y = g(x) is infinite, then so is the area under y = f(x).

**EXAMPLE 9** Show that  $\int_0^\infty e^{-x^2} dx$  is convergent.

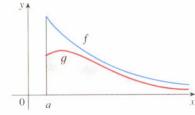
**SOLUTION** We cannot evaluate the integral directly because the antiderivative of  $e^{-x^2}$  is not an elementary function (as explained in Section 5.7). We write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

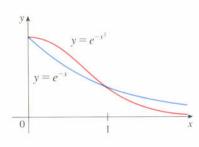
and observe that the first integral on the right-hand side is just an ordinary definite integral. In the second integral we use the fact that for  $x \ge 1$  we have  $x^2 \ge x$ , so  $-x^2 \le -x$  and therefore  $e^{-x^2} \le e^{-x}$  (see Figure 13). The integral of  $e^{-x}$  is easy to evaluate:

$$\int_{1}^{\infty} e^{-x} dx = \lim_{t \to \infty} \int_{1}^{t} e^{-x} dx = \lim_{t \to \infty} \left( e^{-1} - e^{-t} \right) = e^{-1}$$

Thus, taking  $f(x) = e^{-x}$  and  $g(x) = e^{-x^2}$  in the Comparison Theorem, we see that  $\int_{1}^{\infty} e^{-x^2} dx$  is convergent. It follows that  $\int_{0}^{\infty} e^{-x^2} dx$  is convergent.









$t^{-}$	$\int_0^t e^{-x^2} dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

In Example 9 we showed that  $\int_0^\infty e^{-x^2} dx$  is convergent without computing its value. In Exercise 56 we indicate how to show that its value is approximately 0.8862. In probability theory it is important to know the exact value of this improper integral; using the methods of multivariable calculus it can be shown that the exact value is  $\sqrt{\pi}/2$ . Table 1 illustrates the definition of an improper integral by showing how the (computer-generated) values of  $\int_0^t e^{-x^2} dx$  approach  $\sqrt{\pi}/2$  as t becomes large. In fact, these values converge quite quickly because  $e^{-x^2} \to 0$  very rapidly as  $x \to \infty$ .

**EXAMPLE 10** The integral  $\int_{1}^{\infty} \frac{1 + e^{-x}}{x} dx$  is divergent by the Comparison Theorem because

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

and  $\int_{1}^{\infty} (1/x) dx$  is divergent by Example 1 [or by (2) with p = 1].

Table 2 illustrates the divergence of the integral in Example 10. Notice that the values do not approach any fixed number.

TABLE 2

t	$\int_{1}^{1} \left[ (1 + e^{-x})/x \right] dx$		
2	0.8636306042		
5	1.8276735512		
10	2.5219648704		
100	4.8245541204		
1000	7.1271392134		
10000	9.4297243064		



.9 Exercises

I. Explain why each of the following integrals is improper.

(a) 
$$\int_{1}^{\infty} x^{4} e^{-x^{4}} dx$$
 (b)  $\int_{0}^{\pi/2} \sec x \, dx$   
(c)  $\int_{0}^{2} \frac{x}{x^{2} - 5x + 6} \, dx$  (d)  $\int_{-\infty}^{0} \frac{1}{x^{2} + 5} \, dx$ 

2. Which of the following integrals are improper? Why?

(a) 
$$\int_{1}^{2} \frac{1}{2x - 1} dx$$
 (b)  $\int_{0}^{1} \frac{1}{2x - 1} dx$   
(c)  $\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^{2}} dx$  (d)  $\int_{1}^{2} \ln(x - 1) dx$ 

- 3. Find the area under the curve y = 1/x<sup>3</sup> from x = 1 to x = t and evaluate it for t = 10, 100, and 1000. Then find the total area under this curve for x ≥ 1.
- 4. (a) Graph the functions  $f(x) = 1/x^{1.1}$  and  $g(x) = 1/x^{0.9}$ in the viewing rectangles [0, 10] by [0, 1] and [0, 100] by [0, 1].

- (b) Find the areas under the graphs of f and g from x = 1 to x = t and evaluate for  $t = 10, 100, 10^4, 10^6, 10^{10}, and 10^{20}$ .
- (c) Find the total area under each curve for x ≥ 1, if it exists.

**5–32** Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5.  $\int_{0}^{\infty} e^{-x} dx$ 6.  $\int_{2}^{\infty} \frac{1}{(x+3)^{3/2}} dx$ 7.  $\int_{-\infty}^{1} \frac{1}{(2x-3)^{2}} dx$ 8.  $\int_{2}^{\infty} \frac{1}{\sqrt{x+3}} dx$ 9.  $\int_{-\infty}^{\infty} x^{3} dx$ 10.  $\int_{-\infty}^{\infty} (2x^{2} - x + 3) dx$ 11.  $\int_{-\infty}^{\infty} xe^{-x^{2}} dx$ 12.  $\int_{-\infty}^{0} e^{3x} dx$ 

13. 
$$\int_{0}^{\infty} \cos x \, dx$$
  
14. 
$$\int_{-\infty}^{\infty} x^{2} e^{-x^{3}} \, dx$$
  
15. 
$$\int_{0}^{\infty} \frac{5}{2x+3} \, dx$$
  
16. 
$$\int_{-\infty}^{3} \frac{1}{x^{2}+9} \, dx$$
  
17. 
$$\int_{-\infty}^{1} x e^{2x} \, dx$$
  
18. 
$$\int_{0}^{\infty} x e^{-x} \, dx$$
  
19. 
$$\int_{1}^{\infty} \frac{\ln x}{x} \, dx$$
  
20. 
$$\int_{e}^{\infty} \frac{1}{x(\ln x)^{2}} \, dx$$
  
21. 
$$\int_{-\infty}^{\infty} \frac{x}{1+x^{2}} \, dx$$
  
22. 
$$\int_{1}^{\infty} \frac{\ln x}{x^{3}} \, dx$$
  
23. 
$$\int_{0}^{3} \frac{1}{\sqrt{x}} \, dx$$
  
24. 
$$\int_{0}^{3} \frac{1}{x\sqrt{x}} \, dx$$
  
25. 
$$\int_{-1}^{0} \frac{1}{x^{2}} \, dx$$
  
26. 
$$\int_{1}^{9} \frac{1}{\sqrt[3]{x-9}} \, dx$$
  
27. 
$$\int_{-2}^{3} \frac{1}{x^{4}} \, dx$$
  
28. 
$$\int_{0}^{2} \frac{1}{4x-5} \, dx$$
  
29. 
$$\int_{4}^{5} \frac{1}{(5-x)^{2/5}} \, dx$$
  
30. 
$$\int_{\pi/4}^{\pi/2} \sec^{2x} \, dx$$
  
31. 
$$\int_{0}^{1} x \ln x \, dx$$
  
32. 
$$\int_{0}^{1} \frac{\ln x}{\sqrt{x}} \, dx$$

**33–38** Sketch the region and find its area (if the area is finite).

**33.** 
$$S = \{(x, y) \mid x \le 1, 0 \le y \le e^x\}$$
  
**34.**  $S = \{(x, y) \mid x \ge -2, 0 \le y \le e^{-x/2}\}$   
**35.**  $S = \{(x, y) \mid x \ge 1, 0 \le y \le (\ln x)/x^2\}$   
**36.**  $S = \{(x, y) \mid x \ge 0, 0 \le y \le 1/\sqrt{x+1}\}$   
**37.**  $S = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le \tan x \sec x\}$ 

₽ ₽

f

**38.** 
$$S = \{(x, y) \mid 3 < x \le 7, 0 \le y \le 1/\sqrt{x} - 3 \}$$

- **39.** (a) If  $g(x) = (\sin^2 x)/x^2$ , use your calculator or computer to make a table of approximate values of  $\int_1^t g(x) dx$  for t = 2, 5, 10, 100, 1000, and 10,000. Does it appear that  $\int_1^\infty g(x) dx$  is convergent?
  - (b) Use the Comparison Theorem with f(x) = 1/x<sup>2</sup> to show that ∫<sub>1</sub><sup>∞</sup> g(x) dx is convergent.
  - (c) Illustrate part (b) by graphing f and g on the same screen for  $1 \le x \le 10$ . Use your graph to explain intuitively why  $\int_{1}^{\infty} g(x) dx$  is convergent.
- **40.** (a) If  $g(x) = 1/(\sqrt{x} 1)$ , use your calculator or computer to make a table of approximate values of  $\int_2^t g(x) dx$  for t = 5, 10, 100, 1000, and 10,000. Does it appear that  $\int_2^\infty g(x) dx$  is convergent or divergent?

- (b) Use the Comparison Theorem with  $f(x) = 1/\sqrt{x}$  to show that  $\int_{2}^{\infty} g(x) dx$  is divergent.
- (c) Illustrate part (b) by graphing f and g on the same screen for  $2 \le x \le 20$ . Use your graph to explain intuitively why  $\int_2^{\infty} g(x) dx$  is divergent.

**41–46** Use the Comparison Theorem to determine whether the integral is convergent or divergent.

**41.** 
$$\int_{1}^{\infty} \frac{\cos^{2} x}{1 + x^{2}} dx$$
**42.** 
$$\int_{1}^{\infty} \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} dx$$
**43.** 
$$\int_{1}^{\infty} \frac{dx}{x + e^{2x}}$$
**44.** 
$$\int_{1}^{\infty} \frac{1}{\sqrt{x^{3} + 1}} dx$$
**45.** 
$$\int_{0}^{\pi/2} \frac{dx}{x \sin x}$$
**46.** 
$$\int_{0}^{1} \frac{e^{-x}}{\sqrt{x}} dx$$

47. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)} \, dx$$

is improper for two reasons: the interval  $[0, \infty)$  is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_{0}^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$$
$$= \int_{0}^{1} \frac{1}{\sqrt{x}(1+x)} dx + \int_{1}^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$$

48. Evaluate

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}} \, dx$$

by the same method as in Exercise 47.

- **49.** Find the values of *p* for which the integral  $\int_{0}^{1} (1/x^p) dx$  converges and evaluate the integral for those values of *p*.
- **50.** (a) Evaluate the integral  $\int_0^\infty x^n e^{-x} dx$  for n = 0, 1, 2, and 3.
  - (b) Guess the value of  $\int_0^\infty x^n e^{-x} dx$  when *n* is an arbitrary positive integer.
  - (c) Prove your guess using mathematical induction.
- (a) Show that ∫<sup>∞</sup><sub>-∞</sub> x dx is divergent.
  (b) Show that

$$\lim_{t \to \infty} \int_{-t}^{t} x \, dx = 0$$

This shows that we can't define

$$\int_{-\infty}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{-t}^{t} f(x) \, dx$$

**52.** If  $\int_{-\infty}^{\infty} f(x) dx$  is convergent and *a* and *b* are real numbers, show that

$$\int_{-\infty}^{a} f(x) \, dx \, + \, \int_{a}^{\infty} f(x) \, dx \, = \, \int_{-\infty}^{b} f(x) \, dx \, + \, \int_{b}^{\infty} f(x) \, dx$$

- **53.** A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let F(t) be the fraction of the company's bulbs that burn out before *t* hours, so F(t) always lies between 0 and 1.
  - (a) Make a rough sketch of what you think the graph of F might look like.
  - (b) What is the meaning of the derivative r(t) = F'(t)?
  - (c) What is the value of  $\int_0^\infty r(t) dt$ ? Why?
- 54. The average speed of molecules in an ideal gas is

$$\overline{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)} dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\overline{v} = \sqrt{\frac{8RT}{\pi M}}$$

**55.** As we will see in Section 7.5, a radioactive substance decays exponentially: The mass at time *t* is  $m(t) = m(0)e^{kt}$ , where m(0) is the initial mass and *k* is a negative constant. The *mean life M* of an atom in the

substance is

$$M = -k \int_0^\infty t e^{kt} dt$$

For the radioactive carbon isotope, <sup>14</sup>C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a <sup>14</sup>C atom.

- **56.** Estimate the numerical value of  $\int_{0}^{\infty} e^{-x^{2}} dx$  by writing it as the sum of  $\int_{0}^{4} e^{-x^{2}} dx$  and  $\int_{4}^{\infty} e^{-x^{2}} dx$ . Approximate the first integral by using Simpson's Rule with n = 8 and show that the second integral is smaller than  $\int_{4}^{\infty} e^{-4x} dx$ , which is less than 0.0000001.
- **57.** Show that  $\int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-x^2} dx$ .
- **58.** Show that  $\int_0^\infty e^{-x^2} dx = \int_0^1 \sqrt{-\ln y} \, dy$  by interpreting the integrals as areas.
- 59. Determine how large the number a has to be so that

$$\int_{a}^{\infty} \frac{1}{x^2 + 1} \, dx < 0.001$$

60. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$$

is convergent. Evaluate the integral for this value of C.



### Chapter 5 Review

### CONCEPT CHECK •

- (a) Write an expression for a Riemann sum of a function *f*. Explain the meaning of the notation that you use.
  - (b) If f(x) ≥ 0, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
  - (c) If f(x) takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
- **2.** (a) Write the definition of the definite integral of a continuous function from *a* to *b*.
  - (b) What is the geometric interpretation of ∫<sub>a</sub><sup>b</sup> f(x) dx if f(x) ≥ 0?
  - (c) What is the geometric interpretation of ∫<sub>a</sub><sup>b</sup> f(x) dx if f(x) takes on both positive and negative values? Illustrate with a diagram.

- 3. (a) State the Evaluation Theorem.(b) State the Total Change Theorem.
- **4.** If r(t) is the rate at which water flows into a reservoir, what does  $\int_{t_1}^{t_2} r(t) dt$  represent?
- 5. Suppose a particle moves back and forth along a straight line with velocity v(t), measured in feet per second, and acceleration a(t).
  - (a) What is the meaning of  $\int_{60}^{120} v(t) dt$ ?
  - (b) What is the meaning of  $\int_{60}^{120} |v(t)| dt$ ?
  - (c) What is the meaning of  $\int_{60}^{120} a(t) dt$ ?
- **6.** (a) Explain the meaning of the indefinite integral  $\int f(x) dx$ .
  - (b) What is the connection between the definite integral  $\int_{a}^{b} f(x) dx$  and the indefinite integral  $\int f(x) dx$ ?

#### 438 CHAPTER 5 INTEGRALS

- **7.** State both parts of the Fundamental Theorem of Calculus.
- **8.** (a) State the Substitution Rule. In practice, how do you use it?
  - (b) State the rule for integration by parts. In practice, how do you use it?
- **9.** State the following rules for approximating the indefinite integral  $\int_{a}^{b} f(x) dx$ .
  - (a) The Midpoint Rule (b) The Trapezoidal Rule
  - (c) Simpson's Rule
- Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.
- **I.** If f and g are continuous on [a, b], then

$$\int_{a}^{b} \left[ f(x) + g(x) \right] dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$$

**2.** If f and g are continuous on [a, b], then

$$\int_{a}^{b} \left[ f(x)g(x) \right] dx = \left( \int_{a}^{b} f(x) \, dx \right) \left( \int_{a}^{b} g(x) \, dx \right)$$

**3.** If f is continuous on [a, b] and  $f(x) \ge 0$ , then

$$\int_{a}^{b} \sqrt{f(x)} \ dx = \sqrt{\int_{a}^{b} f(x) \ dx}$$

**4.** If f' is continuous on [1, 3], then

$$\int_{1}^{3} f'(v) \, dv = f(3) - f(1)$$

10. Define the following improper integrals.

(a) 
$$\int_a^\infty f(x) dx$$
 (b)  $\int_{-\infty}^b f(x) dx$  (c)  $\int_{-\infty}^\infty f(x) dx$ 

- **11.** Define the improper integral  $\int_a^b f(x) dx$  for each of the following cases.
  - (a) f has an infinite discontinuity at a.
  - (b) f has an infinite discontinuity at b.
  - (c) f has an infinite discontinuity at c, where a < c < b.
- 12. State the Comparison Theorem for improper integrals.

### 🔺 TRUE-FALSE QUIZ 🔺

**5.** If f and g are continuous and  $f(x) \ge g(x)$  for  $a \le x \le b$ , then

$$\int_{a}^{b} f(x) \, dx \ge \int_{a}^{b} g(x) \, dx$$

 If f and g are differentiable and f(x) ≥ g(x) for a < x < b, then f'(x) ≥ g'(x) for a < x < b.</li>

7. 
$$\int_{-1}^{1} \left( x^5 - 6x^9 + \frac{\sin x}{(1+x^4)^2} \right) dx = 0$$

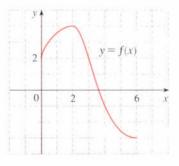
$$8. \int_0^4 \frac{x}{x^2 - 1} \, dx = \frac{1}{2} \ln 15$$

9. 
$$\int_{1}^{\infty} \frac{1}{x^{\sqrt{2}}} dx$$
 is convergent

- 10.  $\int_0^2 (x x^3) dx$  represents the area under the curve  $y = x x^3$  from 0 to 2.
- II. All continuous functions have derivatives.
- **12.** All continuous functions have antiderivatives.

### EXERCISES 🔶

1. Use the given graph of *f* to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for

$$f(x) = x^2 - x \qquad 0 \le x \le 2$$

with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral

$$\int_{0}^{2} (x^{2} - x) dx$$

- (c) Use the Evaluation Theorem to check your answer to part (b).
- (d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left( x \, + \, \sqrt{1 \, - \, x^2} \, \right) \, dx$$

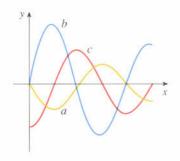
by interpreting it in terms of areas.

4. Express

$$\lim_{n \to \infty} \sum_{i=1}^n \sin x_i \, \Delta x_i$$

as a definite integral on the interval  $[0, \pi]$  and then evaluate the integral.

- **5.** If  $\int_{0}^{6} f(x) dx = 10$  and  $\int_{0}^{4} f(x) dx = 7$ , find  $\int_{4}^{6} f(x) dx$ .
- **6.** (a) Write  $\int_0^2 e^{3x} dx$  as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.
  - (b) Use the Evaluation Theorem to check your answer to part (a).
  - **7.** The figure shows the graphs of f, f', and  $\int_0^x f(t) dt$ . Identify each graph, and explain your choices.



8. Evaluate:

(a) 
$$\int_0^1 \frac{d}{dx} (e^{\arctan x}) dx$$
 (b)  $\frac{d}{dx} \int_0^1 e^{\arctan x} dx$   
(c)  $\frac{d}{dx} \int_0^x e^{\arctan t} dt$ 

dx

dx

9–26 ■ Evaluate the integral.

9. 
$$\int_{0}^{1} (1 - x^{9}) dx$$
  
10.  $\int_{0}^{1} (1 - x)^{9} dx$   
11.  $\int_{0}^{1} \frac{x}{x^{2} + 1} dx$   
12.  $\int_{0}^{1} \frac{1}{x^{2} + 1} dx$   
13.  $\int_{1}^{8} \sqrt[3]{x}(x - 1) dx$   
14.  $\int_{1}^{4} \frac{x^{2} - x + 1}{\sqrt{x}} dx$   
15.  $\int_{3}^{11} \frac{dx}{\sqrt{2x + 3}}$   
16.  $\int_{1}^{2} x^{3} \ln x dx$   
17.  $\int_{0}^{1} e^{\pi t} dt$   
18.  $\int_{1}^{2} \frac{1}{2 - 3x} dx$   
19.  $\int x \sec x \tan x dx$   
20.  $\int \sin x \cos(\cos x) dx$ 

**21.**  $\int \frac{e^x}{e^x + 1} dx$ **22.**  $\int x^2 e^{-3x} dx$ 

**23.**  $\int e^x \cos x \, dx$  **24.**  $\int \tan^{-1} x \, dx$ 

**25.** 
$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$
 **26.** 
$$\int \frac{\cos(\ln x)}{x} dx$$

- 🚰 27–28 🔳 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take C = 0).
  - **27.**  $\int \frac{\cos x}{\sqrt{1+\sin x}} dx$  **28.**  $\int \frac{x^3}{\sqrt{x^2+1}} dx$
- **29.** Use a graph to give a rough estimate of the area of the region that lies under the curve  $y = x\sqrt{x}, 0 \le x \le 4$ . Then find the exact area.
- **30.** Graph the function  $f(x) = \cos^2 x \sin^3 x$  and use the graph to guess the value of the integral  $\int_0^{2\pi} f(x) dx$ . Then evaluate the integral to confirm your guess.

31–34 ■ Find the derivative of the function.

**31.** 
$$F(x) = \int_{1}^{x} \sqrt{1 + t^4} dt$$
 **32.**  $g(x) = \int_{1}^{\cos x} \sqrt[3]{1 - t^2} dt$   
**33.**  $g(x) = \int_{0}^{x^3} \frac{t}{\sqrt{1 + t^3}} dt$  **34.**  $y = \int_{2x}^{3x+1} \sin(t^4) dt$ 

**35–38** ■ Use the Table of Integrals on the back endpapers to evaluate the integral.

**35.**  $\int e^x \sqrt{1 - e^{2x}} \, dx$  **36.**  $\int \tan^5 x \, dx$ **37.**  $\int \sqrt{x^2 + x + 1} \, dx$  **38.**  $\int \frac{\cot x}{\sqrt{1 + 2\sin x}} \, dx$ 

**39-40** ■ Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with n = 10 to approximate the given integral. Round your answers to six decimal places. Can you say whether your answers are underestimates or overestimates?

- **39.**  $\int_0^1 \sqrt{1 + x^4} \, dx$  **40.**  $\int_0^{\pi/2} \sqrt{\sin x} \, dx$
- 41. Estimate the errors involved in Exercise 39, parts (a) and (b).
- **42.** Use Simpson's Rule with n = 6 to estimate the area under the curve  $y = e^{x}/x$  from x = 1 to x = 4.
- **CAS** 43. (a) If  $f(x) = \sin(\sin x)$ , use a graph to find an upper bound for  $|f^{(4)}(x)|$ .
  - (b) Use Simpson's Rule with n = 10 to approximate  $\int_{0}^{\pi} f(x) dx$  and use part (a) to estimate the error.
  - (c) How large should *n* be to guarantee that the size of the error in using  $S_n$  is less than 0.00001?

- **(AS)** 44. (a) How would you evaluate  $\int x^5 e^{-2x} dx$  by hand? (Don't actually carry out the integration.)
  - (b) How would you evaluate  $\int x^5 e^{-2x} dx$  using tables? (Don't actually do it.)
  - (c) Use a CAS to evaluate  $\int x^5 e^{-2x} dx$ .
  - (d) Graph the integrand and the indefinite integral on the same screen. Does your answer to part (c) appear to be reasonable?

45-50 = Evaluate the improper integral or show that it is divergent.

**45.** 
$$\int_{0}^{\infty} \frac{1}{(x+2)^{4}} dx$$
**46.** 
$$\int_{1}^{\infty} \frac{\ln x}{x^{2}} dx$$
**47.** 
$$\int_{-\infty}^{0} e^{-2x} dx$$
**48.** 
$$\int_{-1}^{1} \frac{1}{2x+1} dx$$
**49.** 
$$\int_{1}^{e} \frac{dx}{x\sqrt{\ln x}}$$
**50.** 
$$\int_{2}^{6} \frac{y}{\sqrt{y-2}} dy$$

Use the Comparison Theorem to determine whether the integral

$$\int_{1}^{\infty} \frac{x^3}{x^5 + 2} \, dx$$

is convergent or divergent.

- **52.** For what values of *a* is  $\int_0^{\infty} e^{ax} \cos x \, dx$  convergent? Evaluate the integral for those values of *a*.
- **53.** A particle moves along a line with velocity function  $v(t) = t^2 t$ . Find (a) the displacement and (b) the distance traveled by the particle during the time interval [0, 5].
- **54.** A radar gun was used to record the speed of a runner at the times in the table. Use Simpson's Rule to estimate the distance the runner covered during those 5 seconds.

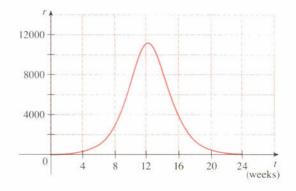
1.(8)	0	0.5	1.0	1.5	2.0	2.5
<i>v</i> (m/s)	0	4.67	7.34	8.86	9.73	10.22

$t_{\rm c}(8)$	3.0	3.5	4,0	4.5	5.0	G
v (m/s)	10.51	10.67	10.76	10.81	10.81	

55. The table gives the rate of increase in the cost of medical care in the United States (as a percentage). Use Simpson's Rule to estimate the total percentage increase in the cost of medical care from 1988 to 1994.

$\langle t \rangle$	1988	1989	199()	[99]	1992	1993	1994
r	6.5	7.7	9.0	8.7	7.4	5.9	4.8

**56.** A population of honeybees increased at a rate of r(t) bees per week, where the graph of r is as shown. Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



**57.** Suppose that the temperature in a long, thin rod placed along the *x*-axis is initially C/(2a) if  $|x| \le a$  and 0 if |x| > a. It can be shown that if the heat diffusivity of the rod is *k*, then the temperature of the rod at the point *x* at time *t* is

$$T(x,t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)} du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a\to 0} T(x,t)$$

Use l'Hospital's Rule to find this limit.

- **58.** The Fresnel function  $S(x) = \int_0^x \sin(\pi t^2/2) dt$  was introduced in Section 5.4. Fresnel also used the function  $C(x) = \int_0^x \cos(\pi t^2/2) dt$  in his theory of the diffraction of light waves.
  - (a) On what intervals is C increasing?
  - (b) On what intervals is C concave upward?
  - (c) Use a graph to solve the following equation correct to one decimal place:

$$\int_0^x \cos(\pi t^2/2) \, dt = 0.7$$

- (d) Plot the graphs of *C* and *S* on the same screen. How are these graphs related?
- **59.** If f is a continuous function such that

$$\int_0^x f(t) \, dt = x e^{2x} + \int_0^x e^{-t} f(t) \, dt$$

for all x, find an explicit formula for f(x).

**60.** Find a function f and a value of the constant a such that

$$2\int_a^x f(t)\,dt = 2\sin x - 1$$

**61.** If f' is continuous on [a, b], show that

$$2 \int_{a}^{b} f(x)f'(x) \, dx = [f(b)]^2 - [f(a)]^2$$

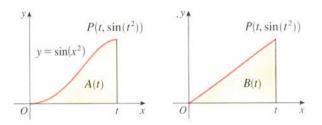
**62.** If *n* is a positive integer, prove that

$$\int_0^1 (\ln x)^n dx = (-1)^n n!$$

**63.** If f' is continuous on  $[0, \infty)$  and  $\lim_{x \to \infty} f(x) = 0$ , show that

$$\int_0^\infty f'(x)\,dx\,=\,-f(0)$$

**64.** The figure shows two regions in the first quadrant: A(t) is the area under the curve  $y = \sin(x^2)$  from 0 to t, and B(t) is the area of the triangle with vertices O, P, and (t, 0). Find  $\lim_{t \to 0^+} A(t)/B(t)$ .





Before you look at the solution of the following example, cover it up and first try to solve the problem yourself.

**Example 1** Evaluate 
$$\lim_{x \to 3} \left( \frac{x}{x-3} \int_{3}^{x} \frac{\sin t}{t} dt \right)$$
.

**Solution** Let's start by having a preliminary look at the ingredients of the function. What happens to the first factor, x/(x - 3), when x approaches 3? The numerator approaches 3 and the denominator approaches 0, so we have

and

$$\frac{x}{x-3} \to \infty \quad \text{as} \quad x \to 3^+$$
$$\frac{x}{x-3} \to -\infty \quad \text{as} \quad x \to 3^-$$

The second factor approaches 
$$\int_3^3 (\sin t)/t \, dt$$
, which is 0. It's not clear what happens to the function as a whole. (One factor is becoming large while the other is becoming small.) So how do we proceed?

One of the principles of problem solving is *recognizing something familiar*. Is there a part of the function that reminds us of something we've seen before? Well, the integral

$$\int_{3}^{x} \frac{\sin t}{t} dt$$

has x as its upper limit of integration and that type of integral occurs in Part 1 of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This suggests that differentiation might be involved.

Once we start thinking about differentiation, the denominator (x - 3) reminds us of something else that should be familiar: One of the forms of the definition of the derivative in Chapter 2 is

$$F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}$$

and with a = 3 this becomes

$$F'(3) = \lim_{x \to 3} \frac{F(x) - F(3)}{x - 3}$$

So what is the function F in our situation? Notice that if we define

$$F(x) = \int_3^x \frac{\sin t}{t} \, dt$$

The principles of problem solving are discussed on page 87.

then F(3) = 0. What about the factor x in the numerator? That's just a red herring, so let's factor it out and put together the calculation:

$$\lim_{x \to 3} \left( \frac{x}{x-3} \int_3^x \frac{\sin t}{t} dt \right) = \left( \lim_{x \to 3} x \right) \cdot \lim_{x \to 3} \frac{\int_3^x \frac{\sin t}{t} dt}{x-3}$$
$$= 3 \lim_{x \to 3} \frac{F(x) - F(3)}{x-3}$$
$$= 3F'(3)$$
$$= 3\frac{\sin 3}{3} \quad \text{(FTC1)}$$
$$= \sin 3$$

### Example 2

(a) Prove that if f is a continuous function, then

$$\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$$

(b) Use part (a) to show that

$$\int_{0}^{\pi/2} \frac{\sin^{n} x}{\sin^{n} x + \cos^{n} x} \, dx = \frac{\pi}{4}$$

for all positive numbers n.

#### Solution

(a) At first sight, the given equation may appear somewhat baffling. How is it possible to connect the left side to the right side? Connections can often be made through one of the principles of problem solving: *Introduce something extra*. Here the extra ingredient is a new variable. We often think of introducing a new variable when we use the Substitution Rule to integrate a specific function. But that technique is still useful in the present circumstance in which we have a general function f.

Once we think of making a substitution, the form of the right side suggests that it should be u = a - x. Then du = -dx. When x = 0, u = a; when x = a, u = 0. So

$$\int_0^a f(a - x) \, dx = -\int_a^0 f(u) \, du = \int_0^a f(u) \, du$$

But this integral on the right side is just another way of writing  $\int_0^a f(x) dx$ . So the given equation is proved.

(b) If we let the given integral be I and apply part (a) with  $a = \pi/2$ , we get

$$I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} \, dx = \int_0^{\pi/2} \frac{\sin^n (\pi/2 - x)}{\sin^n (\pi/2 - x) + \cos^n (\pi/2 - x)} \, dx$$

The computer graphs in Figure I make it seem plausible that all of the integrals in Example 2 have the same value. The graph of each integrand is labeled with the corresponding value of n.



A well-known trigonometric identity tells us that  $\sin(\pi/2 - x) = \cos x$  and  $\cos(\pi/2 - x) = \sin x$ , so we get

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} \, dx$$

Notice that the two expressions for I are very similar. In fact, the integrands have the same denominator. This suggests that we should add the two expressions. If we do so, we get

$$2I = \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \, dx = \int_0^{\pi/2} \, dx = \frac{\pi}{2}$$

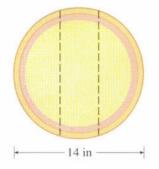
Figure 1

Problems

Therefore,  $I = \pi/4$ .

 $\mathcal{A}$ 

1. Three mathematics students have ordered a 14-inch pizza. Instead of slicing it in the traditional way, they decide to slice it by parallel cuts, as shown in the figure. Being mathematics majors, they are able to determine where to slice so that each gets the same amount of pizza. Where are the cuts made?



- 2. In this problem we approximate the sine function on the interval  $[0,\pi]$  by three quadratic functions, each of which has the same zeros as the sine function on this interval.
  - (a) Find a quadratic function f such that  $f(0) = f(\pi) = 0$  and which has the same maximum value as sin on  $[0,\pi]$ .
  - (b) Find a quadratic function g such that  $g(0) = g(\pi) = 0$  and which has the same rate of change as the sine function at 0 and  $\pi$ .
  - (c) Find a quadratic function h such that h(0) = h(π) = 0 and the area under h from 0 to π is the same as for the sine function.
- (d) Illustrate by graphing f, g, h, and the sine function in the same viewing rectangle  $[0, \pi]$  by [0, 1]. Identify which graph belongs to each function.

**3.** If  $x \sin \pi x = \int_0^{x^2} f(t) dt$ , where *f* is a continuous function, find f(4).

- 4. (a) Graph several members of the family of functions  $f(x) = (2cx x^2)/c^3$  for c > 0 and look at the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.
  - (b) Prove your conjecture in part (a).

- (c) Take another look at the graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. Can you guess what kind of curve this is?
- (d) Find the equation of the curve you sketched in part (c).
- 5. Suppose the curve y = f(x) passes through the origin and the point (1, 1). Find the value of the integral  $\int_{0}^{1} f'(x) dx$ .
- 6. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wetted area of the disk, show that the center of the disk should be positioned at a height  $r/\sqrt{1 + \pi^2}$  above the surface of the liquid.
- 7. Evaluate  $\lim_{x \to 0} \frac{1}{x} \int_0^x (1 \tan 2t)^{1/t} dt$ .

8. If 
$$f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} dt$$
, where  $g(x) = \int_0^{\cos x} [1 + \sin(t^2)] dt$ , find  $f'(\pi/2)$ .

- **9.** Find a function f such that f(1) = -1, f(4) = 7, and f'(x) > 3 for all x, or prove that such a function cannot exist.
- 10. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.
- **11.** Find the interval [a, b] for which the value of the integral  $\int_a^b (2 + x x^2) dx$  is a maximum.
- 12. Suppose f is continuous, f(0) = 0, f(1) = 1, f'(x) > 0, and  $\int_0^1 f(x) dx = \frac{1}{3}$ . Find the value of the integral  $\int_0^1 f^{-1}(y) dy$ .

**13.** Find 
$$\frac{d^2}{dx^2} \int_0^x \left( \int_1^{\sin t} \sqrt{1+u^4} \, du \right) dt.$$

14. Use an integral to estimate the sum  $\sum_{i=1}^{10000} \sqrt{i}$ .

- **15.** Evaluate  $\int_0^1 (\sqrt[3]{1-x^7} \sqrt[7]{1-x^3}) dx$ .
- 16. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q. At what height above the diameter should the horizontal line be placed so as to minimize the shaded area?
- 17. Show that

$$\int_0^1 (1 - x^2)^n dx = \frac{2^{2n} (n!)^2}{(2n+1)!}$$

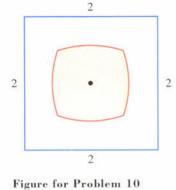
*Hint*: Start by showing that if  $I_n$  denotes the integral, then

$$I_{k+1} = \frac{2k+2}{2k+3} I_k$$

18. For any number c, we let  $f_c(x)$  be the smaller of the two numbers  $(x - c)^2$  and  $(x - c - 2)^2$ . Then we define

$$g(c) = \int_0^1 f_c(x) \, dx$$

Find the maximum and minimum values of g(c) if  $-2 \le c \le 2$ .



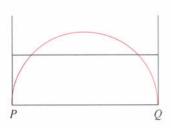
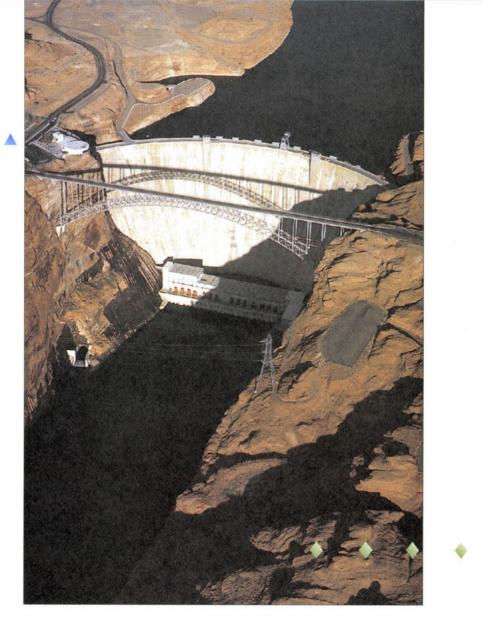


Figure for Problem 16





Applications of Integration



• • • • These photographs illustrate three uses of integrals that we consider in this chapter: calculating the force exerted by water on a dam; deciding where to sit in a movie theater; finding the point where a flat object balances horizontally.



In this chapter we explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, lengths of curves, the average value of a function, the work done by a varying force, the center of gravity of a plate, the force on a dam, as well as quantities of interest in biology, economics, and statistics. The common theme in most of these applications is the following general method, which is similar to the one we used to find areas under curves. We break up a quantity Q into a large number of small parts. We then approximate each small part by a quantity of the form  $f(x_i^*) \Delta x$  and thus approximate Q by a Riemann sum. Then we take the limit and express Q as an integral. Finally we evaluate the integral by using the Evaluation Theorem or Simpson's Rule.



# More about Areas

In Chapter 5 we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of more general regions. First we consider regions that lie between the graphs of two functions. Then we look at regions enclosed by parametric curves.

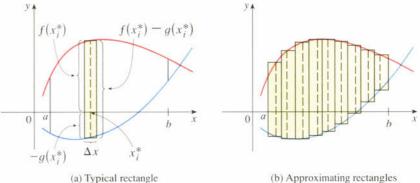
### **Areas between Curves**

Consider the region S that lies between two curves y = f(x) and y = q(x) and between the vertical lines x = a and x = b, where f and g are continuous functions and  $f(x) \ge g(x)$  for all x in [a, b] (see Figure 1).

Just as we did for areas under curves in Section 5.1, we divide S into n strips of equal width and then we approximate the *i*th strip by a rectangle with base  $\Delta x$  and height  $f(x_i^*) - g(x_i^*)$ . (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case  $x_i^* = x_i$ ). The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

is therefore an approximation to what we intuitively think of as the area of S.



#### FIGURE 2



(b) Approximating rectangles

This approximation appears to become better and better as  $n \to \infty$ . Therefore, we define the area A of S as the limiting value of the sum of the areas of these approximating rectangles.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ f(x_i^*) - g(x_i^*) \right] \Delta x$$

We recognize the limit in (1) as the definite integral of f - g. Therefore:

**2** The area A of the region bounded by the curves y = f(x), y = g(x), and the lines x = a, x = b, where f and g are continuous and  $f(x) \ge g(x)$ for all x in [a, b], is

$$A = \int_a^b \left[ f(x) - g(x) \right] dx$$

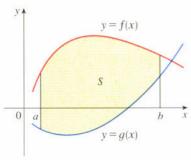
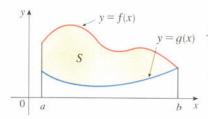
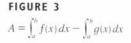
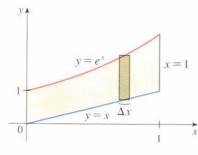


FIGURE I  $S = \{(x, y) \mid a \le x \le b, g(x) \le y \le f(x)\}$ 









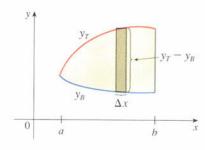
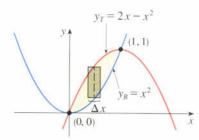


FIGURE 5



Notice that in the special case where g(x) = 0, S is the region under the graph of f and our general definition of area (1) reduces to our previous definition (Definition 4 in Section 5.1).

In the case where both f and g are positive, you can see from Figure 3 why (2) is true:

$$A = [area under y = f(x)] - [area under y = g(x)]$$

$$= \int_{a}^{b} f(x) \, dx \, - \, \int_{a}^{b} g(x) \, dx \, = \, \int_{a}^{b} \left[ f(x) \, - \, g(x) \right] dx$$

**EXAMPLE 1** Find the area of the region bounded above by  $y = e^x$ , bounded below by y = x, and bounded on the sides by x = 0 and x = 1.

**SOLUTION** The region is shown in Figure 4. The upper boundary curve is  $y = e^x$  and the lower boundary curve is y = x. So we use the area formula (2) with  $f(x) = e^x$ , g(x) = x, a = 0, and b = 1:

$$A = \int_0^1 (e^x - x) \, dx = e^x - \frac{1}{2}x^2 \Big]_0^1$$
$$= e - \frac{1}{2} - 1 = e - 1.5$$

In Figure 4 we drew a typical approximating rectangle with width  $\Delta x$  as a reminder of the procedure by which the area is defined in (1). In general, when we set up an integral for an area, it is helpful to sketch the region to identify the top curve  $y_T$ , the bottom curve  $y_B$ , and a typical approximating rectangle as in Figure 5. Then the area of a typical rectangle is  $(y_T - y_B)\Delta x$  and the equation

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_{T} - y_{B}) \Delta x = \int_{a}^{b} (y_{T} - y_{B}) dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point. In the next example both of the side boundaries reduce to a point, so the first step is to find a and b.

**EXAMPLE 2** Find the area of the region enclosed by the parabolas  $y = x^2$  and  $y = 2x - x^2$ .

**SOLUTION** We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives  $x^2 = 2x - x^2$ , or  $2x^2 = 2x$ . Thus, x(x - 1) = 0, so x = 0 or 1. The points of intersection are (0, 0) and (1, 1).

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \qquad y_B = x^2$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = (2x - x^2 - x^2)\Delta x$$

FIGURE 6

and the region lies between x = 0 and x = 1. So the total area is

$$A = \int_0^1 (2x - 2x^2) \, dx = 2 \int_0^1 (x - x^2) \, dx$$
$$= 2 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}$$

Sometimes it is difficult, or even impossible, to find the points of intersection of two curves exactly. As shown in the following example, we can use a graphing calculator or computer to find approximate values for the intersection points and then proceed as before.

**EXAMPLE 3** Find the approximate area of the region bounded by the curves  $y = x/\sqrt{x^2 + 1}$  and  $y = x^4 - x$ .

**SOLUTION** If we were to try to find the exact intersection points, we would have to solve the equation

$$\frac{x}{\sqrt{x^2+1}} = x^4 - x$$

This looks like a very difficult equation to solve exactly (in fact, it's impossible), so instead we use a graphing device to draw the graphs of the two curves in Figure 7. One intersection point is the origin. We zoom in toward the other point of intersection and find that  $x \approx 1.18$ . (If greater accuracy is required, we could use Newton's method or a root-finder, if available on our graphing device.) Thus, an approximation to the area between the curves is

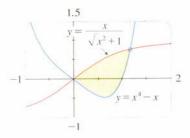
$$A \approx \int_0^{1.18} \left[ \frac{x}{\sqrt{x^2 + 1}} - (x^4 - x) \right] dx$$

To integrate the first term we use the substitution  $u = x^2 + 1$ . Then du = 2x dx, and when x = 1.18, we have  $u \approx 2.39$ . So

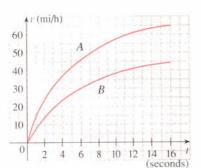
$$A \approx \frac{1}{2} \int_{1}^{2.39} \frac{du}{\sqrt{u}} - \int_{0}^{1.18} (x^{4} - x) dx$$
$$= \sqrt{u} \Big]_{1}^{2.39} - \left[ \frac{x^{5}}{5} - \frac{x^{2}}{2} \right]_{0}^{1.18}$$
$$= \sqrt{2.39} - 1 - \frac{(1.18)^{5}}{5} + \frac{(1.18)^{2}}{2} \approx 0.785$$

**EXAMPLE 4** Figure 8 shows velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use Simpson's Rule to estimate it.

**SOLUTION** We know from Section 5.3 that the area under the velocity curve *A* represents the distance traveled by car *A* during the first 16 seconds. Similarly, the area under curve *B* is the distance traveled by car *B* during that time period. So the area between these curves, which is the difference of the areas under the curves, is the distance between the cars after 16 seconds. We read the velocities from the graph and convert them to feet per second  $(1 \text{ mi/h} = \frac{5280}{3600} \text{ ft/s}).$ 









t	0	2	4	6	8	10	12	14	16
$v_A$	0	34	54	67	76	84	89	92	95
$v_{\scriptscriptstyle B}$	0	21	34	44	51	56	60	63	65
$v_A = v_B$	0	13	20	23	25	28	29	29	30

Using Simpson's Rule with n = 8 intervals, so that  $\Delta t = 2$ , we estimate the distance between the cars after 16 seconds:

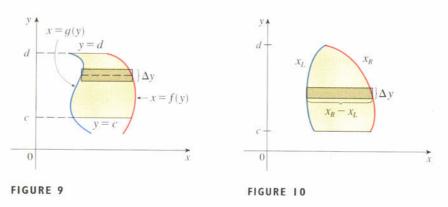
$$\int_{0}^{16} (v_A - v_B) dt$$

$$= \frac{2}{3} [0 + 4(13) + 2(20) + 4(23) + 2(25) + 4(28) + 2(29) + 4(29) + 30]$$

$$\approx 367 \text{ ft}$$

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations x = f(y), x = g(y), y = c, and y = d, where f and g are continuous and  $f(y) \ge g(y)$  for  $c \le y \le d$  (see Figure 9), then its area is

$$A = \int_{c}^{d} \left[ f(y) - g(y) \right] dy$$



If we write  $x_R$  for the right boundary and  $x_L$  for the left boundary, then, as Figure 10 illustrates, we have

$$A = \int_c^d \left( x_R - x_L \right) dy$$

Here a typical approximating rectangle has dimensions  $x_R - x_L$  and  $\Delta y$ .

**EXAMPLE 5** Find the area enclosed by the line y = x - 1 and the parabola  $y^2 = 2x + 6$ .

**SOLUTION** By solving the two equations we find that the points of intersection are (-1, -2) and (5, 4). We solve the equation of the parabola for x and notice from Figure 11 that the left and right boundary curves are

$$x_L = \frac{1}{2}y^2 - 3$$
  $x_R = y + 1$ 

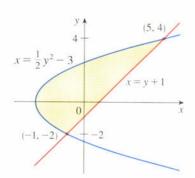


FIGURE II

We must integrate between the appropriate y-values, y = -2 and y = 4. Thus

$$A = \int_{-2}^{4} (x_R - x_L) \, dy$$
  
=  $\int_{-2}^{4} \left[ (y+1) - \left(\frac{1}{2}y^2 - 3\right) \right] dy$   
=  $\int_{-2}^{4} \left( -\frac{1}{2}y^2 + y + 4 \right) dy$   
=  $-\frac{1}{2} \left( \frac{y^3}{3} \right) + \frac{y^2}{2} + 4y \Big]_{-2}^{4}$   
=  $-\frac{1}{6} (64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18$ 

We could have found the area in Example 5 by integrating with respect to x instead of y, but the calculation is much more involved. It would have meant splitting the region in two and computing the areas labeled  $A_1$  and  $A_2$  in Figure 12. The method we used in Example 5 is *much* easier.

### Areas Enclosed by Parametric Curves

We know that the area under a curve y = F(x) from *a* to *b* is  $A = \int_{a}^{b} F(x) dx$ , where  $F(x) \ge 0$ . If the curve is given by the parametric equations x = f(t) and y = g(t),  $\alpha \le t \le \beta$ , then we can calculate an area by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_{a}^{b} y \, dx = \int_{a}^{\beta} g(t) f'(t) \, dt \qquad \left[ \text{ or } \int_{\beta}^{\alpha} g(t) f'(t) \, dt \right]$$

**EXAMPLE 6** Find the area under one arch of the cycloid

$$x = r(\theta - \sin \theta)$$
  $y = r(1 - \cos \theta)$ 

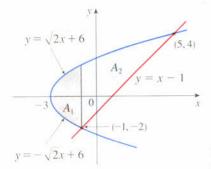
(See Figure 13.)

**SOLUTION** One arch of the cycloid is given by  $0 \le \theta \le 2\pi$ . Using the Substitution Rule with  $y = r(1 - \cos \theta)$  and  $dx = r(1 - \cos \theta) d\theta$ , we have

$$A = \int_{0}^{2\pi r} y \, dx = \int_{0}^{2\pi} r(1 - \cos\theta)r(1 - \cos\theta) \, d\theta$$
  
=  $r^2 \int_{0}^{2\pi} (1 - \cos\theta)^2 \, d\theta = r^2 \int_{0}^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$   
=  $r^2 \int_{0}^{2\pi} \left[ 1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta$   
=  $r^2 \left[ \frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_{0}^{2\pi}$   
=  $r^2 (\frac{3}{2} \cdot 2\pi) = 3\pi r^2$ 

**EXAMPLE 7** Estimate the area enclosed by the loop of the curve with parametric equations

$$x = t^{2} + t + 1$$
  $y = 3t^{4} - 8t^{3} - 18t^{2} + 25$ 





The limits of integration for t are found as usual with the Substitution Rule. When x = a, t is either  $\alpha$  or  $\beta$ . When x = b, t is the remaining value.

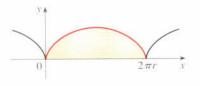
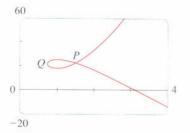


FIGURE 13

The result of Example 6 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 6 in Section I.4). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.



**SOLUTION** The curve is shown in Figure 14. (We graphed it in Example 6 in Section 4.4.) To find the loop we need to know the parameter values that correspond to the rightmost point P on the loop, where the curve crosses itself. Zooming in toward P and using the cursor, we find that its coordinates are approximately (1.497, 22.2). The corresponding parameter values are the solutions of the equation

$$x(t) = t^2 + t + 1 = 1.497$$

The quadratic formula gives  $t \approx -1.36$  and 0.36. The leftmost point Q, where the tangent is vertical, corresponds to the parameter value t = -0.5.

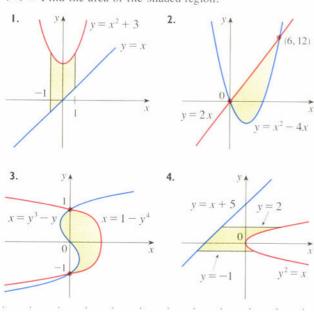
We find the area of the loop by subtracting the area under the bottom part of the loop from the area under the top part of the loop. So the approximate area of the loop is

$$A \approx \int_{-0.5}^{0.36} (3t^4 - 8t^3 - 18t^2 + 25)(2t+1) dt$$
$$- \int_{-0.5}^{-1.36} (3t^4 - 8t^3 - 18t^2 + 25)(2t+1) dt$$

Combining these two integrals, we get

$$A \approx \int_{-1.36}^{0.36} \left(3t^4 - 8t^3 - 18t^2 + 25\right)\left(2t + 1\right) dt \approx 3.6$$

FIGURE 14



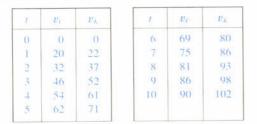
**5-16** Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

5. 
$$y = x$$
,  $y = x^2$   
6.  $y = 1/x$ ,  $y = 1/x^2$ ,  $x = 1$ ,  $x = 2$   
7.  $y = e^x$ ,  $y = e^{3x}$ ,  $x = 1$   
8.  $y = x^2$ ,  $y^2 = x$   
9.  $y = 4x^2$ ,  $y = x^2 + 3$   
10.  $y = x^4 - x^2$ ,  $y = 1 - x^2$   
11.  $y^2 = x$ ,  $x - 2y = 3$   
12.  $x + y^2 = 2$ ,  $x + y = 0$   
13.  $x = 1 - y^2$ ,  $x = y^2 - 1$   
14.  $y = \cos x$ ,  $y = \sec^2 x$ ,  $x = -\pi/4$ ,  $x = \pi/4$   
15.  $y = x^2$ ,  $y = 2/(x^2 + 1)$   
16.  $y = 1/x$ ,  $x = 0$ ,  $y = 1$ ,  $y = 2$ 

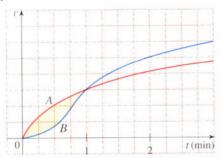
**I**−4 ■ Find the area of the shaded region.

Exercises ·

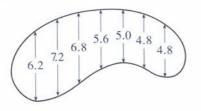
- 17-20 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region enclosed by the curves.
  - **17.**  $y = x^2$ ,  $y = 2 \cos x$  **18.**  $y = x^4 - 1$ ,  $y = x \sin(x^2)$  **19.**  $y = x^2$ ,  $y = xe^{-x/2}$ **20.**  $y = x^2 - 5$ ,  $y = \ln x$
  - **21.** Racing cars driven by Chris and Kelly are side by side at the start of a race. The table shows the velocities of each car (in miles per hour) during the first ten seconds of the race. Use Simpson's Rule to estimate how much farther Kelly travels than Chris does during the first ten seconds.



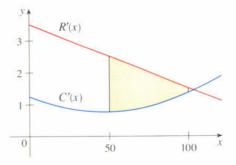
- **22.** Two cars, *A* and *B*, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.
  - (a) Which car is ahead after one minute? Explain.
  - (b) What is the meaning of the area of the shaded region?
  - (c) Which car is ahead after two minutes? Explain.
  - (d) Estimate the time at which the cars are again side by side.



**23.** The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use Simpson's Rule to estimate the area of the pool.



24. The figure shows graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer. [Recall from Section 4.7 that R(x) and C(x) represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.



- 25. Sketch the region that lies between the curves  $y = \cos x$ and  $y = \sin 2x$  and between x = 0 and  $x = \pi/2$ . Notice that the region consists of two separate parts. Find the area of this region.
- **26.** Graph the curves  $y = x^2 x$  and  $y = x^3 4x^2 + 3x$ on a common screen and observe that the region between them consists of two parts. Find the area of this region.
  - 27. Use the parametric equations of an ellipse,  $x = a \cos \theta$ ,  $y = b \sin \theta$ ,  $0 \le \theta \le 2\pi$ , to find the area that it encloses.
- **28.** Graph the parametric curve x = t 1/t, y = t + 1/t. Find the area enclosed between this curve and the line y = 2.5.
- **29.** Graph the region bounded by the curve  $x = \cos t$ ,  $y = e^t$ ,  $0 \le t \le \pi/2$ , and the lines y = 1 and x = 0. Set up an integral that represents the area of this region. Then use a computer algebra system to evaluate the integral.
- **30.** Graph the astroid  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$  and set up an integral for the area that it encloses. Then use a computer algebra system to evaluate the integral.
  - **31.** Find the area enclosed by the loop of the curve with parametric equations  $x = t^2$ ,  $y = t^3 3t$ .
- **32.** Estimate the area of the region enclosed by the loop of the curve  $x = t^3 12t$ ,  $y = 3t^2 + 2t + 5$ .
  - **33.** Find the values of c such that the area of the region enclosed by the parabolas  $y = x^2 c^2$  and  $y = c^2 x^2$  is 576.

- **34.** Find the area of the region bounded by the parabola  $y = x^2$ , the tangent line to this parabola at (1, 1), and the *x*-axis.
- 35. Find the number b such that the line y = b divides the region bounded by the curves y = x<sup>2</sup> and y = 4 into two regions with equal area.
- **36.** (a) Find the number *a* such that the line x = a bisects the area under the curve  $y = 1/x^2$ ,  $1 \le x \le 4$ .

Volumes

(b) Find the number b such that the line y = b bisects the area in part (a).

- **37.** Find a positive continuous function *f* such that the area under the graph of *f* from 0 to *t* is  $A(t) = t^3$  for all t > 0.
- 38. Suppose that 0 < c < π/2. For what value of c is the area of the region enclosed by the curves y = cos x, y = cos(x c), and x = 0 equal to the area of the region enclosed by the curves y = cos(x c), x = π, and y = 0?</li>
- **39.** For what values of *m* do the line y = mx and the curve  $y = x/(x^2 + 1)$  enclose a region? Find the area of the region.



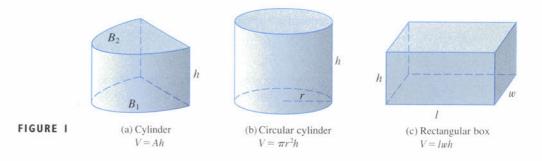
In trying to find the volume of a solid we face the same type of problem as in finding areas. We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

contract (m)

We start with a simple type of solid called a **cylinder** (or, more precisely, a *right cylinder*). As illustrated in Figure 1(a), a cylinder is bounded by a plane region  $B_1$ , called the **base**, and a congruent region  $B_2$  in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join  $B_1$  to  $B_2$ . If the area of the base is A and the height of the cylinder (the distance from  $B_1$  to  $B_2$ ) is h, then the volume V of the cylinder is defined as

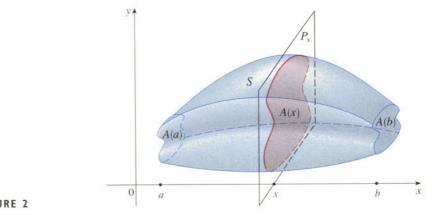
$$V = Ah$$

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume  $V = \pi r^2 h$  [see Figure 1(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume V = lwh [see Figure 1(c)].



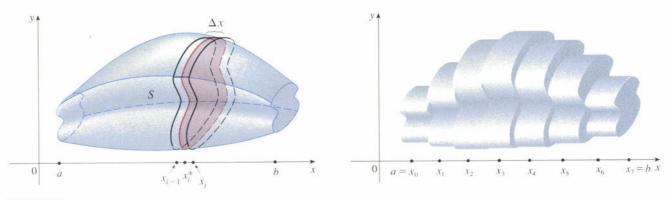
Now let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Let A(x) be the area of the cross-section of S in a plane  $P_x$  perpendicular to the x-axis and passing through the point x, where

 $a \le x \le b$ . (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.) The cross-sectional area A(x) will vary as x increases from a to b.





Let's divide S into n "slabs" of equal width  $\Delta x$  by using the planes  $P_{x_1}, P_{x_2}, \ldots$ to slice the solid. (Think of slicing a loaf of bread.) If we choose sample points  $x_i^*$ in  $[x_{i-1}, x_i]$ , we can approximate the *i*th slab  $S_i$  (the part of S that lies between the planes  $P_{x_{i-1}}$  and  $P_{x_i}$ ) by a cylinder with base area  $A(x_i^*)$  and height  $\Delta x$  (see Figure 3).





The volume of this cylinder is  $A(x_i^*)\Delta x$ , so an approximation to our intuitive conception of the volume of the *i*th slab  $S_i$  is

$$V(S_i) \approx A(x_i^*) \Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*) \, \Delta x$$

This approximation appears to become better and better as  $n \to \infty$ . (Think of the slices as becoming thinner and thinner.) Therefore, we define the volume as the limit of these sums as  $n \to \infty$ . But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

**Definition of Volume** Let S be a solid that lies between x = a and x = b. If the cross-sectional area of S in the plane  $P_x$ , through x and perpendicular to the x-axis, is A(x), where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_a^b A(x) \, dx$$

When we use the volume formula  $V = \int_a^b A(x) dx$  it is important to remember that A(x) is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

**EXAMPLE 1** Show that the volume of a sphere of radius r is

$$V = \frac{4}{2}\pi r^{3}$$

**SOLUTION** If we place the sphere so that its center is at the origin (see Figure 4), then the plane  $P_x$  intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is  $y = \sqrt{r^2 - x^2}$ . So the cross-sectional area is

$$A(x) = \pi y^2 = \pi (r^2 - x^2)$$

Using the definition of volume with a = -r and b = r, we have

 $V = \int_{-r}^{r} A(x) dx = \int_{-r}^{r} \pi (r^{2} - x^{2}) dx$ =  $2\pi \int_{0}^{r} (r^{2} - x^{2}) dx$  (The integrand is even.) =  $2\pi \left[ r^{2}x - \frac{x^{3}}{3} \right]_{0}^{r} = 2\pi \left( r^{3} - \frac{r^{3}}{3} \right)$ =  $\frac{4}{3}\pi r^{3}$ 

Figure 5 illustrates the definition of volume when the solid is a sphere with radius r = 1. From the result of Example 1, we know that the volume of the sphere is  $\frac{4}{3}\pi \approx 4.18879$ . Here the slabs are circular cylinders and the three parts of

(b) Using 10 cylinders,  $V \approx 4.2097$ 

(c) Using 20 cylinders,  $V \approx 4.1940$ 

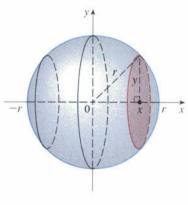


FIGURE 4



(a) Using 5 cylinders,  $V \approx 4.2726$ 

FIGURE 5 Approximating the volume of a sphere with radius 1

Figure 5 show the geometric interpretations of the Riemann sums

$$\sum_{i=1}^{n} A(\bar{x}_{i}) \Delta x = \sum_{i=1}^{n} \pi (1^{2} - \bar{x}_{i}^{2}) \Delta x$$

when n = 5, 10, and 20 if we choose the sample points  $x_i^*$  to be the midpoints  $\bar{x}_i$ . Notice that as we increase the number of approximating cylinders, the corresponding Riemann sums become closer to the true volume.

**EXAMPLE 2** Find the volume of the solid obtained by rotating the region bounded by  $y = x^3$ , y = 8, and x = 0 about the y-axis.

**SOLUTION** The region is shown in Figure 6(a) and the resulting solid is shown in Figure 6(b). Because the region is rotated about the *y*-axis, it makes sense to slice the solid perpendicular to the *y*-axis. If we slice at height *y*, we get a circular disk with radius *x*, where  $x = \sqrt[3]{y}$ . So the area of a cross-section through *y* is

$$A(y) = \pi x^2 = \pi (\sqrt[3]{y})^2 = \pi y^{2/3}$$

and the volume of the approximating cylinder pictured in Figure 6(b) is

$$A(y)\,\Delta y = \pi y^{2/3}\,\Delta y$$

Since the solid lies between y = 0 and y = 8, its volume is

$$V = \int_0^8 A(y) \, dy = \int_0^8 \pi y^{2/3} \, dy$$
$$= \pi \left[\frac{3}{5} y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

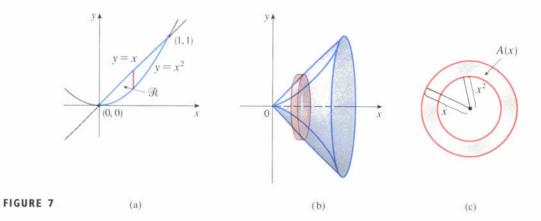
y = 8  $x = 0 \longrightarrow y = x^{3}$ or  $x = \sqrt[3]{y}$   $0 \longrightarrow x$   $y = x^{3}$   $y = x^{3}$ 

Did we get a reasonable answer in Example 2? As a check on our work, let's replace the given region by a rectangle with base [0, 2] and height 8. If we rotate this rectangle, we get a cylinder with radius 2, height 8, and volume  $\pi \cdot 2^2 \cdot 8 = 32\pi$ . We computed that the given solid has a little more than half this volume  $(19.2\pi)$ . That seems about right.



**EXAMPLE 3** The region  $\Re$  enclosed by the curves y = x and  $y = x^2$  is rotated about the *x*-axis. Find the volume of the resulting solid.

**SOLUTION** The curves y = x and  $y = x^2$  intersect at the points (0,0) and (1,1). The region between them, the solid of rotation, and a cross-section perpendicular



to the *x*-axis are shown in Figure 7. A cross-section in the plane  $P_x$  has the shape of an annulus (a ring) with inner radius  $x^2$  and outer radius *x*, so the cross-sectional area is

$$A(x) = \pi x^{2} - \pi (x^{2})^{2} = \pi (x^{2} - x^{4})$$

Therefore, we have

$$V = \int_0^1 A(x) \, dx = \int_0^1 \pi (x^2 - x^4) \, dx$$
$$= \pi \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{2\pi}{15}$$

**EXAMPLE 4** Find the volume of the solid obtained by rotating the region in Example 3 about the line y = 2.

**SOLUTION** The solid and a cross-section are shown in Figure 8. Again a cross-section is an annulus, but this time the inner radius is 2 - x and the outer radius is  $2 - x^2$ . The cross-sectional area is

$$A(x) = \pi (2 - x^2)^2 - \pi (2 - x)^2$$

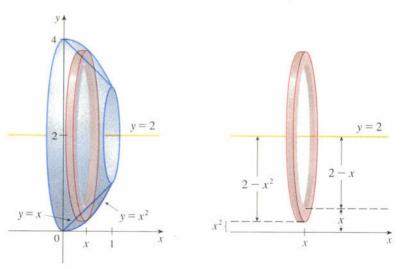


FIGURE 8

and so the volume of S is

$$V = \int_0^1 A(x) \, dx = \pi \int_0^1 \left[ (2 - x^2)^2 - (2 - x)^2 \right] dx$$
  
=  $\pi \int_0^1 \left( x^4 - 5x^2 + 4x \right) dx$   
=  $\pi \left[ \frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2} \right]_0^1 = \frac{8\pi}{15}$ 

**EXAMPLE 5** A solid has a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

**SOLUTION** Let's take the circle to be  $x^2 + y^2 = 1$ . The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 9. Since B lies on the circle, we have  $y = \sqrt{1 - x^2}$  and so the base of the triangle ABC is  $|AB| = 2\sqrt{1 - x^2}$ . Since the triangle is equilateral, we see from Figure 9(c) that its height is  $\sqrt{3} y = \sqrt{3} \sqrt{1 - x^2}$ . The cross-sectional area is therefore

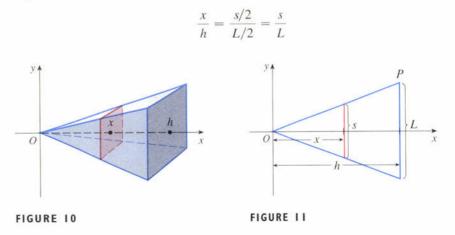
$$A(x) = \frac{1}{2} \cdot 2\sqrt{1 - x^2} \cdot \sqrt{3}\sqrt{1 - x^2} = \sqrt{3}(1 - x^2)$$

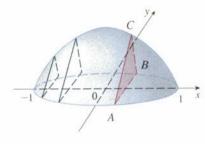
and the volume of the solid is

$$V = \int_{-1}^{1} A(x) dx = \int_{-1}^{1} \sqrt{3} (1 - x^2) dx$$
$$= 2 \int_{0}^{1} \sqrt{3} (1 - x^2) dx = 2 \sqrt{3} \left[ x - \frac{x^3}{3} \right]_{0}^{1} = \frac{4\sqrt{3}}{3}$$

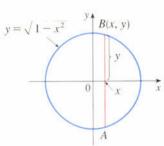
**EXAMPLE 6** Find the volume of a pyramid whose base is a square with side L and whose height is h.

**SOLUTION** We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 10. Any plane  $P_x$  that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 11 that

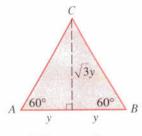
















and so s = Lx/h. [Another method is to observe that the line *OP* has slope L/(2h) and so its equation is y = Lx/(2h).] Thus, the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2} x^2$$

The pyramid lies between x = 0 and x = h, so its volume is

$$V = \int_0^h A(x) \, dx = \int_0^h \frac{L^2}{h^2} x^2 \, dx$$
$$= \frac{L^2}{h^2} \frac{x^3}{3} \bigg]_0^h = \frac{L^2 h}{3}$$

**NOTE** • It was unnecessary to place the vertex of the pyramid at the origin in Example 6. We did so merely to make the equations simple. If, instead, we had placed the center of the base at the origin and the vertex on the positive y-axis, as in Figure 12, you can verify that we would have obtained the integral

$$V = \int_0^h \frac{L^2}{h^2} (h - y)^2 dy = \frac{L^2 h}{3}$$

**EXAMPLE 7** Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curve  $y = 2x^2 - x^3$  and the x-axis.

**SOLUTION** The region is shown in Figure 13. If we try to find the volume using the method of Example 2, then we run into a severe problem. To compute the inner radius and the outer radius of a cross-section, we would have to solve the cubic equation  $y = 2x^2 - x^3$  for x in terms of y; that's not easy.

Instead of slicing, we use a different method, called **cylindrical shells**, to solve this problem. Figure 14 shows a typical approximating rectangle with width  $\Delta x$ . If we rotate this rectangle about the y-axis, we get a cylindrical shell whose average radius is  $\overline{x}_i$ , the midpoint of the *i*th subinterval.

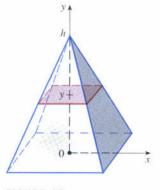
 $y = 2x^2 - x^3$   $x = 2x^2 - \overline{x_i^3}$   $2\overline{x_i^2} - \overline{x_i^3}$   $2\pi \overline{x_i}$ 

FIGURE 14 A cylindrical shell

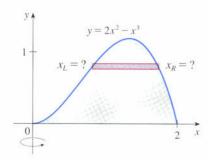
FIGURE 15 The flattened shell

Imagine this shell to be cut and flattened, as in Figure 15. The resulting rectangular slab has dimensions  $2\pi \bar{x}_i$ ,  $\Delta x$ , and  $2\bar{x}_i^2 - \bar{x}_i^3$ , so the volume of the shell is

$$2\pi \overline{x}_i (2\overline{x}_i^2 - \overline{x}_i^3) \Delta x$$









If we do this for every subinterval and then add the results, we get an approximation to the volume of the solid:

$$V \approx \sum_{i=1}^{n} 2\pi \overline{x}_i (2\overline{x}_i^2 - \overline{x}_i^3) \Delta x$$

This approximation improves as n increases, so it seems plausible that

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \overline{x}_i (2\overline{x}_i^2 - \overline{x}_i^3) \Delta x$$
  
=  $\int_0^2 2\pi x (2x^2 - x^3) dx = 2\pi \int_0^2 (2x^3 - x^4) dx$   
=  $2\pi [\frac{1}{2}x^4 - \frac{1}{5}x^5]_0^2 = 2\pi (8 - \frac{32}{5}) = \frac{16}{5}\pi$ 

Notice from Figure 14 that we obtain all shells if we let x increase from 0 to 2.

It can be verified that the method of shells gives the same answer as slicing.

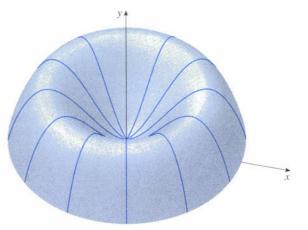


Figure 16 shows a computer-generated picture of the solid whose volume we computed in Example 7.

Exercises





1-10 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. Sketch the region, the solid, and a typical approximating cylinder.

- **I.**  $y = x^2$ , x = 1, y = 0; about the *x*-axis
- **2.**  $y = e^x$ , y = 0, x = 0, x = 1; about the x-axis
- **3.**  $y = x^2$ , y = 4, x = 0, x = 2; about the y-axis
- **4.**  $x = y y^2$ , x = 0; about the y-axis
- **5.**  $y = x^2$ ,  $y^2 = x$ ; about the *x*-axis
- 6.  $y = \cos x$ ,  $y = \sin x$ , x = 0,  $x = \pi/4$ ; about the x-axis
- 7.  $y^2 = x$ , x = 2y; about the y-axis
- **CAS** 8.  $y = e^x$ , y = 1, x = 1; about the y-axis
  - **9.**  $y = x^4$ , y = 1; about y = 2

- **10.** y = x, y = 0, x = 2, x = 4; about x = 1
- **11.** The region enclosed by the curves x = 4y and  $y = \sqrt[3]{x}$  in the first quadrant is rotated about the line x = 8. Find the volume of the resulting solid.
- 12. Find the volume of the solid obtained by rotating the region in Exercise 11 about the line y = 2.
- CAS 13-14 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

**13.** 
$$y = x^2$$
,  $y = \ln(x + 1)$ 

14.  $y = 3\sin(x^2)$ ,  $y = e^{x/2} + e^{-2x}$ 

**15.** A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about

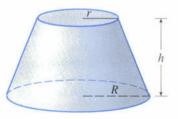
the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 1.5 cm apart. The liver is 15 cm long and the cross-sectional areas, in square centimeters, are 0, 18, 58, 79, 94, 106, 117, 128, 63, 39, and 0. Use Simpson's Rule to estimate the volume of the liver.

16. A log 10 m long is cut at 1-meter intervals and its crosssectional areas A (at a distance x from the end of the log) are listed in the table. Use the Midpoint Rule with n = 5 to estimate the volume of the log.

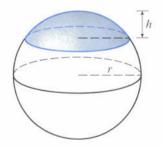
x (m)	$A_{\rm c}({\rm m}^2)$	<i>x</i> (m)	A (m <sup>2</sup> )
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0,48
5	0.59		

**17–29** ■ Find the volume of the described solid *S*.

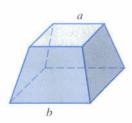
- 17. A right circular cone with height h and base radius r
- **18.** A frustum of a right circular cone with height h, lower base radius R, and top radius r



**19.** A cap of a sphere with radius r and height h



**20.** A frustum of a pyramid with square base of side *b*, square top of side *a*, and height *h* 



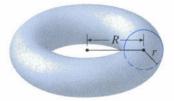
- **21.** A pyramid with height h and rectangular base with dimensions b and 2b
- **22.** A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)



- 23. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm
- **24.** The base of *S* is a circular disk with radius *r*. Parallel cross-sections perpendicular to the base are squares.
- **25.** The base of S is an elliptical region with boundary curve  $9x^2 + 4y^2 = 36$ . Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.
- 26. The base of S is the parabolic region  $\{(x, y) \mid x^2 \le y \le 1\}$ . Cross-sections perpendicular to the y-axis are equilateral triangles.
- **27.** *S* has the same base as in Exercise 26, but crosssections perpendicular to the *y*-axis are squares.
- **28.** The base of S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles.
- **29.** *S* has the same base as in Exercise 28, but crosssections perpendicular to the *x*-axis are isosceles triangles with height equal to the base.
- 30. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.(a) Set up an integral for the volume of S.
  - (b) By interpreting the integral as an area, find the

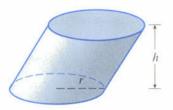
volume of S.

- **31.** (a) Set up an integral for the volume of a solid torus (the donut-shaped solid shown in the figure) with radii *r* and *R*.
  - (b) By interpreting the integral as an area, find the volume of the torus.



### 464 CHAPTER 6 APPLICATIONS OF INTEGRATION

- **32.** A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.
- 33. (a) Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S1 and S2, then the volumes of S1 and S2 are equal. Prove this principle.
  - (b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.



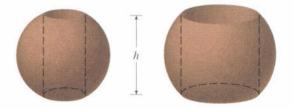
**34.** Find the volume common to two circular cylinders, each with radius *r*, if the axes of the cylinders intersect at right angles.



- **35.** Find the volume common to two spheres, each with radius *r*, if the center of each sphere lies on the surface of the other sphere.
- **36.** A bowl is shaped like a hemisphere with diameter 30 cm. A ball with diameter 10 cm is placed in the

bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

- **37.** A hole of radius *r* is bored through a cylinder of radius R > r at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume cut out.
- **38.** A hole of radius r is bored through the center of a sphere of radius R > r. Find the volume of the remaining portion of the sphere.
- **39.** Let S be the solid obtained by rotating about the y-axis the region bounded by  $y = x(x 1)^2$  and y = 0. Explain why it is awkward to use slicing to find the volume V of S. Then find V using cylindrical shells.
- **40.** Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by y = x and  $y = x^2$ . Find V both by slicing and by cylindrical shells. In both cases draw a diagram to explain your method.
- **41.** Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by  $y = x x^2$  and y = 0 about the line x = 2. Sketch the region and a typical shell. Explain why this method is preferable to slicing.
- **42.** Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both rings have the same height *h*, as shown in the figure.
  - (a) Guess which ring has more wood in it.
  - (b) Check your guess: Use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius *r* through the center of a sphere of radius *R* and express the answer in terms of *h*.





# Arc Length



FIGURE I

What do we mean by the length of a curve? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve.

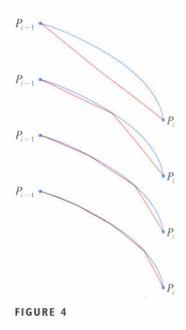
It's easy to find the length of a polygon; we just add the lengths of the line segments that form the polygon. We are going to define the length of a curve by first approximating it by a polygon and then taking a limit. This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

Suppose that a curve C is described by the parametric equations

x = f(t) y = g(t)  $a \le t \le b$ 

 $\begin{array}{c} & & \\$ 





Let's assume that *C* is **smooth** in the sense that the derivatives f'(t) and g'(t) are continuous and not simultaneously zero for a < t < b. (This ensures that *C* has no sudden change in direction.) We divide the parameter interval [a, b] into *n* subintervals of equal width  $\Delta t$ . If  $t_0, t_1, t_2, \ldots, t_n$  are the endpoints of these subintervals, then  $x_i = f(t_i)$  and  $y_i = g(t_i)$  are the coordinates of points  $P_i(x_i, y_i)$  that lie on *C* and the polygon with vertices  $P_0, P_1, \ldots, P_n$  approximates *C* (see Figure 3). The length *L* of *C* is approximately the length of this polygon and the approximation gets better as we let *n* increase (see Figure 4). Therefore, we define the **length** of *C* to be the limit of the lengths of these inscribed polygons:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume. We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as  $n \to \infty$ .

For computational purposes we need a more convenient expression for *L*. If we let  $\Delta x_i = x_i - x_{i-1}$  and  $\Delta y_i = y_i - y_{i-1}$ , then the length of the *i*th line segment of the polygon is

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

But from the definition of a derivative we know that

$$f'(t_i) \approx \frac{\Delta x_i}{\Delta t}$$

if  $\Delta t$  is small. (We could have used any sample point  $t_i^*$  in place of  $t_i$ .) Therefore

217 X 4

$$\Delta x_i \approx f'(t_i) \Delta t \qquad \Delta y_i \approx g'(t_i) \Delta t$$
$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$
$$\approx \sqrt{[f'(t_i) \Delta t]^2 + [g'(t_i) \Delta t]^2}$$
$$= \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2} \Delta t$$
$$L \approx \sum_{i=1}^n \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2} \Delta t$$

Thus

and so

This is a Riemann sum for the function  $\sqrt{[f'(t)]^2 + [g'(t)]^2}$  and so our argument suggests that

$$L = \int_{a}^{b} \sqrt{[f'(t)]^{2} + [g'(t)]^{2}} dt$$

In fact, our reasoning can be made precise; this formula is correct, provided that we rule out situations where a portion of the curve is traced out more than once.

Arc Length Formula If a smooth curve with parametric equations  $x = f(t), y = g(t), a \le t \le b$ , is traversed exactly once as t increases from a to b, then its length is

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^{2} + \left(\frac{dy}{dt}\right)^{2}} dt$$

**EXAMPLE 1** Find the length of the arc of the curve  $x = t^2$ ,  $y = t^3$  that lies between the points (1, 1) and (4, 8) (see Figure 5).

**SOLUTION** First we notice from the equations  $x = t^2$  and  $y = t^3$  that the portion of the curve between (1, 1) and (4, 8) corresponds to the parameter interval  $1 \le t \le 2$ . So the arc length formula (1) gives

$$L = \int_{1}^{2} \sqrt{\left(\frac{dx}{dt}\right)^{2} + \left(\frac{dy}{dt}\right)^{2}} dt = \int_{1}^{2} \sqrt{(2t)^{2} + (3t^{2})^{2}} dt$$
$$= \int_{1}^{2} \sqrt{4t^{2} + 9t^{4}} dt = \int_{1}^{2} t \sqrt{4 + 9t^{2}} dt$$

As a check on our answer to Example I. If we substitute  $u = 4 + 9t^2$ , then du = 18t dt. When t = 1, u = 13; when t = 2, u = 40. Therefore

$$L = \frac{1}{18} \int_{13}^{40} \sqrt{u} \, du = \frac{1}{18} \cdot \frac{2}{3} u^{3/2} \Big]_{13}^{40}$$
$$= \frac{1}{27} \Big[ 40^{3/2} - 13^{3/2} \Big] = \frac{1}{27} \Big( 80 \sqrt{10} - 13 \sqrt{13} \Big)$$

If we are given a curve with equation y = f(x),  $a \le x \le b$ , then we can regard x as a parameter. Then parametric equations are x = x, y = f(x), and Formula 1 becomes

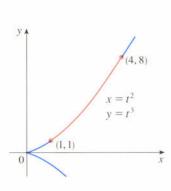
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$$L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Similarly, if a curve has the equation x = f(y),  $a \le y \le b$ , we regard y as the parameter and the length is

 $L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dy}\right)^{2} + 1} \, dy$ 3

Because of the presence of the square root sign in Formulas 1, 2, and 3, the calculation of an arc length often leads to an integral that is very difficult or even impossible to evaluate explicitly. Thus, we often have to be content with finding an approximation to the length of a curve as in the following example.





notice from Figure 5 that it ought to be slightly larger than the distance from (1,1)to (4, 8), which is

 $\sqrt{58} \approx 7.615773$ According to our calculation in Example I, we have

 $L = \frac{1}{27} (80\sqrt{10} - 13\sqrt{13}) \approx 7.633705$ Sure enough, this is a bit greater than

the length of the line segment.

**EXAMPLE 2** Estimate the length of the portion of the hyperbola xy = 1 from the point (1, 1) to the point  $(2, \frac{1}{2})$ .

**SOLUTION** We have

$$y = \frac{1}{x} \qquad \qquad \frac{dy}{dx} = -\frac{1}{x^2}$$

and so, from Formula 2, the length is

$$L = \int_{1}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^{2}} \, dx = \int_{1}^{2} \sqrt{1 + \frac{1}{x^{4}}} \, dx$$

It is impossible to evaluate this integral exactly, so let's use Simpson's Rule (see Section 5.8) with a = 1, b = 2, n = 10,  $\Delta x = 0.1$ , and  $f(x) = \sqrt{1 + 1/x^4}$ . Thus

$$L = \int_{1}^{2} \sqrt{1 + \frac{1}{x^{4}}} dx$$
  

$$\approx \frac{\Delta x}{3} [f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \dots + 2f(1.8) + 4f(1.9) + f(2)]$$
  

$$\approx 1.1321$$

Checking the value of the definite integral with a more accurate approximation produced by a computer algebra system, we see that the approximation using Simpson's Rule is accurate to four decimal places.

**EXAMPLE 3** Find the length of the arc of the parabola  $y^2 = x$  from (0,0) to (1,1).

**SOLUTION** Since  $x = y^2$ , we have dx/dy = 2y, and Formula 3 gives

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy = \int_0^1 \sqrt{4y^2 + 1} \, dy$$

Using either a computer algebra system or the Table of Integrals (use Formula 21 after substituting u = 2y), we find that

$$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4}$$

Figure 6 shows the arc of the parabola whose length is computed in Example 3, together with polygonal approximations having n = 1 and n = 2 line segments, respectively. For n = 1 the approximate length is  $L_1 = \sqrt{2}$ , the diagonal of a square. The table shows the approximations  $L_n$  that we get by dividing [0, 1] into n equal subintervals. Notice that each time we double the number of sides of the polygon, we get closer to the exact length, which is

$$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4} \approx 1.478943$$

n	$L_n$
1	1.414
2	1.445
4	1.464
8	1.472
16	1.476
32	1.478
64	1.479

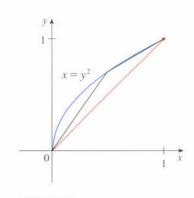


FIGURE 6

**EXAMPLE 4** Find the length of one arch of the cycloid  $x = r(\theta - \sin \theta)$ ,  $y = r(1 - \cos \theta).$ 

**SOLUTION** From Example 6 in Section 1.4 we see that one arch is described by the parameter interval  $0 \le \theta \le 2\pi$ . Since

$$\frac{dx}{d\theta} = r(1 - \cos \theta)$$
 and  $\frac{dy}{d\theta} = r \sin \theta$ 

we have

$$L = 8r$$

$$C = 2\pi r$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \ d\theta = \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta} \ d\theta$$
$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} \ d\theta = r \int_0^{2\pi} \sqrt{2(1 - \cos\theta)} \ d\theta$$

This integral could be evaluated after using further trigonometric identities. Instead we use a computer algebra system:

$$L = r \int_0^{2\pi} \sqrt{2(1 - \cos \theta)} \, d\theta = 8r$$



FIGURE 7

1. Use the arc length formula (2) to find the length of the curve y = 2x + 1,  $-1 \le x \le 3$ . Check your answer by noting that the curve is a line segment and calculating

its length by the distance formula. 2. (a) In Example 2 in Section 1.4 we showed that the parametric equations  $x = \cos t$ ,  $y = \sin t$ ,  $0 \le t \le 2\pi$ , represent the unit circle. Use these equations to show that the length of the unit circle has the expected value.

(b) In Example 3 in Section 1.4 we showed that the equations  $x = \sin 2t$ ,  $y = \cos 2t$ ,  $0 \le t \le 2\pi$ , also represent the unit circle. What value does the integral in Formula 1 give? How do you explain the discrepancy?

**3–6** Graph the curve and find its exact length.

**3.** 
$$x = e^t \cos t$$
,  $y = e^t \sin t$ ,  $0 \le t \le \pi$ 

**4.** 
$$x = 3t - t^3$$
,  $y = 3t^2$ ,  $0 \le t \le 2$ 

**5.** 
$$x = y^{3/2}, \quad 0 \le y \le 1$$

**6.**  $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta),$  $0 \leq \theta \leq \pi$ 

**7-9** Use Simpson's Rule with n = 10 to estimate the length of the curve.

**7.**  $y = x^3$ ,  $0 \le x \le 1$ 

**8.**  $y = \tan x$ ,  $0 \le x \le \pi/4$ 9.  $y = \sin x$ ,  $0 \le x \le \pi$ 

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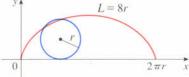
- 10. In Exercise 31 in Section 1.4 you were asked to derive the parametric equations  $x = 2a \cot \theta$ ,  $y = 2a \sin^2 \theta$  for the curve called the witch of Maria Agnesi. Use Simpson's Rule with n = 4 to estimate the length of the arc of this curve given by the parameter interval  $\pi/4 \leq \theta \leq \pi/2.$
- **11.** (a) Graph the curve  $y = x\sqrt[3]{4-x}$ ,  $0 \le x \le 4$ .
  - (b) Compute the lengths of inscribed polygons with n = 1, 2, and 4 sides. (Divide the interval into equal subintervals.) Illustrate by sketching these polygons (as in Figure 6).
    - (c) Set up an integral for the length of the curve.
    - (d) If your calculator (or CAS) evaluates definite integrals, use it to find the length of the curve to four decimal places. If not, use Simpson's Rule. Compare with the approximations in part (b).

**12.** Repeat Exercise 11 for the curve

$$y = x + \sin x$$
  $0 \le x \le 2\pi$ 

**CAS** 13–15 Use either a CAS or a table of integrals to find the exact length of the curve.

**13.** 
$$x = t^3$$
,  $y = t^4$ ,  $0 \le t \le 1$ 



Exercises ·

The result of Example 4 says that the length of one arch of a cycloid is eight times the radius of the generating circle (see Figure 7). This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's

Cathedral in London.

14. 
$$x = \ln(1 - y^2), \quad 0 \le y \le \frac{1}{2}$$

- **15.**  $y = \ln(\cos x), \quad 0 \le x \le \pi/4$
- **CAS** 16. Use either a computer algebra system or a table of integrals to find the exact length of the arc of the exponential curve  $y = e^x$  that lies between the points (0, 1) and (1, e). If your CAS has trouble evaluating the integral, make a substitution that changes the integral into one that the CAS can evaluate.
  - 17. A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation  $y = \sin(\pi x/7)$  and find the width *w* of a flat metal sheet that is needed to make a 28-inch panel. (If your calculator or CAS evaluates definite integrals, use it. Otherwise, use Simpson's Rule.)



- **18.** Find the total length of the astroid  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$ .
- **19.** Show that the total length of the ellipse  $x = a \sin \theta$ ,  $y = b \cos \theta$ , a > b > 0, is

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \ d\theta$$

where *e* is the eccentricity of the ellipse (e = c/a,where  $c = \sqrt{a^2 - b^2}$ ).

**20.** The curves with equations  $x^n + y^n = 1$ , n = 4, 6, 8, ..., are called **fat circles**. Graph the curves with n = 2, 4, 6, 8, and 10 to see why. Set up an integral for the length  $L_{2k}$  of the fat circle with n = 2k. Without attempting to evaluate this integral, state the value of

$$\lim_{k\to\infty}L_2$$

**CAS 21.** (a) Graph the epitrochoid with equations

$$x = 11\cos t - 4\cos(11t/2)$$

 $y = 11 \sin t - 4 \sin(11t/2)$ 

What parameter interval gives the complete curve?

- (b) Use your CAS to find the approximate length of this curve.
- **CAS 22.** A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du$$
$$y = S(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$$

where C and S are the Fresnel functions that were introduced in Section 5.4.

- (a) Graph this curve. What happens as  $t \to \infty$  and as  $t \to -\infty$ ?
- (b) Find the length of Cornu's spiral from the origin to the point with parameter value *t*.



## Average Value of a Function

It is easy to calculate the average value of finitely many numbers  $y_1, y_2, \ldots, y_n$ :

$$y_{\rm ave} = \frac{y_1 + y_2 + \dots + y_n}{n}$$

 $\begin{array}{c} T \\ 15 \\ 10 \\ 5 \\ \hline 0 \\ 12 \\ 18 \\ 24 \end{array}$ 

But how do we compute the average temperature during a day if infinitely many temperature readings are possible? Figure 1 shows the graph of a temperature function T(t) (where t is measured in hours, T in °C) and a guess at the average temperature,  $T_{ave}$ .

In general, let's try to compute the average value of a function y = f(x),  $a \le x \le b$ . We start by dividing the interval [a, b] into *n* equal subintervals, each with length  $\Delta x = (b - a)/n$ . Then we choose points  $x_1^*, \ldots, x_n^*$  in successive subintervals and calculate the average of the numbers  $f(x_1^*), \ldots, f(x_n^*)$ :

$$\frac{f(x_1^*) + \dots + f(x_n^*)}{n}$$

FIGURE I

(For example, if f represents a temperature function and n = 24, this means that we take temperature readings every hour and then average them.) Because  $\Delta x = (b - a)/n$ , we can write  $n = (b - a)/\Delta x$  and the average value becomes

$$\frac{f(x_1^*) + \dots + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a} \left[ f(x_1^*) \Delta x + \dots + f(x_n^*) \Delta x \right]$$
$$= \frac{1}{b-a} \sum_{i=1}^n f(x_i^*) \Delta x$$

If we let *n* increase, we would be computing the average value of a large number of closely spaced values. (For example, we would be averaging temperature readings taken every minute or even every second.) The limiting value is

$$\lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} f(x_i^*) \Delta x = \frac{1}{b - a} \int_a^b f(x) \, dx$$

by the definition of a definite integral.

Therefore, we define the average value of f on the interval [a, b] as

$$f_{\rm ave} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

**EXAMPLE I** Find the average value of the function  $f(x) = 1 + x^2$  on the interval [-1, 2].

**SOLUTION** With a = -1 and b = 2 we have

$$f_{\text{ave}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx = \frac{1}{2-(-1)} \int_{-1}^{2} (1+x^2) \, dx$$
$$= \frac{1}{3} \left[ x + \frac{x^3}{3} \right]_{-1}^{2} = 2$$

The question arises: Is there a number c at which the value of f is exactly equal to the average value of the function, that is,  $f(c) = f_{ave}$ ? The following theorem says that this is true for continuous functions.

**Mean Value Theorem for Integrals** If f is continuous on [a, b], then there exists a number c in [a, b] such that

$$\int_a^b f(x) \, dx = f(c) \, (b - a)$$

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus. The proof is outlined in Exercise 17. The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base [a, b] and height f(c) has the same area as the region under the graph of f from a to b (see Figure 2 and the more picturesque interpretation in the margin note).

You can always chop off the top of a (two-dimensional) mountain at a certain height and use it to fill in the valleys so that the mountain becomes completely flat.

#### FIGURE 2

(2, 5)

**EXAMPLE 2** Since  $f(x) = 1 + x^2$  is continuous on the interval [-1, 2], the Mean Value Theorem for Integrals says there is a number c in [-1, 2] such that

$$\int_{-1}^{2} (1 + x^2) \, dx = f(c) \left[ 2 - (-1) \right]$$

In this particular case we can find *c* explicitly. From Example 1 we know that  $f_{ave} = 2$ , so the value of *c* satisfies

$$f(c) = f_{ave} = 2$$
  
1 +  $c^2 = 2$  so  $c^2 = 1$ 

Therefore

Thus, in this case there happen to be two numbers  $c = \pm 1$  in the interval [-1, 2] that work in the Mean Value Theorem for Integrals.

Examples 1 and 2 are illustrated by Figure 3.

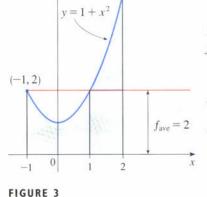
**EXAMPLE 3** Show that the average velocity of a car over a time interval  $[t_1, t_2]$  is the same as the average of its velocities during the trip.

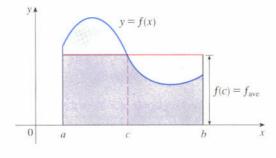
**SOLUTION** If s(t) is the displacement of the car at time t, then, by definition, the average velocity of the car over the interval is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

On the other hand, the average value of the velocity function on the interval is

$$v_{\text{ave}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t) \, dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t) \, dt$$
$$= \frac{1}{t_2 - t_1} [s(t_2) - s(t_1)] \quad \text{(by the Total Change Theorem)}$$
$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{average velocity}$$





Exercises · ·

6.4

1-4 Find the average value of f on the given interval.

1.  $f(x) = x^2 - 2x$ , [0,3]2.  $f(x) = \sin x$ ,  $[0,\pi]$ 3.  $f(x) = e^x$ , [0,2]4. f(x) = 1/x, [1,4]

#### 5-8 🔳

(a) Find the average value of f on the given interval.

(b) Find c such that  $f_{ave} = f(c)$ .

(c) Sketch the graph of *f* and a rectangle whose area is the same as the area under the graph of *f*.

**5.** 
$$f(x) = 4 - x^2$$
, [0, 2]

**6.** 
$$f(x) = 4x - x^2$$
, [0,3]

**7.**  $f(x) = x^3 - x + 1$ , [0,2]

**8.** 
$$f(x) = x \sin(x^2), [0, \sqrt{\pi}]$$

- **9.** If f is continuous and  $\int_{1}^{3} f(x) dx = 8$ , show that f takes on the value 4 at least once on the interval [1, 3].
- 10. Find the numbers b such that the average value of  $f(x) = 2 + 6x 3x^2$  on the interval [0, b] is equal to 3.
- **11.** In a certain city the temperature (in  $^{\circ}$ F) *t* hours after 9 A.M. was approximated by the function

$$T(t) = 50 + 14\sin\frac{\pi t}{12}$$

Find the average temperature during the period from 9  $_{A.M.}$  to 9  $_{P.M.}$ 

**12.** The temperature of a metal rod, 5 m long, is 4x (in °C) at a distance *x* meters from one end of the rod. What is the average temperature of the rod?

- 13. The linear density in a rod 8 m long is  $12/\sqrt{x+1}$  kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.
- 14. If a freely falling body starts from rest, then its displacement is given by  $s = \frac{1}{2}gt^2$ . Let the velocity after a time *T* be  $v_T$ . Show that if we compute the average of the velocities with respect to *t* we get  $v_{ave} = \frac{1}{2}v_T$ , but if we compute the average of the velocities with respect to *s* we get  $v_{ave} = \frac{2}{3}v_T$ .
- **15.** Use the result of Exercise 65 in Section 5.5 to compute the average volume of inhaled air in the lungs in one respiratory cycle.
- 16. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l} \left(R^2 - r^2\right)$$

where *P* is the pressure difference between the ends of the vessel and  $\eta$  is the viscosity of the blood (see Example 7 in Section 3.3). Find the average velocity (with respect to *r*) over the interval  $0 \le r \le R$ . Compare the average velocity with the maximum velocity.

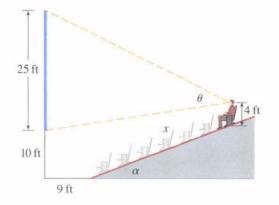
- 17. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives (see Section 4.3) to the function  $F(x) = \int_a^x f(t) dt$ .
- **18.** If  $f_{ave}[a, b]$  denotes the average value of f on the interval [a, b] and a < c < b, show that

$$f_{\text{ave}}[a,b] = \frac{c-a}{b-a} f_{\text{ave}}[a,c] + \frac{b-c}{b-a} f_{\text{ave}}[c,b]$$

# Applied Project

## Where to Sit at the Movies

A movie theater has a screen that is positioned 10 ft off the floor and is 25 ft high. The first row of seats is placed 9 ft from the screen and the rows are set 3 ft apart. The floor of the seating area is inclined at an angle of  $\alpha = 20^{\circ}$  above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so  $0 \le x \le 60$ . Suppose you decide that the best place to sit is in the row where the angle  $\theta$  subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 4 ft above the floor, as shown in the figure. (In Exercise 32 in Section 4.6 we



looked at a simpler version of this problem, where the floor is horizontal, but this project involves a more complicated situation and requires technology.)

I. Show that

 $\theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$  $a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$ 

where

and

 $b^{2} = (9 + x \cos \alpha)^{2} + (x \sin \alpha - 6)^{2}$ 

- **2.** Use a graph of  $\theta$  as a function of x to estimate the value of x that maximizes  $\theta$ . In which row should you sit? What is the viewing angle  $\theta$  in this row?
- **3.** Use your computer algebra system to differentiate  $\theta$  and find a numerical value for the root of the equation  $d\theta/dx = 0$ . Does this value confirm your result in Problem 2?
- **4.** Use the graph of  $\theta$  to estimate the average value of  $\theta$  on the interval  $0 \le x \le 60$ . Then use your CAS to compute the average value. Compare with the maximum and minimum values of  $\theta$ .



# Applications to Physics and Engineering

As a consequence of a calculation of work, you will be able to compute the velocity needed for a rocket to escape Earth's gravitational field. (See Exercise 18.) Among the many applications of integral calculus to physics and engineering, we consider three: work, force due to water pressure, and centers of mass. As with our previous applications to geometry (areas, volumes, and lengths), our strategy is to break up the physical quantity into a large number of small parts, approximate each small part, add the results, take the limit, and then evaluate the result-ing integral.



The term *work* is used in everyday language to mean the total amount of effort required to perform a task. In physics it has a technical meaning that depends on the idea of a *force*. Intuitively, you can think of a force as describing a push or pull

on an object—for example, a horizontal push of a book across a table or the downward pull of Earth's gravity on a ball. In general, if an object moves along a straight line with position function s(t), then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration:

$$F = m \frac{d^2 s}{dt^2}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ( $N = kg \cdot m/s^2$ ). Thus, a force of 1 N acting on a mass of 1 kg produces an acceleration of 1  $m/s^2$ . In the U.S. Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves:

W = Fd work = force × distance

If *F* is measured in newtons and *d* in meters, then the unit for *W* is a newton-meter, which is called a joule (J). If *F* is measured in pounds and *d* in feet, then the unit for *W* is a foot-pound (ft-lb), which is about 1.36 J.

For instance, suppose you lift a 1.2-kg book off the floor to put it on a desk that is 0.7 m high. The force you exert is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8) = 11.76 \,\mathrm{N}$$

and then Equation 2 gives the work done as

2

$$W = Fd = (11.76)(0.7) \approx 8.2 \text{ J}$$

But if a 20-lb weight is lifted 6 ft off the ground, then the force is given as F = 20 lb, so the work done is

$$W = Fd = 20 \cdot 6 = 120 \, \text{ft-lb}$$

Here we didn't multiply by g because we were given the *weight* (a force) and not the mass.

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the x-axis in the positive direction, from x = a to x = b, and at each point x between a and b a force f(x) acts on the object, where f is a continuous function. We divide the interval [a, b] into n subintervals with endpoints  $x_0, x_1, \ldots, x_n$  and equal width  $\Delta x$ . We choose a sample point  $x_i^*$  in the *i*th subinterval  $[x_{i-1}, x_i]$ . Then the force at that point is  $f(x_i^*)$ . If n is large, then  $\Delta x$  is small, and since f is continuous, the values of f don't change very much over the interval  $[x_{i-1}, x_i]$ . In other words, f is almost constant on the interval and so the work  $W_i$  that is done in moving the particle from  $x_{i-1}$  to  $x_i$  is approximately given by Equation 2:

$$W_i \approx f(x_i^*) \Delta x$$

Thus, we can approximate the total work by

3

$$W \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

It seems that this approximation becomes better as we make *n* larger. Therefore, we define the **work done in moving the object from** *a* **to** *b* as the limit of this quantity as  $n \rightarrow \infty$ . Since the right side of (3) is a Riemann sum, we recognize its limit as being a definite integral and so

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \int_a^b f(x) \, dx$$

**EXAMPLE 1** When a particle is located at a distance x feet from the origin, a force of  $x^2 + 2x$  pounds acts on it. How much work is done in moving it from x = 1 to x = 3?

SOLUTION 
$$W = \int_{1}^{3} (x^{2} + 2x) dx = \frac{x^{3}}{3} + x^{2} \Big|_{1}^{3} = \frac{50}{3}$$

The work done is  $16\frac{2}{3}$  ft-lb.

In the next example we use a law from physics: Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant (called the **spring constant**). Hooke's Law holds provided that x is not too large (see Figure 1).

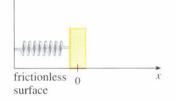
**EXAMPLE 2** A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

**SOLUTION** According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is f(x) = kx. When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that f(0.05) = 40, so

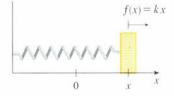
$$0.05k = 40$$
  $k = \frac{40}{0.05} = 800$ 

Thus, f(x) = 800x and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x \, dx = 800 \frac{x^2}{2} \bigg]_{0.05}^{0.08}$$
$$= 400[(0.08)^2 - (0.05)^2] = 1.56 \text{ J}$$

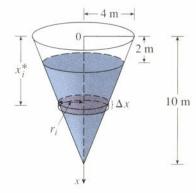


(a) Natural position of spring



(b) Stretched position of spring

FIGURE I Hooke's Law





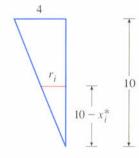


FIGURE 3

**SOLUTION** Let's measure depths from the top of the tank by introducing a vertical coordinate line as in Figure 2. The water extends from a depth of 2 m to a depth of 10 m and so we divide the interval [2, 10] into *n* subintervals with endpoints  $x_0, x_1, \ldots, x_n$  and choose  $x_i^*$  in the *i*th subinterval. This divides the water into *n* layers. The *i*th layer is approximated by a circular cylinder with radius  $r_i$  and height  $\Delta x$ . We can compute  $r_i$  from similar triangles, using Figure 3, as follows:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \qquad r_i = \frac{2}{5}(10 - x_i^*)$$

Thus, an approximation to the volume of the *i*th layer of water is

$$V_i \approx \pi r_i^2 \Delta x = \frac{4\pi}{25} (10 - x_i^*)^2 \Delta x$$

and so its mass is

 $m_i = \text{density} \times \text{volume}$ 

$$\approx 1000 \cdot \frac{4\pi}{25} (10 - x_i^*)^2 \Delta x = 160\pi (10 - x_i^*)^2 \Delta x$$

The force required to raise this layer must overcome the force of gravity and so

$$F_i = m_i g \approx (9.8)160\pi (10 - x_i^*)^2 \Delta x$$
  
$$\approx 1570\pi (10 - x_i^*)^2 \Delta x$$

Each particle in the layer must travel a distance of approximately  $x_i^*$ . The work  $W_i$  done to raise this layer to the top is approximately the product of the force  $F_i$  and the distance  $x_i^*$ :

$$W_i \approx F_i x_i^* \approx 1570 \pi x_i^* (10 - x_i^*)^2 \Delta x$$

To find the total work done in emptying the entire tank, we add the contributions of each of the *n* layers and then take the limit as  $n \to \infty$ :

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 1570 \, \pi x_i^* (10 - x_i^*)^2 \, \Delta x$$
  
=  $\int_2^{10} 1570 \, \pi x (10 - x)^2 \, dx$   
=  $1570 \, \pi \int_2^{10} (100x - 20x^2 + x^3) \, dx$   
=  $1570 \, \pi \bigg[ 50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \bigg]_2^{10}$   
=  $1570 \, \pi \Big( \frac{2048}{3} \Big) \approx 3.4 \times 10^6 \, \mathrm{J}$ 

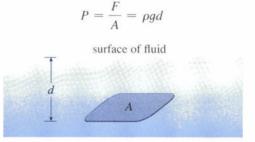
#### Hydrostatic Pressure and Force

Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density  $\rho$  kilograms per cubic meter at a depth d meters below the surface of the fluid as in Figure 4. The fluid directly above the plate has volume V = Ad, so its mass is  $m = \rho V = \rho Ad$ . The force exerted by the fluid on the plate is therefore

$$F = mg = \rho gAd$$

where g is the acceleration due to gravity. The pressure P on the plate is defined to be the force per unit area:



#### FIGURE 4

4

The SI unit for measuring pressure is newtons per square meter, which is called a pascal (abbreviation:  $1 \text{ N/m}^2 = 1 \text{ Pa}$ ). Since this is a small unit, the kilopascal (kPa) is often used. For instance, because the density of water is  $\rho = 1000 \text{ kg/m}^3$ , the pressure at the bottom of a swimming pool 2 m deep is

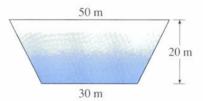
$$P = \rho g d = 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2 \text{ m}$$
  
= 19,600 Pa = 19.6 kPa

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions*. (A diver feels the same pressure on nose and both ears.) Thus, the pressure in *any* direction at a depth *d* in a fluid with mass density  $\rho$  is given by

$$P = \rho g d = \delta d$$

This helps us determine the hydrostatic force against a *vertical* plate or wall or dam in a fluid. This is not a straightforward problem, because the pressure is not constant but increases as the depth increases.

**EXAMPLE 4** • A dam has the shape of the trapezoid shown in Figure 5. The height is 20 m, and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.



When using U.S. Customary units, we write  $P = \rho g d = \delta d$ , where  $\delta = \rho g$  is the weight density (as opposed to  $\rho$ , which is the mass density). For instance, the weight density of water is  $\delta = 62.5 \text{ lb/ft}^3$ .

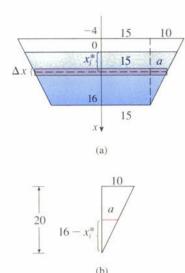


FIGURE 6

**SOLUTION** We choose a vertical *x*-axis with origin at the surface of the water as in Figure 6(a). The depth of the water is 16 m, so we divide the interval [0, 16] into subintervals with endpoints  $x_i$  and we choose  $x_i^* \in [x_{i-1}, x_i]$ . The *i*th horizontal strip of the dam is approximated by a rectangle with height  $\Delta x$  and width  $w_i$ , where, from similar triangles in Figure 6(b),

$$\frac{a}{16 - x_i^*} = \frac{10}{20} \qquad a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$
$$w_i = 2(15 + a) = 2\left(15 + 8 - \frac{x_i^*}{2}\right) = 46 - x_i^*$$

and so

If 
$$A_i$$
 is the area of the *i*th strip, then

$$A_i \approx w_i \,\Delta x = (46 - x_i^*) \,\Delta x$$

If  $\Delta x$  is small, then the pressure  $P_i$  on the *i*th strip is almost constant and we can use Equation 4 to write

$$P_i \approx 1000 g x_i^*$$

The hydrostatic force  $F_i$  acting on the *i*th strip is the product of the pressure and the area:

$$F_i = P_i A_i \approx 1000 g x_i^* (46 - x_i^*) \Delta x$$

Adding these forces and taking the limit as  $n \to \infty$ , we obtain the total hydrostatic force on the dam:

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 1000 gx_i^* (46 - x_i^*) \Delta x$$
  
=  $\int_0^{16} 1000 gx (46 - x) dx$   
=  $1000(9.8) \int_0^{16} (46x - x^2) dx$   
=  $9800 \left[ 23x^2 - \frac{x^3}{3} \right]_0^{16}$   
 $\approx 4.43 \times 10^7 \text{ N}$ 

### Moments and Centers of Mass

5

Our main objective is to find the point P on which a thin plate of any given shape balances horizontally as in Figure 7. This point is called the **center of mass** (or center of gravity) of the plate.

We first consider the simpler situation illustrated in Figure 8, where two masses  $m_1$  and  $m_2$  are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances  $d_1$  and  $d_2$  from the fulcrum. The rod will balance if

$$m_1d_1 = m_2d_2$$

This is an experimental fact discovered by Archimedes and called the Law of the

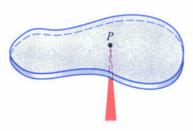


FIGURE 7

Lever. (Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

Now suppose that the rod lies along the x-axis with  $m_1$  at  $x_1$  and  $m_2$  at  $x_2$  and the center of mass at  $\bar{x}$ . If we compare Figures 8 and 9, we see that  $d_1 = \bar{x} - x_1$  and  $d_2 = x_2 - \bar{x}$  and so Equation 5 gives

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$
$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$
$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

The numbers  $m_1x_1$  and  $m_2x_2$  are called the **moments** of the masses  $m_1$  and  $m_2$  (with respect to the origin), and Equation 6 says that the center of mass  $\bar{x}$  is obtained by adding the moments of the masses and dividing by the total mass  $m = m_1 + m_2$ .



FIGURE 9

In general, if we have a system of *n* particles with masses  $m_1, m_2, \ldots, m_n$  located at the points  $x_1, x_2, \ldots, x_n$  on the *x*-axis, it can be shown similarly that the center of mass of the system is located at

n n

7

8

6

$$\overline{x} = \frac{\sum\limits_{i=1}^{n} m_i x_i}{\sum\limits_{i=1}^{n} m_i} = \frac{\sum\limits_{i=1}^{n} m_i x_i}{m}$$

where  $m = \sum m_i$  is the total mass of the system, and the sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$

is called the moment of the system with respect to the origin. Then Equation 7 could be rewritten as  $m\bar{x} = M$ , which says that if the total mass were considered as being concentrated at the center of mass  $\bar{x}$ , then its moment would be the same as the moment of the system.

Now we consider a system of *n* particles with masses  $m_1, m_2, \ldots, m_n$  located at the points  $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$  in the *xy*-plane as shown in Figure 10. By analogy with the one-dimensional case, we define the **moment of the system about the y-axis** to be

$$M_y = \sum_{i=1}^n m_i x_i$$

and the moment of the system about the x-axis as

 $M_x = \sum_{i=1}^n m_i y_i$ 

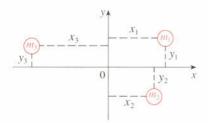


FIGURE 10

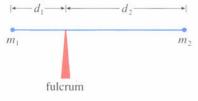


FIGURE 8

Then  $M_y$  measures the tendency of the system to rotate about the y-axis and  $M_x$  measures the tendency to rotate about the x-axis.

As in the one-dimensional case, the coordinates  $(\bar{x}, \bar{y})$  of the center of mass are given in terms of the moments by the formulas

$$\overline{x} = \frac{M_y}{m} \qquad \overline{y} = \frac{M_x}{m}$$

where  $m = \sum m_i$  is the total mass. Since  $m\bar{x} = M_y$  and  $m\bar{y} = M_x$ , the center of mass  $(\bar{x}, \bar{y})$  is the point where a single particle of mass m would have the same moments as the system.

**EXAMPLE 5** Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (-1, 1), (2, -1), and (3, 2).

**SOLUTION** We use Equations 8 and 9 to compute the moments:

$$M_y = 3(-1) + 4(2) + 8(3) = 29$$

 $M_x = 3(1) + 4(-1) + 8(2) = 15$ 

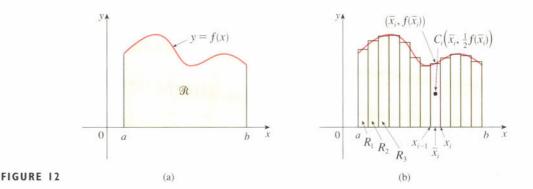
Since m = 3 + 4 + 8 = 15, we use Equations 10 to obtain

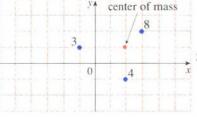
$$\bar{x} = \frac{M_y}{m} = \frac{29}{15}$$
  $\bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$ 

Thus, the center of mass is  $(1\frac{14}{15}, 1)$  (see Figure 11).

Next we consider a flat plate (called a *lamina*) with uniform density  $\rho$  that occupies a region  $\mathcal{R}$  of the plane. We wish to locate the center of mass of the plate, which is called the **centroid** of  $\mathcal{R}$ . In doing so we use the following physical principles: The **symmetry principle** says that if  $\mathcal{R}$  is symmetric about a line *l*, then the centroid of  $\mathcal{R}$  lies on *l*. (If  $\mathcal{R}$  is reflected about *l*, then  $\mathcal{R}$  remains the same so its centroid remains fixed. But the only fixed points lie on *l*.) Thus, the centroid of a rectangle is its center. Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged. Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region  $\Re$  is of the type shown in Figure 12(a); that is,  $\Re$  lies between the lines x = a and x = b, above the *x*-axis, and beneath the graph of *f*,







where f is a continuous function. We divide the interval [a, b] into n subintervals with endpoints  $x_0, x_1, \ldots, x_n$  and equal width  $\Delta x$ . We choose the sample point  $x_i^*$ to be the midpoint  $\overline{x}_i$  of the *i*th subinterval, that is,  $\overline{x}_i = (x_{i-1} + x_i)/2$ . This determines the polygonal approximation to  $\Re$  shown in Figure 12(b). The centroid of the *i*th approximating rectangle  $R_i$  is its center  $C_i(\overline{x}_i, \frac{1}{2}f(\overline{x}_i))$ . Its area is  $f(\overline{x}_i) \Delta x$ , so its mass is

$$\rho f(\overline{x}_i) \Delta x$$

The moment of  $R_i$  about the y-axis is the product of its mass and the distance from  $C_i$  to the y-axis, which is  $\bar{x}_i$ . Thus

$$M_{\rm v}(R_i) = \left[\rho f(\bar{x}_i) \,\Delta x\right] \bar{x}_i = \rho \bar{x}_i f(\bar{x}_i) \,\Delta x$$

Adding these moments, we obtain the moment of the polygonal approximation to  $\mathcal{R}$ , and then by taking the limit as  $n \to \infty$  we obtain the moment of  $\mathcal{R}$  itself about the *y*-axis:

$$M_{y} = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \bar{x}_{i} f(\bar{x}_{i}) \Delta x = \rho \int_{a}^{b} x f(x) dx$$

In a similar fashion we compute the moment of  $R_i$  about the x-axis as the product of its mass and the distance from  $C_i$  to the x-axis:

$$M_x(R_i) = \left[\rho f(\overline{x}_i) \Delta x\right] \frac{1}{2} f(\overline{x}_i) = \rho \cdot \frac{1}{2} \left[ f(\overline{x}_i) \right]^2 \Delta x$$

Again we add these moments and take the limit to obtain the moment of  $\mathcal{R}$  about the *x*-axis:

$$M_{x} = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2} [f(\bar{x}_{i})]^{2} \Delta x = \rho \int_{a}^{b} \frac{1}{2} [f(x)]^{2} dx$$

Just as for systems of particles, the center of mass of the plate is defined so that  $m\bar{x} = M_y$  and  $m\bar{y} = M_x$ . But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x) \, dx$$

and so

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b xf(x) \, dx}{\rho \int_a^b f(x) \, dx} = \frac{\int_a^b xf(x) \, dx}{\int_a^b f(x) \, dx}$$
$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2} [f(x)]^2 \, dx}{\rho \int_a^b f(x) \, dx} = \frac{\int_a^b \frac{1}{2} [f(x)]^2 \, dx}{\int_a^b f(x) \, dx}$$

Notice the cancellation of the  $\rho$ 's. The location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of  $\Re$ ) is located at the point  $(\bar{x}, \bar{y})$ , where

$$\overline{x} = \frac{1}{A} \int_a^b x f(x) \, dx \qquad \overline{y} = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)]^2 \, dx$$

**EXAMPLE 6** Find the center of mass of a semicircular plate of radius r.

 $y = \sqrt{r^2 - x^2}$   $(0, \frac{4r}{3\pi})$  -r 0 r x

**SOLUTION** In order to use (11) we place the semicircle as in Figure 13 so that  $f(x) = \sqrt{r^2 - x^2}$  and a = -r, b = r. Here there is no need to use the formula to calculate  $\bar{x}$  because, by the symmetry principle, the center of mass must lie on the y-axis, so  $\bar{x} = 0$ . The area of the semicircle is  $A = \pi r^2/2$ , so

$$\overline{y} = \frac{1}{A} \int_{-r}^{r} \frac{1}{2} [f(x)]^2 dx$$

$$= \frac{1}{\pi r^2 / 2} \cdot \frac{1}{2} \int_{-r}^{r} (\sqrt{r^2 - x^2})^2 dx$$

$$= \frac{2}{\pi r^2} \int_{0}^{r} (r^2 - x^2) dx = \frac{2}{\pi r^2} \left[ r^2 x - \frac{x^3}{3} \right]_{0}^{r}$$

$$= \frac{2}{\pi r^2} \frac{2r^3}{3} = \frac{4r}{3\pi}$$

The center of mass is located at the point  $(0, 4r/(3\pi))$ .



Exercises

- 1. A particle is moved along the x-axis by a force that measures  $5x^2 + 1$  pounds at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance of 10 ft.
- When a particle is located at a distance x meters from the origin, a force of cos(πx/3) newtons acts on it. How much work is done in moving the particle from x = 1 to x = 2? Interpret your answer by considering the work done from x = 1 to x = 1.5 and from x = 1.5 to x = 2.
- **3.** A force of 10 lb is required to hold a spring stretched 4 in. beyond its natural length. How much work is done in stretching it from its natural length to 6 in. beyond its natural length?
- **4.** A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

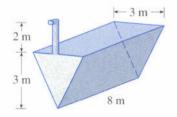
- Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.
   (a) How much work is needed to stretch it from 35 cm
  - to 40 cm?(b) How far beyond its natural length will a force of 30 N keep the spring stretched?
- **6.** If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

**7–12** Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it.

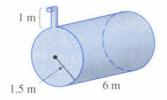
**7.** A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high. How much work is done in pulling the rope to the top of the building?



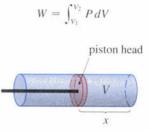
- **8.** A uniform cable hanging over the edge of a tall building is 40 ft long and weighs 60 lb. How much work is required to pull 10 ft of the cable to the top?
- **9.** A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mineshaft 500 ft deep. Find the work done.
- 10. A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket starts with 40 lb of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done in pulling the bucket to the top of the well.
- 11. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m<sup>3</sup>.)
- 12. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 62.5 lb/ft<sup>3</sup>.)
- **13.** The tank shown is full of water.
  - (a) Find the work required to pump the water out of the outlet.
- (b) Suppose that the pump breaks down after  $4.7 \times 10^5$  J of work has been done. What is the depth of the water remaining in the tank?



14. The tank shown is half full of oil that has a density of 920 kg/m<sup>3</sup>. Find the work required to pump the oil out of the outlet.



**15.** When gas expands in a cylinder with radius *r*, the pressure at any given time is a function of the volume: P = P(V). The force exerted by the gas on the piston (see the figure) is the product of the pressure and the area:  $F = \pi r^2 P$ . Show that the work done by the gas when the volume expands from volume  $V_1$  to volume  $V_2$  is



- 16. In a steam engine the pressure P and volume V of steam satisfy the equation  $PV^{1.4} = k$ , where k is a constant. (This is true for adiabatic expansion, that is, expansion in which there is no heat transfer between the cylinder and its surroundings.) Use Exercise 15 to calculate the work done by the engine during a cycle when the steam starts at a pressure of 160 lb/in<sup>2</sup> and a volume of 100 in<sup>3</sup> and expands to a volume of 800 in<sup>3</sup>.
- **17.** (a) Newton's Law of Gravitation states that two bodies with masses  $m_1$  and  $m_2$  attract each other with a force

$$F = G \, \frac{m_1 m_2}{r^2}$$

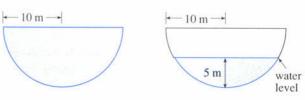
where r is the distance between the bodies and G is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from r = a to r = b.

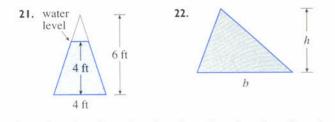
- (b) Compute the work required to launch a 1000-kg satellite vertically to an orbit 1000 km high. You may assume that Earth's mass is  $5.98 \times 10^{24}$  kg and is concentrated at its center. Take the radius of Earth to be  $6.37 \times 10^6$  m and  $G = 6.67 \times 10^{-11}$  N·m<sup>2</sup>/kg<sup>2</sup>.
- 18. (a) Use an improper integral and information from Exercise 17 to find the work needed to propel a 1000-kg satellite out of Earth's gravitational field.
  - (b) Find the *escape velocity*  $v_0$  that is needed to propel a rocket of mass *m* out of the gravitational field of a planet with mass *M* and radius *R*. (Use the fact that the initial kinetic energy of  $\frac{1}{2}mv_0^2$  supplies the needed work.)

**19–22** The end of a tank containing water is vertical and has the indicated shape. Explain how to approximate the hydrostatic force against the end of the tank by a Riemann sum. Then express the force as an integral and evaluate it.

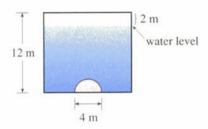
20.







- 23. A swimming pool is 20 ft wide and 40 ft long and its bottom is an inclined plane, the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, find the hydrostatic force on (a) the shallow end, (b) the deep end, (c) one of the sides, and (d) the bottom of the pool.
- **24.** A vertical dam has a semicircular gate as shown in the figure. Find the hydrostatic force against the gate.



**25–26** The masses  $m_i$  are located at the points  $P_i$ . Find the moments  $M_x$  and  $M_y$  and the center of mass of the system.

**25.** 
$$m_1 = 4$$
,  $m_2 = 8$ ;  $P_1(-1, 2)$ ,  $P_2(2, 4)$   
**26.**  $m_1 = 3$ ,  $m_2 = 3$ ,  $m_3 = 8$ ,  $m_4 = 6$ ;

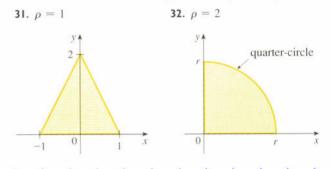
$$P_1(0,0), P_2(1,8), P_3(3,-4), P_4(-6,-5)$$

**27–30** Find the centroid of the region bounded by the given curves. Sketch the region and plot the centroid to see if your answer is reasonable.

**27.**  $y = e^x$ , y = 0, x = 0, x = 1

**28.** 
$$y = 1 - x^2$$
,  $y = 0$   
**29.**  $y = \cos 2x$ ,  $y = 0$ ,  $x = -\pi/4$ ,  $x = \pi/4$   
**30.**  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 4$ 

**31–32** Calculate the moments  $M_x$  and  $M_y$  and the center of mass of a lamina with the given density and shape.



$$\bar{x} = \frac{1}{A} \int_{a}^{b} x[f(x) - g(x)] dx$$
$$\bar{y} = \frac{1}{A} \int_{a}^{b} \frac{1}{2} \{[f(x)]^{2} - [g(x)]^{2}\} dx$$

- (b) Find the centroid of the region bounded by the line y = x and the parabola y = x<sup>2</sup>.
- **34.** Let  $\mathcal{R}$  be the region that lies between the curves  $y = x^m$  and  $y = x^n$ ,  $0 \le x \le 1$ , where *m* and *n* are integers with  $0 \le n < m$ .
  - (a) Sketch the region  $\mathcal{R}$ .
  - (b) Find the coordinates of the centroid of  $\mathcal{R}$ .
  - (c) Try to find values of m and n such that the centroid lies *outside*  $\mathcal{R}$ .



## Applications to Economics and Biology

In this section we consider some applications of integration to economics (consumer surplus) and biology (blood flow, cardiac output). Others are found in the exercises.



### **Consumer Surplus**

Recall from Section 4.7 that the demand function p(x) is the price that a company has to charge in order to sell x units of a commodity. Usually, selling larger quantities requires lowering prices, so the demand function is a decreasing function.

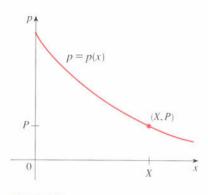
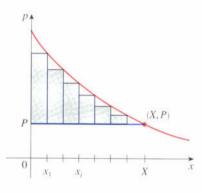
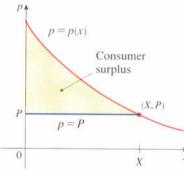


FIGURE I A typical demand curve



1







The graph of a typical demand function, called a **demand curve**, is shown in Figure 1. If X is the amount of the commodity that is currently available, then P = p(X) is the current selling price.

We divide the interval [0, X] into *n* subintervals, each of length  $\Delta x = X/n$ , and let  $x_i^* = x_i$  be the right endpoint of the *i*th subinterval, as in Figure 2. If, after the first  $x_{i-1}$  units were sold, a total of only  $x_i$  units had been available and the price per unit had been set at  $p(x_i)$  dollars, then the additional  $\Delta x$  units could have been sold (but no more). The consumers who would have paid  $p(x_i)$  dollars placed a high value on the product; they would have paid what it was worth to them. So, in paying only *P* dollars they have saved an amount of

(savings per unit) (number of units) =  $[p(x_i) - P]\Delta x$ 

Considering similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

$$\sum_{i=1}^{n} \left[ p(x_i) - P \right] \Delta x$$

If we let  $n \to \infty$ , this Riemann sum approaches the integral

$$\int_0^X \left[ p(x) - P \right] dx$$

which economists call the consumer surplus for the commodity.

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X. Figure 3 shows the interpretation of the consumer surplus as the area under the demand curve and above the line p = P.

**EXAMPLE I** The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2$$

Find the consumer surplus when the sales level is 500.

**SOLUTION** Since the number of products sold is X = 500, the corresponding price is

$$P = 1200 - (0.2)(500) - (0.0001)(500)^2 = 1075$$

x Therefore, from Definition 1, the consumer surplus is

$$\int_{0}^{500} [p(x) - P] dx = \int_{0}^{500} (1200 - 0.2x - 0.0001x^{2} - 1075) dx$$
$$= \int_{0}^{500} (125 - 0.2x - 0.0001x^{2}) dx$$
$$= 125x - 0.1x^{2} - (0.0001) \left(\frac{x^{3}}{3}\right) \Big]_{0}^{500}$$
$$= (125) (500) - (0.1) (500)^{2} - \frac{(0.0001) (500)^{3}}{3}$$
$$= \$33.333.33$$

### Blood Flow

In Example 7 in Section 3.3 we discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l} \left( R^2 - r^2 \right)$$

which gives the velocity v of blood that flows along a blood vessel with radius R and length l at a distance r from the central axis, where P is the pressure difference between the ends of the vessel and  $\eta$  is the viscosity of the blood. Now, in order to compute the flux (volume per unit time) we consider smaller, equally spaced radii  $r_1, r_2, \ldots$  The approximate area of the annulus with inner radius  $r_{i-1}$  and outer radius  $r_i$  is

$$2\pi r_i \Delta r$$
 where  $\Delta r = r_i - r_{i-1}$ 

(See Figure 4.) If  $\Delta r$  is small, then the velocity is almost constant throughout this annulus and can be approximated by  $v(r_i)$ . Thus, the volume of blood per unit time that flows across the annulus is approximately

$$(2\pi r_i \,\Delta r) \,v(r_i) = 2\pi r_i \,v(r_i) \,\Delta r$$

and the total volume of blood that flows across a cross-section per unit time is approximately

$$\sum_{i=1}^{n} 2\pi r_i v(r_i) \ \Delta r$$

This approximation is illustrated in Figure 5. Notice that the velocity (and hence the volume per unit time) increases toward the center of the blood vessel. The approximation gets better as n increases. When we take the limit we get the exact value of the *flux* (or *discharge*), which is the volume of blood that passes a cross-section per unit time:

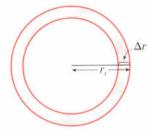
$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi r_{i} v(r_{i}) \Delta r$$
  
=  $\int_{0}^{R} 2\pi r v(r) dr$   
=  $\int_{0}^{R} 2\pi r \frac{P}{4\eta l} (R^{2} - r^{2}) dr$   
=  $\frac{\pi P}{2\eta l} \int_{0}^{R} (R^{2}r - r^{3}) dr = \frac{\pi P}{2\eta l} \left[ R^{2} \frac{r^{2}}{2} - \frac{r^{4}}{4} \right]_{r=0}^{r=R}$   
=  $\frac{\pi P}{2\eta l} \left[ \frac{R^{4}}{2} - \frac{R^{4}}{4} \right] = \frac{\pi P R^{4}}{8\eta l}$ 

The resulting equation

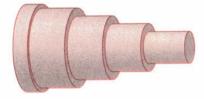
2

$$F = \frac{\pi PR}{8nl}$$

is called **Poiseuille's Law**; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.









#### Cardiac Output

Figure 6 shows the human cardiovascular system. Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation. It then flows back into the left atrium through the pulmonary veins and then out to the rest of the body through the aorta. The **cardiac output** of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

The dye dilution method is used to measure the cardiac output. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval [0, T] until the dye has cleared. Let c(t) be the concentration of the dye at time t. If we partition [0, T] into subintervals of equal length  $\Delta t$ , then the amount of dye that flows past the measuring point during the subinterval from  $t = t_{i-1}$  to  $t = t_i$  is approximately

(concentration) (volume) =  $c(t_i)(F \Delta t)$ 

where F is the rate of flow that we are trying to determine. Thus, the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i) F \, \Delta t = F \sum_{i=1}^{n} c(t_i) \, \Delta t$$

and, letting  $n \to \infty$ , we find that the amount of dye is

$$A = F \int_0^T c(t) \, dt$$

Thus, the cardiac output is given by

$$F = \frac{A}{\int_0^T c(t) \, dt}$$

where the amount of dye *A* is known and the integral can be approximated from the concentration readings.

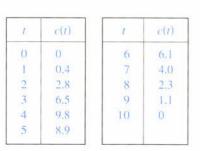
**EXAMPLE 2** • A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the chart. Estimate the cardiac output.

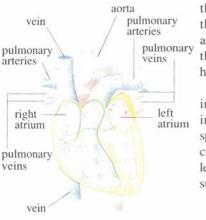
**SOLUTION** Here A = 5,  $\Delta t = 1$ , and T = 10. We use Simpson's Rule to approximate the integral of the concentration:

$$\int_{0}^{10} c(t) dt \approx \frac{1}{3} [0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9)]$$

$$+ 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0$$

$$\approx 41.87$$







Exercises

Thus, Formula 3 gives the cardiac output to be

$$F = \frac{A}{\int_0^{10} c(t) dt} \approx \frac{5}{41.87}$$
  
\$\approx 0.12 L/s = 7.2 L/min

6.6

- 1. The marginal cost function C'(x) was defined to be the derivative of the cost function. (See Sections 3.3 and 4.7.) If the marginal cost of manufacturing x units of a product is  $C'(x) = 0.006x^2 1.5x + 8$  (measured in dollars per unit) and the fixed start-up cost is C(0) = \$1,500,000, use the Total Change Theorem to find the cost of producing the first 2000 units.
- The marginal revenue from selling x items is 90 - 0.02x. The revenue from the sale of the first 100 items is \$8800. What is the revenue from the sale of the first 200 items?
- **3.** The marginal cost of producing x units of a certain product is  $140 0.5x + 0.012x^2$  (in dollars per unit). Find the increase in cost if the production level is raised from 3000 units to 5000 units.
- 4. The demand function for a certain commodity is p = 5 x/10. Find the consumer surplus when the sales level is 30. Illustrate by drawing the demand curve and identifying the consumer surplus as an area.
- 5. A demand curve is given by p = 1000/(x + 20). Find the consumer surplus when the selling price is \$20.
- 6. The supply function  $p_s(x)$  for a commodity gives the relation between the selling price and the number of units that manufacturers will produce at that price. For a higher price, manufacturers will produce more units, so  $p_s$  is an increasing function of x. Let X be the amount of the commodity currently produced and let  $P = p_s(X)$  be the current price. Some producers would be willing to make and sell the commodity for a lower selling price and are therefore receiving more than their minimal price. The excess is called the **producer surplus**. An argument similar to that for consumer surplus shows that the surplus is given by the integral

$$\int_0^x \left[ P - p_s(x) \right] dx$$

Calculate the producer surplus for the supply function  $p_S(x) = 3 + 0.01x^2$  at the sales level X = 10. Illustrate by drawing the supply curve and identifying the producer surplus as an area.

- 7. A supply curve is given by  $p = 5 + \frac{1}{10}\sqrt{x}$ . Find the producer surplus when the selling price is \$10.
- 8. For a given commodity and pure competition, the number of units produced and the price per unit are determined as the coordinates of the point of intersection of the supply and demand curves. Given the demand curve p = 50 x/20 and the supply curve p = 20 + x/10, find the consumer surplus and the producer surplus. Illustrate by sketching the supply and demand curves and identifying the surpluses as areas.
- **9.** A manufacturer has been selling 1000 television sets a week at \$450 each. A market survey indicates that for every \$10 that the price is reduced, the number of sets sold will increase by 100 a week. Find the demand function and calculate the consumer surplus when the selling price is set at \$400.
- 10. If the amount of capital that a company has at time t is f(t), then the derivative, f'(t), is called the *net investment flow*. Suppose that the net investment flow is  $\sqrt{t}$  million dollars per year (where t is measured in years). Find the increase in capital (the capital formation) from the fourth year to the eighth year.
- 11. Use Poiseuille's Law to calculate the rate of flow in a small human artery where we can take  $\eta = 0.027$ , R = 0.008 cm, l = 2 cm, and P = 4000 dynes/cm<sup>2</sup>.
- 12. High blood pressure results from constriction of the arteries. To maintain a normal flow rate (flux), the heart has to pump harder, thus increasing the blood pressure. Use Poiseuille's Law to show that if  $R_0$  and  $P_0$  are normal values of the radius and pressure in an artery and the constricted values are R and P, then for the flux to remain constant, P and R are related by the equation

$$\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4$$

Deduce that if the radius of an artery is reduced to three-fourths of its former value, then the pressure is more than tripled.

#### SECTION 6.7 PROBABILITY # 489

c(t)

2.8

1.4

0.6

0.2

0

1

12

14

16

18

20

- 13. The dye dilution method is used to measure cardiac output with 8 mg of dye. The dye concentrations, in mg/L, are modeled by  $c(t) = \frac{1}{4}t(12 t), 0 \le t \le 12$ , where t is measured in seconds. Find the cardiac output.
- 14. After a 6-mg injection of dye, the readings of dye concentrations at two-second intervals are as shown in the table at the right. Use Simpson's Rule to estimate the cardiac output.



Calculus plays a role in the analysis of random behavior. Suppose we consider the cholesterol level of a person chosen at random from a certain age group, or the height of an adult female chosen at random, or the lifetime of a randomly chosen battery of a certain type. Statisticians call such quantities **continuous random variables** because their values actually range over an interval of real numbers, although they might be measured or recorded only to the nearest integer. We might want to know the probability that a blood cholesterol level is greater than 250, or the probability that the height of an adult female lies between 60 and 70 inches, or the probability that the battery we are buying lasts between 100 and 200 hours. If *X* represents the lifetime of that type of battery, we denote this last probability as follows:

t

0

2

4

6

8

10

0

2.1

4.5

7.3

5.8

3.6

$$P(100 \le X \le 200)$$

According to the frequency interpretation of probability, this number is the longrun proportion of all batteries of the specified type whose lifetimes lie between 100 and 200 hours. Since it represents a proportion, the probability naturally falls between 0 and 1.

Every continuous random variable X has a **probability density function** f. This means that the probability that X lies between a and b is found by integrating f from a to b:

П

$$P(a \le X \le b) = \int_a^b f(x) \, dx$$

For example, Figure 1 shows the graph of a model of the probability density function f for a random variable X defined to be the height in inches of an adult female

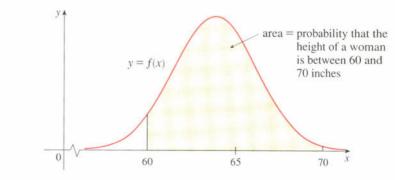


FIGURE I Probability density function for the height of an adult female

in the United States (according to data from the National Health Survey). The probability that the height of a woman chosen at random from this population is between 60 and 70 inches is equal to the area under the graph of f from 60 to 70.

In general, the probability density function f of a random variable X satisfies  $f(x) \ge 0$  for all x. Because probabilities are measured on a scale from 0 to 1, it follows that

$$\int_{-\infty}^{\infty} f(x) \, dx = 1$$

**EXAMPLE 1** Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

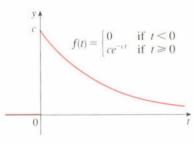
**SOLUTION** Think of the random variable as being the time you wait on hold before an agent of a company you're telephoning answers your call. So instead of x, let's use t to represent time, in minutes. If f is the probability density function and you call at time t = 0, then, from Definition 1,  $\int_0^2 f(t) dt$  represents the probability that an agent answers within the first two minutes and  $\int_4^5 f(t) dt$  is the probability that your call is answered during the fifth minute.

It's clear that f(t) = 0 for t < 0 (the agent can't answer before you place the call). For t > 0 we are told to use an exponentially decreasing function, that is, a function of the form  $f(t) = Ae^{-ct}$ , where A and c are positive constants. Thus

$$f(t) = \begin{cases} 0 & \text{if } t < 0\\ Ae^{-ct} & \text{if } t \ge 0 \end{cases}$$

We use Condition 2 to determine the value of A:

$$1 = \int_{-\infty}^{\infty} f(t) dt = \int_{-\infty}^{0} f(t) dt + \int_{0}^{\infty} f(t) dt$$
$$= \int_{0}^{\infty} Ae^{-ct} dt = \lim_{x \to \infty} \int_{0}^{x} Ae^{-ct} dt$$
$$= \lim_{x \to \infty} \left[ -\frac{A}{c} e^{-ct} \right]_{0}^{x} = \lim_{x \to \infty} \frac{A}{c} (1 - e^{-cx})$$
$$= \frac{A}{c}$$



Therefore, A/c = 1 and so A = c. Thus, every exponential density function has the form

$$S(t) = \begin{cases} 0 & \text{if } t < 0\\ ce^{-ct} & \text{if } t \ge 0 \end{cases}$$

f

A typical graph is shown in Figure 2.

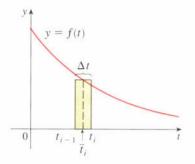
FIGURE 2 An exponential density function



#### **Average Values**

Suppose you're waiting for a company to answer your phone call and you wonder how long, on the average, you could expect to wait. Let f(t) be the corresponding density function, where t is measured in minutes, and think of a sample of N people

.





who have called this company. Most likely, none of them had to wait more than an hour, so let's restrict our attention to the interval  $0 \le t \le 60$ . Let's divide that interval into *n* intervals of length  $\Delta t$  and endpoints 0,  $t_1, t_2, \ldots$  (Think of  $\Delta t$  as lasting a minute, or half a minute, or 10 seconds, or even a second.) The probability that somebody's call gets answered during the time period from  $t_{i-1}$  to  $t_i$  is the area under the curve y = f(t) from  $t_{i-1}$  to  $t_i$ , which is approximately equal to  $f(\bar{t}_i) \Delta t$ . (This is the area of the approximating rectangle in Figure 3, where  $\bar{t}_i$  is the midpoint of the interval.)

Since the long-run proportion of calls that get answered in the time period from  $t_{i-1}$  to  $t_i$  is  $f(\bar{t}_i) \Delta t$ , we expect that, out of our sample of N callers, the number whose call was answered in that time period is approximately  $Nf(\bar{t}_i) \Delta t$  and the time that each waited is about  $\bar{t}_i$ . Therefore, the total time they waited is the product of these numbers: approximately  $\bar{t}_i [Nf(\bar{t}_i) \Delta t]$ . Adding over all such intervals, we get the approximate total of everybody's waiting times:

$$\sum_{i=1}^n N\,\bar{t}_i f(\bar{t}_i)\,\,\Delta t$$

If we now divide by the number of callers *N*, we get the approximate *average* waiting time:

$$\sum_{i=1}^{n} \bar{t}_i f(\bar{t}_i) \ \Delta t$$

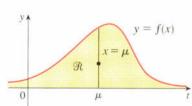
We recognize this as a Riemann sum for the function tf(t). As the time interval shrinks (that is,  $\Delta t \rightarrow 0$  and  $n \rightarrow \infty$ ), this Riemann sum approaches the integral

$$\int_0^{60} tf(t) \, dt$$

This integral is called the mean waiting time.

In general, the **mean** of any probability density function f is defined to be

It is traditional to denote the mean by the Greek letter  $\mu$  (mu).



**FIGURE 4**  $\mathscr{R}$  balances at a point on the line  $x = \mu$ 

$$\mu = \int_{-\infty}^{\infty} x f(x) \, dx$$

The mean can be interpreted as the long-run average value of the random variable X. It can also be interpreted as a measure of centrality of the probability density function.

The expression for the mean resembles an integral we have seen before. If  $\mathcal{R}$  is the region that lies under the graph of f, we know from Formula 11 in Section 6.5 that the *x*-coordinate of the centroid of  $\mathcal{R}$  is

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xf(x) \, dx}{\int_{-\infty}^{\infty} f(x) \, dx} = \int_{-\infty}^{\infty} xf(x) \, dx = \mu$$

because of Equation 2. So a thin plate in the shape of  $\Re$  balances at a point on the vertical line  $x = \mu$  (see Figure 4).

**EXAMPLE 2** Find the mean of the exponential distribution of Example 1:

$$f(t) = \begin{cases} 0 & \text{if } t < 0\\ ce^{-ct} & \text{if } t \ge 0 \end{cases}$$

SOLUTION According to the definition of a mean, we have

$$\mu = \int_{-\infty}^{\infty} tf(t) \, dt = \int_{0}^{\infty} tc e^{-ct} \, dt$$

To evaluate this integral we use integration by parts, with u = t and  $dv = ce^{-ct} dt$ :

$$\int_0^\infty tce^{-ct} dt = \lim_{x \to \infty} \int_0^x tce^{-ct} dt$$
$$= \lim_{x \to \infty} \left( -te^{-ct} \right]_0^x + \int_0^x e^{-ct} dt \right)$$
$$= \lim_{x \to \infty} \left( -xe^{-cx} + \frac{1}{c} - \frac{e^{-cx}}{c} \right)$$
$$= \frac{1}{c}$$

The limit of the first term is 0 by l'Hospital's Rule.

The mean is  $\mu = 1/c$ , so we can rewrite the probability density function as

$$f(t) = \begin{cases} 0 & \text{if } t < 0\\ \mu^{-1} e^{-t/\mu} & \text{if } t \ge 0 \end{cases}$$

**EXAMPLE 3** Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.

(a) Find the probability that a call is answered during the first minute.

(b) Find the probability that a customer waits more than five minutes to be answered.

#### SOLUTION

(a) We are given that the mean of the exponential distribution is  $\mu = 5$  min and so, from the result of Example 2, we know that the probability density function is

$$f(t) = \begin{cases} 0 & \text{if } t < 0\\ 0.2e^{-t/5} & \text{if } t \ge 0 \end{cases}$$

Thus, the probability that a call is answered during the first minute is

$$P(0 \le T \le 1) = \int_0^1 f(t) dt = \int_0^1 0.2e^{-t/5} dt$$
$$= 0.2(-5)e^{-t/5} \Big]_0^1 = 1 - e^{-1/5} \approx 0.1813$$

So about 18% of customers' calls are answered during the first minute.

(b) The probability that a customer waits more than five minutes is

$$P(T > 5) = \int_{5}^{\infty} f(t) dt = \int_{5}^{\infty} 0.2e^{-t/5} dt$$
$$= \lim_{x \to \infty} \int_{5}^{x} 0.2e^{-t/5} dt = \lim_{x \to \infty} (e^{-1} - e^{-x/5})$$
$$= \frac{1}{e} \approx 0.368$$

About 37% of customers wait more than five minutes before their calls are answered.

Notice the result of Example 3(b): Even though the mean waiting time is 5 minutes, only 37% of callers wait more than 5 minutes. The reason is that some callers have to wait much longer (maybe 10 or 15 minutes), and they bring up the average.

Another measure of centrality of a probability density function is the *median*. That is a number m such that half the callers have a waiting time less than m and the other callers have a waiting time longer than m. In general, the median of a probability density function is the number *m* such that

$$\int_{m}^{\infty} f(x) \, dx = \frac{1}{2}$$

This means that half the area under the graph of f lies to the right of m. In Exercise 5 you are asked to show that the median waiting time for the company described in Example 3 is approximately 3.5 minutes.

3

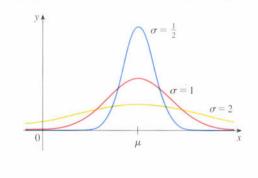
#### **Normal Distributions**

Many important random phenomena-such as test scores on aptitude tests, heights and weights of individuals from a homogeneous population, annual rainfall in a given location-are modeled by a normal distribution. That means that the probability density function of the random variable X is a member of the family of functions

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

You can verify that the mean for this function is  $\mu$ . The positive constant  $\sigma$  is called the standard deviation; it measures how spread out the values of X are. From the bell-shaped graphs of members of the family in Figure 5, we see that for





The standard deviation is denoted by

the lowercase Greek letter  $\sigma$  (sigma).

small values of  $\sigma$  the values of X are clustered about the mean, whereas for larger values of  $\sigma$  the values of X are more spread out. Statisticians have methods for using sets of data to estimate  $\mu$  and  $\sigma$ .

The factor  $1/(\sigma \sqrt{2\pi})$  is needed to make f a probability density function. In fact, it can be verified using the methods of multivariable calculus that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} dx = 1$$

**EXAMPLE 4** Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (Figure 6 shows the corresponding probability density function.)

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?

#### SOLUTION

(a) Since IQ scores are normally distributed, we use the probability density function given by Equation 3 with  $\mu = 100$  and  $\sigma = 15$ :

$$P(85 \le X \le 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2+15^2)} dx$$

Recall from Section 5.7 that the function  $y = e^{-x^2}$  doesn't have an elementary antiderivative, so we can't evaluate the integral exactly. But we can use the numerical integration capability of a calculator or computer (or the Midpoint Rule or Simpson's Rule) to estimate the integral. Doing so, we find that

$$P(85 \le X \le 115) \approx 0.68$$

So about 68% of the population has an IQ between 85 and 115, that is, within one standard deviation of the mean.

(b) The probability that the IQ score of a person chosen at random is more than 140 is

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450} dx$$

To avoid the improper integral we could approximate it by the integral from 140 to 200. (It's quite safe to say that people with an IQ over 200 are extremely rare.) Then

$$P(X > 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450} dx \approx 0.0038$$

Therefore, about 0.4% of the population has an IQ over 140.



Exercises

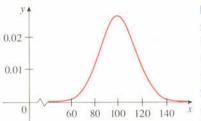
1. If *f*(*t*) is the probability density function for the lifetime of a type of battery, where *t* is measured in hours, what is the meaning of each integral?

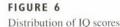
(a) 
$$\int_{100}^{200} f(t) dt$$
 (b)  $\int_{200}^{\infty} f(t) dt$ 

**2.** If f(x) is the probability density function for the blood cholesterol level of men over the age of 40, where x is

measured in milligrams per deciliter, express the following probabilities as integrals.

- (a) The probability that the cholesterol level of such a man lies between 180 and 240
- (b) The probability that the cholesterol level of such a man is less than 200
- **3.** A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the





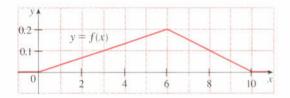
sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.

(a) Explain why the function

$$f(x) = \begin{cases} 0.1 & \text{if } 0 \le x \le 10\\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function for the spinner's values.

- (b) What does your intuition tell you about the value of the mean? Check your guess by evaluating an integral.
- **4.** (a) Explain why the function whose graph is shown is a probability density function.
  - (b) Use the graph to find the following probabilities.
    (i) P(X < 3)</li>
    (ii) P(3 ≤ X ≤ 8)
  - (c) Calculate the mean.



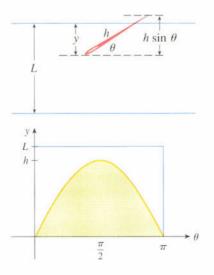
- Show that the median waiting time for a phone call to the company described in Example 3 is about 3.5 minutes.
- 6. (a) A type of lightbulb is labeled as having an average lifetime of 1000 hours. It's reasonable to model the probability of failure of these bulbs by an exponential density function with mean  $\mu = 1000$ . Use this model to find the probability that a bulb
  - (i) fails within the first 200 hours,
  - (ii) burns for more than 800 hours.
  - (b) What is the median lifetime of these lightbulbs?
- **7.** The manager of a fast-food restaurant determines that the average time that her customers wait for service is 2.5 minutes.
  - (a) Find the probability that a customer has to wait for more than 4 minutes.
  - (b) Find the probability that a customer is served within the first 2 minutes.
  - (c) The manager wants to advertise that anybody who isn't served within a certain number of minutes gets a free hamburger. But she doesn't want to give away free hamburgers to more than 2% of her customers. What should the advertisement say?
- **8.** According to the National Health Survey, the heights of adult males in the United States are normally distributed with mean 69.0 inches and standard deviation 2.8 inches.
  - (a) What is the probability that an adult male chosen at random is between 65 inches and 73 inches tall?

- (b) What percentage of the adult male population is more than 6 feet tall?
- **9.** The "Garbage Project" at the University of Arizona reports that the amount of paper discarded by house-holds per week is normally distributed with mean 9.4 lb and standard deviation 4.2 lb. What percentage of households throw out at least 10 lb of paper a week?
- 10. Boxes are labeled as containing 500 g of cereal. The machine filling the boxes produces weights that are normally distributed with standard deviation 12 g.
  - (a) If the target weight is 500 g, what is the probability that the machine produces a box with less than 480 g of cereal?
  - (b) Suppose a law states that no more than 5% of a manufacturer's cereal boxes can contain less than the stated weight of 500 g. At what target weight should the manufacturer set its filling machine?
- **11.** For any normal distribution, find the probability that the random variable lies within two standard deviations of the mean.
- 12. The standard deviation for a random variable with probability density function f and mean  $\mu$  is defined by

$$\sigma = \left[ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \right]^{1/2}$$

Find the standard deviation for an exponential density function with mean  $\mu$ .

13. In a famous 18th-century problem, known as *Buffon's* needle problem, a needle of length h is dropped onto a flat surface (for example, a table) on which parallel lines L units apart,  $L \ge h$ , have been drawn. The problem is to determine the probability that the needle will come to rest intersecting one of the lines. Assume that the lines run east-west, parallel to the x-axis in a rectangular coordinate system (as in the figure). Let y be



the distance from the "southern" end of the needle to the nearest line to the north. (If the needle's southern end lies on a line, let y = 0. If the needle happens to lie east-west, let the "western" end be the "southern" end.) Let  $\theta$  be the angle that the needle makes with a ray extending eastward from the "southern" end. Then  $0 \le y \le L$  and  $0 \le \theta < \pi$ . Note that the needle intersects one of the lines only when  $y < h \sin \theta$ . Now, the total set of possibilities for the needle can be identified with the rectangular region  $0 \le y \le L$ ,  $0 \le \theta < \pi$ , and the proportion of times that the needle intersects a line is the ratio

$$\frac{\text{area under } y = h \sin \theta}{\text{area of rectangle}}$$

This ratio is the probability that the needle intersects a line. Find the probability that the needle will intersect a line if h = L. What if h = L/2?

14. The hydrogen atom is composed of one proton in the nucleus and one electron, which moves about the nucleus. In the quantum theory of atomic structure, it is assumed that the electron does not move in a welldefined orbit. Instead, it occupies a state known as an *orbital*, which may be thought of as a "cloud" of negative charge surrounding the nucleus. At the state of lowest energy, called the *ground state*, or *ls-orbital*, the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability density function

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \qquad r \ge 0$$

where  $a_0$  is the *Bohr radius* ( $a_0 \approx 5.59 \times 10^{-11}$  m). The integral

$$P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds$$

gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.

- (a) Verify that p(r) is a probability density function.
- (b) Find lim<sub>r→∞</sub> p(r). For what value of r does p(r) have its maximum value?
- (c) Graph the density function.
- (d) Find the probability that the electron will be within the sphere of radius  $4a_0$  centered at the nucleus.
- (e) Calculate the mean distance of the electron from the nucleus in the ground state of the hydrogen atom.



### **Chapter 6 Review**

#### CONCEPT CHECK •

- (a) Draw two typical curves y = f(x) and y = g(x), where f(x) ≥ g(x) for a ≤ x ≤ b. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.
  - (b) Explain how the situation changes if the curves have equations x = f(y) and x = g(y), where f(y) ≥ g(y) for c ≤ y ≤ d.
- **2.** Suppose that Sue leads Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?
- **3.** Suppose *S* is a solid with known cross-sectional areas. Explain how to approximate the volume of *S* by a Riemann sum. Then write an expression for the exact volume.

- 4. (a) How is the length of a curve defined?
  - (b) Write an expression for the length of a smooth curve with parametric equations x = f(t), y = g(t), a ≤ t ≤ b.
  - (c) How does the expression in part (b) simplify if the curve is described by giving y terms of x, that is, y = f(x), a ≤ x ≤ b? What if x is given as a function of y?
- 5. (a) What is the average value of a function f on an interval [a, b]?
  - (b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?
- 6. Suppose that you push a book across a 6-meter-long table by exerting a force f(x) at each point from x = 0 to x = 6. What does \$\int\_0^6 f(x)\$ dx represent? If f(x) is measured in newtons, what are the units for the integral?

- **7.** (a) What is the physical significance of the center of mass of a thin plate?
  - (b) If the plate lies between y = f(x) and y = 0, where a ≤ x ≤ b, write expressions for the coordinates of the center of mass.
- 8. Given a demand function p(x), explain what is meant by the consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate with a sketch.
- 9. (a) What is the cardiac output of the heart?
  - (b) Explain how the cardiac output can be measured by the dye dilution method.
- 10. Suppose f(x) is the probability density function for the weight of female college students, where x is measured in pounds.
  - (a) What is the meaning of the integral  $\int_{0}^{100} f(x) dx$ ?
  - (b) Write an expression for the mean of this density function.

#### 🔶 EXERCISES 🧄

I-2 Find the area of the region bounded by the given curves.

1. 
$$y = x^2 - 6x$$
,  $y = 12x - 2x^2$ 

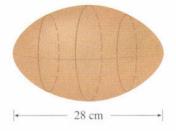
2. 
$$x - 2y + 7 = 0$$
,  $y^2 - 6y - x = 0$ 

- Let R be the region bounded by the curves y = tan(x<sup>2</sup>), x = 1, and y = 0. Use the Midpoint Rule with n = 4 to estimate the following:
  - (a) The area of  $\mathcal{R}$
  - (b) The volume obtained by rotating  $\mathcal{R}$  about the x-axis
- **4.** Let  $\Re$  be the region in the first quadrant bounded by the curves  $y = x^3$  and  $y = 2x x^2$ . Calculate the following quantities:
  - (a) The area of  $\mathcal{R}$
  - (b) The volume obtained by rotating  $\mathcal R$  about the *x*-axis
- 5. Find the volumes of the solids obtained by rotating the region bounded by the curves y = x and  $y = x^2$  about the following lines:
  - (a) The x-axis (b) The y-axis (c) y = 2
- 6. Let  $\Re$  be the region bounded by the curves  $y = 1 x^2$ and  $y = x^6 - x + 1$ . Estimate the following:
  - (a) The *x*-coordinates of the points of intersection of the curves
  - (b) The area of  $\mathcal{R}$
  - (c) The volume generated when  $\mathcal{R}$  is rotated about the *x*-axis
- 7. Use a graph to find the coordinates of the point where the curve with parametric equations

$$x = t^3 - 3t$$
  $y = t^2 + t + t$ 

intersects itself. Then find the area enclosed by the loop of this curve.

 Suppose you are asked to estimate the volume of a football. You measure and find that a football is 28 cm long. You use a piece of string and measure the circumference at its widest point to be 53 cm. The circumference 7 cm from each end is 45 cm. Use Simpson's Rule to make your estimate.



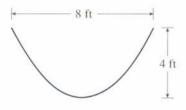
- **9.** The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.
- 10. The base of a solid is the region bounded by the parabolas  $y = x^2$  and  $y = 2 x^2$ . Find the volume of the solid if the cross-sections perpendicular to the *x*-axis are squares with one side lying along the base.
- **11.** The height of a monument is 20 m. A horizontal crosssection at a distance x meters from the top is an equilateral triangle with side x/4 meters. Find the volume of the monument.
- 12. (a) The base of a solid is a square with vertices at (1,0), (0,1), (-1,0), and (0, -1). Each cross-section perpendicular to the *x*-axis is a semicircle. Find the volume of the solid.
  - (b) Show that by cutting the solid of part (a), we can rearrange it to form a cone. Thus compute its volume more simply.
- **13.** Find the length of the curve with parametric equations  $x = 3t^2$ ,  $y = 2t^3$ ,  $0 \le t \le 2$ .
- 14. Use Simpson's Rule with n = 10 to estimate the length of the arc of the curve  $y = 1/x^2$  from (1, 1) to  $(2, \frac{1}{4})$ .

- **15.** A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?
- **16.** A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?
- **17.** A tank full of water has the shape of a paraboloid of revolution as in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
  - (a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.
- (b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?

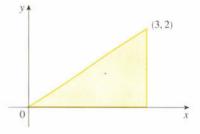
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**18.** A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure. Find the hydrostatic force on one end of the trough.



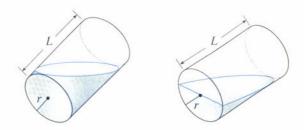
- **19.** A gate in an irrigation canal is in the form of a trapezoid 3 ft wide at the bottom, 5 ft wide at the top, and 2 ft high. It is placed vertically in the canal, with the water extending to its top. Find the hydrostatic force on one side of the gate.
- 20. Find the centroid of the region shown.



- **21.** The demand function for a commodity is given by  $p = 2000 0.1x 0.01x^2$ . Find the consumer surplus when the sales level is 100.
- **22.** Find the average value of the function  $f(x) = x^3$  on the interval [2, 4].
- **23.** If f is a continuous function, what is the limit as  $h \rightarrow 0$  of the average value of f on the interval [x, x + h]?
- 24. Lengths of human pregnancies are normally distributed with mean 268 days and standard deviation 15 days. What percentage of pregnancies last between 250 and 280 days?
- **25.** The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.
  - (a) What is the probability that a customer is served in the first 3 minutes?
  - (b) What is the probability that a customer has to wait more than 10 minutes?
  - (c) What is the median waiting time?



- 1. A solid is generated by rotating about the *x*-axis the region bounded by the *x*-axis, the *y*-axis, and the curve y = f(x), where *f* is a positive function and  $x \ge 0$ . The volume generated by the part of the curve from x = 0 to x = b is  $b^2$  for all b > 0. Find the function *f*.
- **2.** A cylindrical glass of radius r and height L is filled with water and then tilted until the water remaining in the glass exactly covers its base.
  - (a) Determine a way to "slice" the water into parallel rectangular cross-sections and then *set up* a definite integral for the volume of the water in the glass.
  - (b) Determine a way to "slice" the water into parallel cross-sections that are trapezoids and then *set up* a definite integral for the volume of the water.
  - (c) Find the volume of water in the glass by evaluating one of the integrals in part (a) or part (b).
  - (d) Find the volume of the water in the glass from purely geometric considerations.
  - (e) Suppose the glass is tilted until the water exactly covers half the base. In what direction can you "slice" the water into triangular cross-sections? Rectangular cross-sections? Cross-sections that are segments of circles? Find the volume of water in the glass.



**3.** (a) Show that the volume of a segment of height h of a sphere of radius r is

$$V = \frac{1}{3}\pi h^2(3r - h)$$

(b) Show that if a sphere of radius 1 is sliced by a plane at a distance x from the center in such a way that the volume of one segment is twice the volume of the other, then x is a solution of the equation

$$3x^3 - 9x + 2 = 0$$

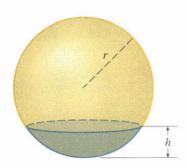
where 0 < x < 1. Use Newton's method to find x accurate to four decimal places.

(c) Using the formula for the volume of a segment of a sphere, it can be shown that the depth x to which a floating sphere of radius r sinks in water is a root of the equation

$$x^{3} - 3rx^{2} + 4r^{3}s = 0$$

where s is the specific gravity of the sphere. Suppose a wooden sphere of radius 0.5 m has specific gravity 0.75. Calculate, to four-decimal-place accuracy, the depth to which the sphere will sink.

- (d) A hemispherical bowl has radius 5 in. and water is running into the bowl at the rate of 0.2 in $^3$ /s.
  - (i) How fast is the water level in the bowl rising at the instant the water is 3 in. deep?
  - (ii) At a certain instant, the water is 4 in. deep. How long will it take to fill the bowl?



**Figure for Problem 3** 

**4.** Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density  $\rho_0$  floating partly submerged in a fluid of density  $\rho_f$ , the buoyant force is given by  $F = \rho_f g \int_{-h}^{0} A(y) dy$ , where g is the acceleration due to gravity and A(y) is the area of a typical cross-section of the object. The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y) \, dy$$

(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100 \ \frac{\rho_f - \rho_0}{\rho_f}$$

- (b) The density of ice is 917 kg/m<sup>3</sup> and the density of seawater is 1030 kg/m<sup>3</sup>. What percentage of the volume of an iceberg is above water?
- (c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?
- (d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is 1000 kg/m<sup>3</sup>.
- **5.** Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.
- **6.** A sphere of radius 1 overlaps a smaller sphere of radius *r* in such a way that their intersection is a circle of radius *r*. (In other words, they intersect in a great circle of the small sphere.) Find *r* so that the volume inside the small sphere and outside the large sphere is as large as possible.
- 7. Suppose that the density of seawater,  $\rho = \rho(z)$ , varies with the depth z below the surface.
  - (a) Show that the hydrostatic pressure is governed by the differential equation

$$\frac{dP}{dz} = \rho(z)g$$

where g is the acceleration due to gravity. Let  $P_0$  and  $\rho_0$  be the pressure and density at z = 0. Express the pressure at depth z as an integral.

- (b) Suppose the density of seawater at depth z is given by  $\rho = \rho_0 e^{z/H}$ , where H is a positive constant. Find the total force, expressed as an integral, exerted on a vertical circular porthole of radius r whose center is located at a distance L > r below the surface.
- 8. A paper drinking cup filled with water has the shape of a cone with height h and semivertical angle  $\theta$  (see the figure). A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?
- **9.** A string is wound around a circle and then unwound while being held taut. The curve traced by the point P at the end of the string is called the **involute** of the circle. If the circle has radius r and center O and the initial position of P is (r, 0),

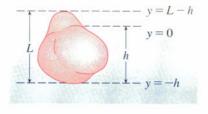


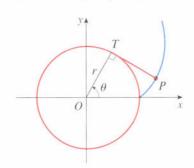
Figure for Problem 4



Figure for Problem 8

and if the parameter  $\theta$  is chosen as in the figure, show that parametric equations of the involute are





- 10. A cow is tied to a silo of radius r by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.
- 11. A curve is defined by the parametric equations

$$x = \int_{1}^{t} \frac{\cos u}{u} \, du \qquad y = \int_{1}^{t} \frac{\sin u}{u} \, du$$

Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

12. Let C be the arc of the curve y = f(x) between the points P(p, f(p)) and Q(q, f(q)) and let R be the region bounded by C, by the line y = mx + b (which lies entirely on one side of C), and by the perpendiculars to the line from P and Q.
(a) Show that the area of R is

$$\frac{1}{1+m^2} \int_p^q \left[ f(x) - mx - b \right] \left[ 1 + mf'(x) \right] dx$$

(b) Find a formula similar to the one in part (a) for the volume of the solid obtained by rotating  $\Re$  about the line y = mx + b.

[*Hint:* The formula in part (a) can be verified by subtracting areas, but it is more instructive to derive it by first approximating the area using rectangles perpendicular to the line, as shown in the figure. This will also help in finding the formula for part (b). Use the figure to help express  $\Delta u$  in terms of  $\Delta x$ .]

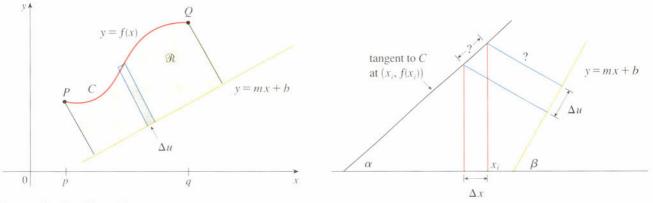


Figure for Problem 12

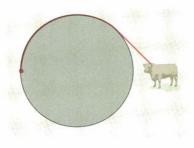


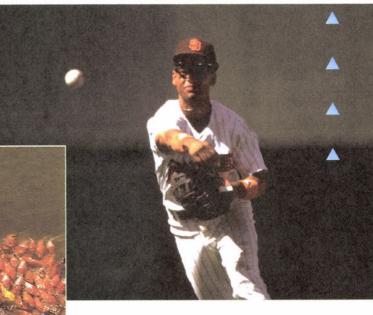
Figure for Problem 10





Differential Equations

• • • Pictured are three situations that are analyzed with differential equations in this chapter: growth of fish populations; interaction of Canada lynx and snowshoe hare populations; positioning a baseball infielder to relay a throw to home plate.





Perhaps the most important of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.

A



## Modeling with Differential Equations

Now is a good time to read (or reread) the discussion of mathematical modeling on page 75.

In describing the process of modeling in Section 1.7, we talked about formulating a mathematical model of a real-world problem either through intuitive reasoning about the phenomenon or from a physical law based on evidence from experiments. The mathematical model often takes the form of a *differential equation*, that is, an equation that contains an unknown function and some of its derivatives. This is not surprising because in a real-world problem we often notice that changes occur and we want to predict future behavior on the basis of how current values change. Let's begin by examining several examples of how differential equations arise when we model physical phenomena.

#### Models of Population Growth

П

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let's identify and name the variables in this model:

t = time (the independent variable)

P = the number of individuals in the population (the dependent variable)

The rate of growth of the population is the derivative dP/dt. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\frac{dP}{dt} = kP$$

where k is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function P and its derivative dP/dt.

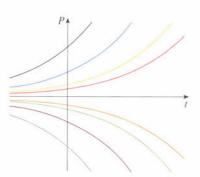
Having formulated a model, let's look at its consequences. If we rule out a population of 0, then P(t) > 0 for all t. So, if k > 0, then Equation 1 shows that P'(t) > 0 for all t. This means that the population is always increasing. In fact, as P(t) increases, Equation 1 shows that dP/dt becomes larger. In other words, the growth rate increases as the population increases.

Let's try to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We know that exponential functions have that property. In fact, if we let  $P(t) = Ce^{kt}$ , then

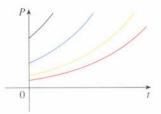
$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

Thus, any exponential function of the form  $P(t) = Ce^{kt}$  is a solution of Equation 1. When we study this equation in detail in Section 7.5, we will see that there is no other solution.

Allowing C to vary through all the real numbers, we get the *family* of solutions  $P(t) = Ce^{kt}$  whose graphs are shown in Figure 1. But populations have only posi-



**FIGURE I** The family of solutions of dP/dt = kP



**FIGURE 2** The family of solutions  $P(t) = Ce^{kt}$  with C > 0 and  $t \ge 0$ 

tive values and so we are interested only in the solutions with C > 0. And we are probably concerned only with values of t greater than the initial time t = 0. Figure 2 shows the physically meaningful solutions. Putting t = 0, we get  $P(0) = Ce^{k(0)} = C$ , so the constant C turns out to be the initial population, P(0).

Equation 1 is appropriate for modeling population growth under ideal conditions, but we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity K* (or decreases toward *K* if it ever exceeds *K*). For a model to take into account both trends, we make two assumptions:

- $\frac{dP}{dt} \approx kP$  if P is small (Initially, the growth rate is proportional to P.)
- $\frac{dP}{dt} < 0 \text{ if } P > K \qquad (P \text{ decreases if it ever exceeds } K.)$

A simple expression that incorporates both assumptions is given by the equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

Notice that if P is small compared with K, then P/K is close to 0 and so  $dP/dt \approx kP$ . If P > K, then 1 - P/K is negative and so dP/dt < 0.

Equation 2 is called the *logistic differential equation* and was proposed by the Dutch mathematical biologist Verhulst in the 1840s as a model for world population growth. We will develop techniques that enable us to find explicit solutions of the logistic equation in Section 7.6, but for now we can deduce qualitative characteristics of the solutions directly from Equation 2. We first observe that the constant functions P(t) = 0 and P(t) = K are solutions because, in either case, one of the factors on the right side of Equation 2 is zero. (This certainly makes physical sense: If the population is ever either 0 or at the carrying capacity, it stays that way.) These two constant solutions are called *equilibrium solutions*.

If the initial population P(0) lies between 0 and K, then the right side of Equation 2 is positive, so dP/dt > 0 and the population increases. But if the population exceeds the carrying capacity (P > K), then 1 - P/K is negative, so dP/dt < 0 and the population decreases. Notice that, in either case, if the population approaches the carrying capacity  $(P \rightarrow K)$ , then  $dP/dt \rightarrow 0$ , which means the population levels off. So we expect that the solutions of the logistic differential equation have graphs that look something like the ones in Figure 3. Notice that the graphs move away from the equilibrium solution P = 0 and move toward the equilibrium solution P = K.

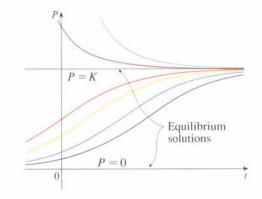


FIGURE 3 Solutions of the logistic equation

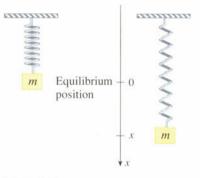


FIGURE 4

#### A Model for the Motion of a Spring

Let's now look at an example of a model from the physical sciences. We consider the motion of an object with mass m at the end of a vertical spring (as in Figure 4). In Section 6.5 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

restoring force = -kx

where k is a positive constant (called the *spring constant*). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$m\frac{d^2x}{dt^2} = -kx$$

This is an example of what is called a *second-order differential equation* because it involves second derivatives. Let's see what we can guess about the form of the solution directly from the equation. We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which says that the second derivative of x is proportional to x but has the opposite sign. We know two functions with this property, the sine and cosine functions. In fact, it turns out that all solutions of Equation 3 can be written as combinations of certain sine and cosine functions. (See Exercise 3.) This is not surprising; we expect the spring to oscillate about its equilibrium position and so it is natural to think that trigonometric functions are involved.

#### General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Thus, Equations 1 and 2 are first-order equations and Equation 3 is a second-order equation. In all three of those equations the independent variable is called t and represents time, but in general the independent variable doesn't have to represent time. For example, when we consider the differential equation

#### 4

3

$$y' = xy$$

it is understood that y is an unknown function of x.

A function *f* is called a **solution** of a differential equation if the equation is satisfied when y = f(x) and its derivatives are substituted into the equation. Thus, *f* is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of x in some interval.

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation. We have already solved some particularly simple

differential equations, namely, those of the form

$$y' = f(x)$$

For instance, we know that the general solution of the differential equation

$$y' = x^3$$

is given by

$$y = \frac{x^4}{4} + C$$

where C is an arbitrary constant.

But, in general, solving a differential equation is not an easy matter. There is no systematic technique that enables us to solve all differential equations. But in Section 7.2 we will see how to draw rough graphs of solutions even when we have no explicit formula. And the methods of Section 7.3 will enable us to find numerical approximations to solutions.

**EXAMPLE I** Show that every member of the family of functions

$$y = \frac{1 + ce'}{1 - ce'}$$

is a solution of the differential equation  $y' = \frac{1}{2}(y^2 - 1)$ .

**SOLUTION** We use the Quotient Rule to differentiate the expression for y:

$$y' = \frac{(1 - ce^{t})(ce^{t}) - (1 + ce^{t})(-ce^{t})}{(1 - ce^{t})^{2}}$$
$$= \frac{ce^{t} - c^{2}e^{2t} + ce^{t} + c^{2}e^{2t}}{(1 - ce^{t})^{2}} = \frac{2ce^{t}}{(1 - ce^{t})^{2}}$$

The right side of the differential equation becomes

$$\frac{1}{2}(y^2 - 1) = \frac{1}{2} \left[ \left( \frac{1 + ce^t}{1 - ce^t} \right)^2 - 1 \right] = \frac{1}{2} \left[ \frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2} \right]$$

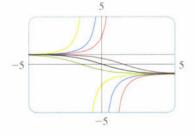
$$= \frac{1}{2} \frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2}$$

Therefore, for every value of c, the given function is a solution of the differential equation.

When applying differential equations we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form  $y(t_0) = y_0$ . This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point  $(t_0, y_0)$ . Physically,

Figure 5 shows graphs of seven members of the family in Example 1. The differential equation shows that if  $y = \pm 1$ , then  $y' \approx 0$ . That is borne out by the flatness of the graphs near y = 1 and y = -1.





this corresponds to measuring the state of a system at time  $t_0$  and using the solution of the initial-value problem to predict the future behavior of the system.

**EXAMPLE 2** Find a solution of the differential equation  $y' = \frac{1}{2}(y^2 - 1)$  that satisfies the initial condition y(0) = 2.

**SOLUTION** Substituting the values t = 0 and y = 2 into the formula

$$y = \frac{1 + ce'}{1 - ce'}$$

from Example 1, we get

$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c}$$

Solving this equation for c, we get 2 - 2c = 1 + c, which gives  $c = \frac{1}{3}$ . So the solution of the initial-value problem is

$$y = \frac{1 + \frac{1}{3}e^{t}}{1 - \frac{1}{3}e^{t}} = \frac{3 + e^{t}}{3 - e^{t}}$$



Æ

1. Show that  $y = 2 + e^{-x^3}$  is a solution of the differential equation  $y' + 3x^2y = 6x^2$ .

2. Verify that  $y = (2 + \ln x)/x$  is a solution of the initialvalue problem

$$x^2y' + xy = 1$$
  $y(1) = 2$ 

- **3.** (a) For what nonzero values of k does the function  $y = \sin kt$  satisfy the differential equation y'' + 9y = 0?
  - (b) For those values of *k*, verify that every member of the family of functions

$$y = A \sin kt + B \cos kt$$

is also a solution.

Exercises

- **4.** For what values of *r* does the function  $y = e^{rt}$  satisfy the differential equation y'' + y' 6y = 0?
- 5. Which of the following functions are solutions of the differential equation y" + 2y' + y = 0?
  (a) y = e'
  (b) y = e^{-t'}

(a) 
$$y = e$$
  
(b)  $y = e$   
(c)  $y = te^{-t}$   
(d)  $y = t^2e^{-t}$ 

- **6.** (a) Show that every member of the family of functions  $y = Ce^{x^2/2}$  is a solution of the differential equation y' = xy.
  - (b) Illustrate part (a) by graphing several members of the family of solutions on a common screen.

- (c) Find a solution of the differential equation y' = xy that satisfies the initial condition y(0) = 5.
- (d) Find a solution of the differential equation y' = xy that satisfies the initial condition y(1) = 2.
- 7. (a) What can you say about a solution of the equation  $y' = -y^2$  just by looking at the differential equation?
  - (b) Verify that all members of the family y = 1/(x + C) are solutions of the equation in part (a).
  - (c) Can you think of a solution of the differential equation  $y' = -y^2$  that is not a member of the family in part (b)?
  - (d) Find a solution of the initial-value problem

$$y' = -y^2$$
  $y(0) = 0.5$ 

- 8. (a) What can you say about the graph of a solution of the equation y' = xy<sup>3</sup> when x is close to 0? What if x is large?
  - (b) Verify that all members of the family  $y = (c - x^2)^{-1/2}$  are solutions of the differential equation  $y' = xy^3$ .
- (c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?
  - (d) Find a solution of the initial-value problem

Æ

$$y' = xy^3 \qquad y(0) = 2$$

9. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P \left(1 - \frac{P}{4200}\right)$$

- (a) For what values of *P* is the population increasing?
- (b) For what values of P is the population decreasing?
- (c) What are the equilibrium solutions?
- **10.** A function y(t) satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$$

- (a) What are the constant solutions of the equation?
- (b) For what values of *y* is *y* increasing?
- (c) For what values of y is y decreasing?
- 11. Psychologists interested in learning theory study learning curves. A learning curve is the graph of a function P(t), the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.
  - (a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.
  - (b) If *M* is the maximum level of performance of which the learner is capable, explain why the differential

**Direction Fields** 

equation

$$\frac{dP}{dt} = k(M - P) \qquad k \text{ a positive constant}$$

is a reasonable model for learning.

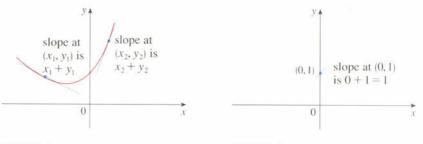
- (c) Make a rough sketch of a possible solution of this differential equation.
- **12.** Suppose you have just poured a cup of freshly brewed coffee with temperature 95 °C in a room where the temperature is 20 °C.
  - (a) When do you think the coffee cools most quickly? What happens to the rate of cooling as time goes by? Explain.
  - (b) Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Write a differential equation that expresses Newton's Law of Cooling for this particular situation. What is the initial condition? In view of your answer to part (a), do you think this differential equation is an appropriate model for cooling?
  - (c) Make a rough sketch of the graph of the solution of the initial-value problem in part (b).

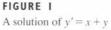


Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y$$
  $y(0) = 1$ 

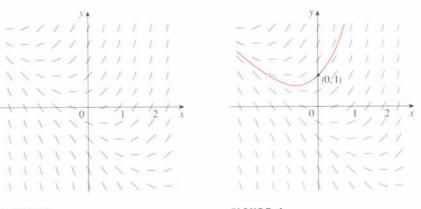
We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation y' = x + y tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the x- and y-coordinates of the point. (See Figure 1.) In particular, because the curve passes through the point (0, 1), its slope there must be 0 + 1 = 1. So a small portion of the solution curve near the point (0, 1) looks like a short line segment through (0, 1) with slope 1. (See Figure 2.)





**FIGURE 2** The beginning of the solution curve through (0, 1)

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope x + y. The result is called a *direction field* and is shown in Figure 3. For instance, the line segment at the point (1, 2) has slope 1 + 2 = 3. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.



### FIGURE 3

Direction field for y' = x + y

**FIGURE 4** The solution curve through (0, 1)

Now we can sketch the solution curve through the point (0, 1) by following the direction field as in Figure 4. Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where F(x, y) is some expression in x and y. The differential equation says that the slope of a solution curve at a point (x, y) on the curve is F(x, y). If we draw short line segments with slope F(x, y) at several points (x, y), the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

#### EXAMPLE I

- (a) Sketch the direction field for the differential equation  $y' = x^2 + y^2 1$ .
- (b) Use part (a) to sketch the solution curve that passes through the origin.

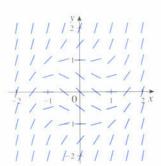
#### SOLUTION

(a) We start by computing the slope at several points in the following chart:

х	-2	- I	0	- E	2	-2	-1	0	1	2	1.4.4
У	0	0	0	0	0	1	I	1	1	1	122
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	177

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

(b) We start at the origin and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it





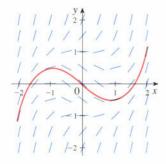
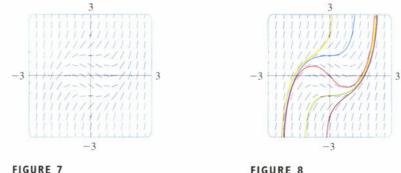


FIGURE 6

moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 6. Returning to the origin, we draw the solution curve to the left as well.

The more line segments we draw in a direction field, the clearer the picture becomes. Of course, it's tedious to compute slopes and draw line segments for a huge number of points by hand, but computers are well suited for this task. Figure 7 shows a more detailed, computer-drawn direction field for the differential equation in Example 1. It enables us to draw, with reasonable accuracy, the solution curves shown in Figure 8 with y-intercepts -2, -1, 0, 1, and 2.





1



Now let's see how direction fields give insight into physical situations. The simple electric circuit shown in Figure 9 contains an electromotive force (usually a battery or generator) that produces a voltage of E(t) volts (V) and a current of I(t)amperes (A) at time t. The circuit also contains a resistor with a resistance of Rohms  $(\Omega)$  and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is L(dI/dt). One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage E(t). Thus, we have

 $L\frac{dI}{dt} + RI = E(t)$ 

which is a first-order differential equation that models the current I at time t.

**EXAMPLE 2** Suppose that in the simple circuit of Figure 9 the resistance is 12  $\Omega$ , the inductance is 4 H, and a battery gives a constant voltage of 60 V.

(a) Draw a direction field for Equation 1 with these values.

(b) What can you say about the limiting value of the current?

(c) Identify any equilibrium solutions.

(d) If the switch is closed when t = 0 so the current starts with I(0) = 0, use the direction field to sketch the solution curve.

#### SOLUTION

(a) If we put L = 4, R = 12, and E(t) = 60 in Equation 1, we get

$$4 \frac{dI}{dt} + 12I = 60$$
 or  $\frac{dI}{dt} = 15 - 3I$ 

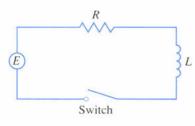


FIGURE 9

The direction field for this differential equation is shown in Figure 10.

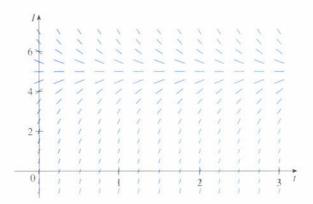


FIGURE 10

(b) It appears from the direction field that all solutions approach the value 5 A, that is,

$$\lim_{t \to \infty} I(t) = 5$$

(c) It appears that the constant function I(t) = 5 is an equilibrium solution. Indeed, we can verify this directly from the differential equation. If I(t) = 5, then the left side is dI/dt = 0 and the right side is 15 - 3(5) = 0.

(d) We use the direction field to sketch the solution curve that passes through (0,0), as shown in red in Figure 11.

> 2 0

FIGURE II

Notice from Figure 10 that the line segments along any horizontal line are parallel. That is because the independent variable t does not occur on the right side of the equation I' = 15 - 3I. In general, a differential equation of the form

$$y' = f(y)$$

in which the independent variable is missing from the right side, is called autonomous. For such an equation, the slopes corresponding to two different points with the same y-coordinate must be equal. This means that if we know one solution to an autonomous differential equation, then we can obtain infinitely many others just by shifting the graph of the known solution to the right or left. In Figure 11 we have shown the solutions that result from shifting the solution curve of Example 2

15 - 31

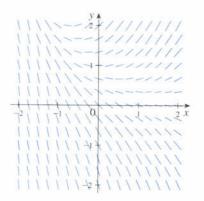


one and two units to the right. They correspond to closing the switch when t = 1 and t = 2. Notice that the system behaves the same at any time.

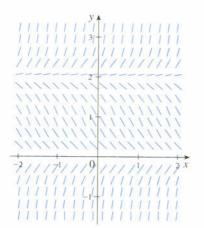


# 7.2 Exercises

I. A direction field for the differential equation y' = y - e<sup>-x</sup> is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.
(a) y(0) = 0 (b) y(0) = 1 (c) y(0) = -1

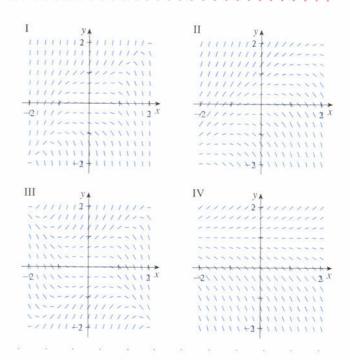


- 2. (a) A direction field for the differential equation y' = 2y(y 2) is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.
  (i) y(0) = 1 (ii) y(0) = 2.5 (iii) y(0) = -1
  - (b) Suppose the initial condition is y(0) = c. For what values of c is lim<sub>t→∞</sub> y(t) finite? What are the equilibrium solutions?



**3–6** ■ Match the differential equation with its direction field (labeled I–IV). Give reasons for your answer.

**3.** y' = y - 1**4.** y' = y - x**5.**  $y' = y^2 - x^2$ **6.**  $y' = y^3 - x^3$ 



**7.** Use the direction field labeled I (for Exercises 3–6) to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) y(0) = 1 (b) y(0) = 0 (c) y(0) = -1

8. Repeat Exercise 7 for the direction field labeled III.

9-10 ■ Sketch a direction field for the differential equation. Then use it to sketch three solution curves.

 $= xy + y^2$ 

**9.** 
$$y' = x - y$$
 **10.**  $y'$ 

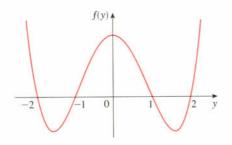
**11–14** ■ Sketch the direction field of the given differential equation. Then use it to sketch a solution curve that passes through the given point.

<b>11.</b> $y' = y^2$ , (0, 1)	i -	<b>12.</b> $y' = x^2 + y$ , (1, 1)					
<b>13.</b> $y' = x^2 + y^2$ ,	(0, 0)	<b>14.</b> $y' = y(4 - y)$ , (0, 1)					
141 A A A	2 2						

CAS 15-16 ■ Use a computer algebra system to draw a direction field for the given differential equation. Get a printout and sketch on it the solution curve that passes through (0, 1). Then use the CAS to draw the solution curve and compare it with your sketch.

**15.** 
$$y' = y \sin 2x$$
 **16.**  $y' = \sin(x + y)$ 

- **CAS** 17. Use a computer algebra system to draw a direction field for the differential equation  $y' = y^3 - 4y$ . Get a printout and sketch on it solutions that satisfy the initial condition y(0) = c for various values of c. For what values of c does  $\lim_{t\to\infty} y(t)$  exist? What are the possible values for this limit?
  - **18.** Make a rough sketch of a direction field for the autonomous differential equation y' = f(y), where the graph of f is as shown. How does the limiting behavior of solutions depend on the value of y(0)?



19. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms ( $\Omega$ ). The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

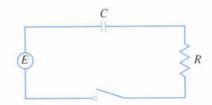
$$RI + \frac{Q}{C} = E(t)$$

But I = dQ/dt, so we have

$$R \, \frac{dQ}{dt} + \frac{1}{C} \, Q = E(t)$$

Suppose the resistance is 5  $\Omega$ , the capacitance is 0.05 F, and a battery gives a constant voltage of 60 V.

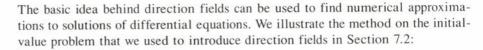
- (a) Draw a direction field for this differential equation.
- (b) What is the limiting value of the charge?
- (c) Is there an equilibrium solution?
- (d) If the initial charge is Q(0) = 0 C, use the direction field to sketch the solution curve.



- **20.** In Exercise 12 in Section 7.1 we considered a 95 °C cup of coffee in a 20 °C room. Suppose it is known that the coffee cools at a rate of 1 °C per minute when its temperature is 70 °C.
  - (a) What does the differential equation become in this case?
  - (b) Sketch a direction field and use it to sketch the solution curve for the initial-value problem. What is the limiting value of the temperature?



#### Euler's Method ·



$$y' = x + y$$
  $y(0) = 1$ 

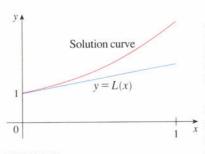
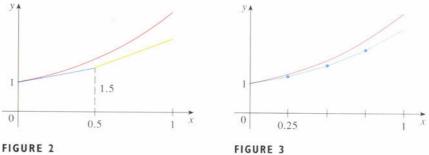
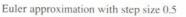


FIGURE I First Euler approximation

The differential equation tells us that y'(0) = 0 + 1 = 1, so the solution curve has slope 1 at the point (0, 1). As a first approximation to the solution we could use the linear approximation L(x) = x + 1. In other words, we could use the tangent line at (0, 1) as a rough approximation to the solution curve (see Figure 1).

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field. Figure 2 shows what happens if we start out along the tangent line but stop when x = 0.5. (This horizontal distance traveled is called the *step size*.) Since L(0.5) = 1.5, we have  $y(0.5) \approx 1.5$  and we take (0.5, 1.5) as the starting point for a new line segment. The differential equa-





Euler approximation with step size 0.25

tion tells us that y'(0.5) = 0.5 + 1.5 = 2, so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for x > 0.5 (the gold-colored segment in Figure 2). If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 3.

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce the exact solution to an initial-value problem—it gives approximations. But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution. (Compare Figures 1, 2, and 3.)

For the general first-order initial-value problem y' = F(x, y),  $y(x_0) = y_0$ , our aim is to find approximate values for the solution at equally spaced numbers  $x_0$ ,  $x_1 = x_0 + h$ ,  $x_2 = x_1 + h$ , ..., where h is the step size. The differential equation tells us that the slope at  $(x_0, y_0)$  is  $y' = F(x_0, y_0)$ , so Figure 4 shows that the approximate value of the solution when  $x = x_1$  is

	$y_1 = y_0 + hF(x_0, y_0)$
Similarly,	$y_2 = y_1 + hF(x_1, y_1)$
In general,	$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$

**EXAMPLE I** Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

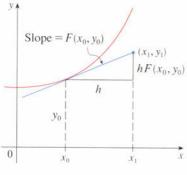
**SOLUTION** We are given that h = 0.1,  $x_0 = 0$ ,  $y_0 = 1$ , and F(x, y) = x + y. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$
  

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$
  

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

This means that if y(x) is the exact solution, then  $y(0.3) \approx 1.362$ .





11	$X_{0}$	Na	<u>n</u>	$X_{B}$	$\mathbb{P}^n$
ĩ	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3,187485

Proceeding with similar calculations, we get the values in the table:

For a more accurate table of values in Example 1 we could decrease the step size. But for a large number of small steps the amount of computation is considerable and so we need to program a calculator or computer to carry out these calculations. The following table shows the results of applying Euler's method with decreasing step size to the initial-value problem of Example 1.

Step size	Euler estimate of y(0.5)	Euler estimate of y(1		
0.500	1.500000	2,500000		
0.250	1.625000	2.882813		
0.100	1.721020	3.187485		
().()5()	1.757789	3.306595		
0.020	1.781212	3.383176		
0.010	1.789264	3,409628		
0.005	1.793337	3.423034		
0.001	1.796619	3.433848		

Notice that the Euler estimates in the table seem to be approaching limits, namely, the true values of y(0.5) and y(1). Figure 5 shows graphs of the Euler approximations with step sizes 0.5, 0.25, 0.1, 0.05, 0.02, 0.01, and 0.005. They are approaching the exact solution curve as the step size *h* approaches 0.

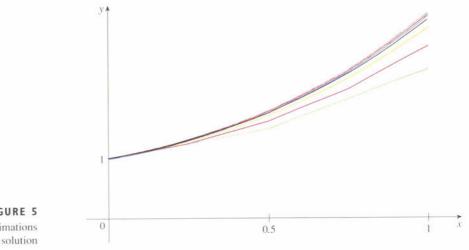


FIGURE 5 Euler approximations approaching the exact solution

**EXAMPLE 2** In Example 2 in Section 7.2 we discussed a simple electric circuit with resistance 12  $\Omega$ , inductance 4 H, and a battery with voltage 60 V. If the switch is closed when t = 0, we modeled the current I at time t by the initial-value problem

$$\frac{dI}{dt} = 15 - 3I$$
  $I(0) = 0$ 

Estimate the current in the circuit half a second after the switch is closed.

**SOLUTION** We use Euler's method with F(t, I) = 15 - 3I,  $t_0 = 0$ ,  $I_0 = 0$ , and step size h = 0.1 second:

$$I_{1} = 0 + 0.1(15 - 3 \cdot 0) = 1.5$$

$$I_{2} = 1.5 + 0.1(15 - 3 \cdot 1.5) = 2.55$$

$$I_{3} = 2.55 + 0.1(15 - 3 \cdot 2.55) = 3.285$$

$$I_{4} = 3.285 + 0.1(15 - 3 \cdot 3.285) = 3.7995$$

$$I_{5} = 3.7995 + 0.1(15 - 3 \cdot 3.7995) = 4.15965$$

. . . . . . . . . . . . .

So the current after 0.5 s is

$$I(0.5) \approx 4.16 \text{ A}$$



1. (a) Use Euler's method with each of the following step sizes to estimate the value of y(0.4), where y is the solution of the initial-value problem y' = y, y(0) = 1.

(i) 
$$h = 0.4$$
 (ii)  $h = 0.2$ 

(iii) 
$$h = 0.1$$

Exercises ·

- (b) We know that the exact solution of the initial-value problem in part (a) is  $y = e^x$ . Draw, as accurately as you can, the graph of  $y = e^x$ ,  $0 \le x \le 0.4$ , together with the Euler approximations using the step sizes in part (a). (Your sketches should resemble Figures 1, 2, and 3.) Use your sketches to decide whether your estimates in part (a) are underestimates or overestimates.
- (c) The error in Euler's method is the difference between the exact value and the approximate value. Find the errors made in part (a) in using Euler's method to estimate the true value of y(0.4), namely  $e^{0.4}$ . What happens to the error each time the step size is halved?
- A direction field for a differential equation is shown. Draw, with a ruler, the graphs of the Euler approximations to the solution curve that passes through the ori-

gin. Use step sizes h = 1 and h = 0.5. Will the Euler estimates be underestimates or overestimates? Explain.

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- **3.** Use Euler's method with step size 0.5 to compute the approximate *y*-values  $y_1$ ,  $y_2$ ,  $y_3$ , and  $y_4$  of the solution of the initial-value problem y' = 1 + 3x 2y, y(1) = 2.
- 4. Use Euler's method with step size 0.2 to estimate y(1), where y(x) is the solution of the initial-value problem  $y' = x + y^2$ , y(0) = 0.

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- where y(x) is the solution of the initial-value problem  $y' = x^2 + y^2$ , y(0) = 1.
- 6. (a) Use Euler's method with step size 0.2 to estimate v(0,4), where v(x) is the solution of the initial-value problem  $y' = 2xy^2$ , y(0) = 1.
  - (b) Repeat part (a) with step size 0.1.
- **7.** (a) Program a calculator or computer to use Euler's method to compute y(1), where y(x) is the solution of the initial-value problem

$$\frac{dy}{dx} + 3x^2y = 6x^2 \qquad y(0) = 3$$

(i) 
$$h = 1$$
 (ii)  $h = 0.1$ 

iii) 
$$h = 0.01$$
 (iv)  $h = 0.001$ 

- (b) Verify that  $y = 2 + e^{-x^3}$  is the exact solution of the differential equation.
- (c) Find the errors in using Euler's method to compute y(1) with the step sizes in part (a). What happens to the error when the step size is divided by 10?



Separable Equations

We have looked at first-order differential equations from a geometric point of view (direction fields) and from a numerical point of view (Euler's method). What about the symbolic point of view? It would be nice to have an explicit formula for a solution of a differential equation. Unfortunately, that is not always possible. But in this section we examine a certain type of differential equation that can be solved explicitly.

A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name separable comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. Equivalently, if  $f(y) \neq 0$ , we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where h(y) = 1/f(y). To solve this equation we rewrite it in the differential form

$$h(y) \, dy = g(x) \, dx$$

so that all y's are on one side of the equation and all x's are on the other side. Then we integrate both sides of the equation:

The technique for solving separable differential equations was first used by James Bernoulli (in 1690) in solving a problem about pendulums and by Leibniz (in a letter to Huygens in 1691). John Bernoulli explained the general method in a paper published in 1694.

$$\int h(y) \, dy = \int g(x) \, dx$$

5. Use Euler's method with step size 0.1 to estimate y(0.5), [AS] 8. (a) Program your computer algebra system, using Euler's method with step size 0.01, to calculate y(2), where y is the solution of the initial-value problem

$$y' = x^3 - y^3$$
  $y(0) = 1$ 

- (b) Check your work by using the CAS to draw the solution curve.
- 9. In Exercise 19 in Section 7.2 we considered the differential equation

$$R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$

for the case of a simple electric circuit with resistance  $R = 5 \Omega$ , capacitance C = 0.05 F, and constant voltage E(t) = 60 V. If the initial charge is Q(0) = 0 C, use Euler's method with step size 0.1 to estimate the charge after half a second.

10. In Exercise 20 in Section 7.2 we considered a 95 °C cup of coffee in a 20 °C room. Use Euler's method with step size h = 2 minutes to estimate the temperature of the coffee after 10 minutes.

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

The justification for the step in Equation 2 comes from the Substitution Rule:

$$\int h(y) \, dy = \int h(y(x)) \frac{dy}{dx} \, dx$$
$$= \int h(y(x)) \frac{g(x)}{h(y(x))} \, dx \quad \text{(from Equation 1)}$$
$$= \int g(x) \, dx$$

#### EXAMPLE 1

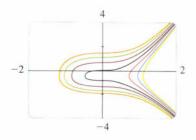
(a) Solve the differential equation  $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$ .

(b) Find the solution of this equation that satisfies the initial condition  $y(1) = \pi$ .

#### SOLUTION

(a) Writing the equation in differential form and integrating both sides, we have

Some computer algebra systems can plot curves defined by implicit equations. Figure I shows the graphs of several members of the family of solutions of the differential equation in Example I. As we look at the curves from left to right, the values of C are 3, 2, 1, 0, -1, -2, and -3.



3

 $(2y + \cos y) dy = 6x^{2} dx$  $\int (2y + \cos y) dy = \int 6x^{2} dx$  $y^{2} + \sin y = 2x^{3} + C$ 

where C is an arbitrary constant. (We could have used a constant  $C_1$  on the left side and another constant  $C_2$  on the right side. But then we could combine these constants by writing  $C = C_2 - C_1$ .)

Equation 3 gives the general solution implicitly. In this case it's impossible to solve the equation to express y explicitly as a function of x.

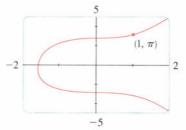
(b) We are given the initial condition  $y(1) = \pi$ , so we substitute x = 1 and  $y = \pi$  in Equation 3:

$$\pi^2 + \sin \pi = 2(1)^3 + C$$
$$C = \pi^2 - 2$$

Therefore, the solution is given implicitly by

$$y^2 + \sin y = 2x^3 + \pi^2 - 2$$

The graph of this solution is shown in Figure 2. (Compare with Figure 1).







**EXAMPLE 2** Solve the equation  $y' = x^2 y$ .

**SOLUTION** First we rewrite the equation using Leibniz notation:

$$\frac{dy}{dx} = x^2 y$$

If  $y \neq 0$ , we can rewrite it in differential notation and integrate:

1

$$\frac{dy}{y} = x^2 dx \qquad y \neq 0$$
$$\int \frac{dy}{y} = \int x^2 dx$$
$$n|y| = \frac{x^3}{3} + C$$

Several solutions of the differential equation in Example 2 are graphed in Figure 3. The values of A are the same as the *y*-intercepts.

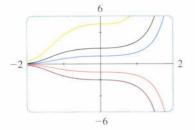


FIGURE 3

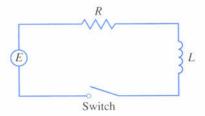


FIGURE 4

This equation defines y implicitly as a function of x. But in this case we can solve explicitly for y as follows:

$$|y| = e^{\ln|y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$
  
 $y = \pm e^C e^{x^3/3}$ 

We note that the function y = 0 is also a solution of the given differential equation. So we can write the general solution in the form

$$v = A e^{x^{3/3}}$$

where A is an arbitrary constant  $(A = e^{C}, \text{ or } A = -e^{C}, \text{ or } A = 0)$ .

**EXAMPLE 3** In Section 7.2 we modeled the current I(t) in the electric circuit shown in Figure 4 by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

Find an expression for the current in a circuit where the resistance is 12  $\Omega$ , the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when t = 0. What is the limiting value of the current?

**SOLUTION** With L = 4, R = 12, and E(t) = 60, the equation becomes

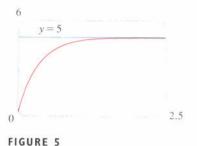
$$4 \frac{dI}{dt} + 12I = 60$$
 or  $\frac{dI}{dt} = 15 - 3I$ 

and the initial-value problem is

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0$$

We recognize this equation as being separable, and we solve it as follows:

Figure 5 shows how the solution in Example 3 (the current) approaches its limiting value. Comparison with Figure II in Section 7.2 shows that we were able to draw a fairly accurate solution curve from the direction field.



 $\int \frac{dI}{15 - 3I} = \int dt$ - $\frac{1}{3} \ln |15 - 3I| = t + C$  $|15 - 3I| = e^{-3(t+C)}$  $15 - 3I = \pm e^{-3C}e^{-3t} = Ae^{-3t}$  $I = 5 - \frac{1}{3}Ae^{-3t}$ 

Since I(0) = 0, we have  $5 - \frac{1}{3}A = 0$ , so A = 15 and the solution is

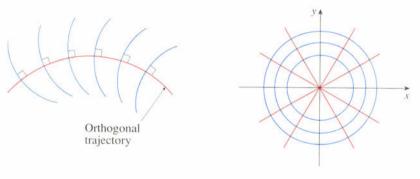
$$I(t) = 5 - 5e^{-3}$$

The limiting current is

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} (5 - 5e^{-3t}) = 5 - 5 \lim_{t \to \infty} e^{-3t} = 5 - 0 = 5$$

#### Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 6). For instance, each member of the family y = mx of straight lines through the origin is an orthogonal trajectory of the family  $x^2 + y^2 = r^2$  of concentric circles with center the origin (see Figure 7). We say that the two families are orthogonal trajectories of each other.



#### FIGURE 6

FIGURE 7

**EXAMPLE 4** Find the orthogonal trajectories of the family of curves  $x = ky^2$ , where k is an arbitrary constant.

**SOLUTION** The curves  $x = ky^2$  form a family of parabolas whose axis of symmetry is the *x*-axis. The first step is to find a single differential equation that is satisfied by all members of the family. If we differentiate  $x = ky^2$ , we get

$$1 = 2ky \frac{dy}{dx}$$
 or  $\frac{dy}{dx} = \frac{1}{2ky}$ 

This is a differential equation, but it depends on k. To eliminate k we note that, from the equation of the given general parabola  $x = ky^2$ , we have  $k = x/y^2$  and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2}y}$$
$$\frac{dy}{dx} = \frac{y}{2x}$$

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is y' = y/(2x). On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

This differential equation is separable, and we solve it as follows:

$$\int y \, dy = -\int 2x \, dx$$
$$\frac{y^2}{2} = -x^2 + C$$
$$x^2 + \frac{y^2}{2} = C$$

where C is an arbitrary positive constant. Thus, the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched in Figure 8.

Orthogonal trajectories occur in various branches of physics. For example, in an electrostatic field the lines of force are orthogonal to the lines of constant potential. Also, the streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

#### Mixing Problems

4

or

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance (say, salt). A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate. If y(t) denotes the amount of substance in the tank at time t, then y'(t) is the rate at which the substance is being added minus the rate at which it is being removed. The mathematical description of this situation often leads to a first-order separable differential equation. We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

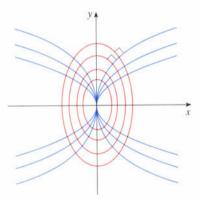


FIGURE 8

**EXAMPLE 5** • A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

**SOLUTION** Let y(t) be the amount of salt (in kilograms) after t minutes. We are given that y(0) = 20 and we want to find y(30). We do this by finding a differential equation satisfied by y(t). Note that dy/dt is the rate of change of the amount of salt, so

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank. We have

rate in = 
$$\left(0.03 \ \frac{\text{kg}}{\text{L}}\right) \left(25 \ \frac{\text{L}}{\text{min}}\right) = 0.75 \ \frac{\text{kg}}{\text{min}}$$

The tank always contains 5000 L of liquid, so the concentration at time t is y(t)/5000 (measured in kilograms per liter). Since the brine flows out at a rate of 25 L/min, we have

rate out 
$$= \left(\frac{y(t)}{5000} \frac{\text{kg}}{\text{L}}\right) \left(25 \frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200} \frac{\text{kg}}{\text{min}}$$

Thus, from Equation 5 we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$
$$-\ln|150 - y| = \frac{t}{200} + C$$

Figure 9 shows the graph of the function 
$$y(t)$$
 of Example 5. Notice that, as time goes by, the amount of salt approaches 150 kg.

Since y(0) = 20, we have  $-\ln 130 = C$ , so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Therefore

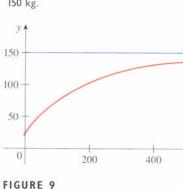
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Since 
$$y(t)$$
 is continuous and  $y(0) = 20$  and the right side is never 0, we deduce that  $150 - y(t)$  is always positive. Thus,  $|150 - y| = 150 - y$  and so

 $|150 - y| = 130e^{-t/200}$ 

$$y(t) = 150 - 130e^{-t/200}$$

 $y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$ 



7.4

□ Solve the differential equation.

Exercises

1. 
$$\frac{dy}{dx} = y^2$$
  
3.  $yy' = x$   
5.  $\frac{du}{dt} = e^{u+2t}$   
2.  $\frac{dy}{dx} = \frac{x + \sin x}{3y^2}$   
4.  $y' = xy$   
6.  $\frac{dx}{dt} = 1 + t - x - tx$ 

Find the solution of the differential equation that satisfies the given initial condition.

7. 
$$\frac{dy}{dx} = y^2 + 1$$
,  $y(1) = 0$   
8.  $\frac{dy}{dx} = \frac{1+x}{xy}$ ,  $x > 0$ ,  $y(1) = -4$   
9.  $xe^{-t}\frac{dx}{dt} = t$ ,  $x(0) = 1$ 

**10.** 
$$x + 2y\sqrt{x^2 + 1} \frac{dy}{dx} = 0, \quad y(0) = 1$$

11. 
$$\frac{du}{dt} = \frac{2t+1}{2(u-1)}, \quad u(0) = -1$$
  
 $\frac{dv}{dt} = \frac{tv+3t}{2}$ 

12. 
$$\frac{dy}{dt} = \frac{ty + 3t}{t^2 + 1}, \quad y(2) = 2$$

13. Find the equation of the curve that satisfies  $dy/dx = 4x^3y$  and whose y-intercept is 7.

- 14. Find an equation of the curve that passes through the point (1, 1) and whose slope at (x, y) is  $y^2/x^3$ .
- **15.** Solve the initial-value problem  $y' = y \sin x$ , y(0) = 1, and graph the solution.
- **16.** Solve the equation  $e^{-y}y' + \cos x = 0$  and graph several members of the family of solutions. How does the solution curve change as the constant *C* varies?
- **17.** Solve the initial-value problem  $y' = (\sin x)/\sin y$ ,  $y(0) = \pi/2$ , and graph the solution (if your CAS does implicit plots).
- **18.** Solve the equation  $y' = x \sqrt{x^2 + 1}/(ye^y)$  and graph several members of the family of solutions (if your CAS does implicit plots). How does the solution curve change as the constant *C* varies?

#### CC 15-20 =

(a) Use a computer algebra system to draw a direction field for the differential equation. Get a printout and use it to sketch some solution curves without solving the differential equation.

- (b) Solve the differential equation.
- (c) Use the CAS to draw several members of the family of solutions obtained in part (b). Compare with the curves from part (a).

**19.** y' = 1/y **20.**  $y' = x^2/y$ 

21-24 Find the orthogonal trajectories of the family of curves. Use a graphing device to draw several members of each family on a common screen.

**21.** 
$$y = kx^2$$
  
**22.**  $x^2 - y^2 = k$   
**23.**  $y = (x + k)^{-1}$   
**24.**  $y = ke^{-x}$ 

- 25. Solve the initial-value problem in Exercise 19 in Section 7.2 to find an expression for the charge at time *t*. Find the limiting value of the charge.
- **26.** In Exercise 20 in Section 7.2 we discussed a differential equation that models the temperature of a 95 °C cup of coffee in a 20 °C room. Solve the differential equation to find an expression for the temperature of the coffee at time *t*.
- In Exercise 11 in Section 7.1 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where P(t) measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for P(t). What is the limit of this expression?

28. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: A + B → C. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[C]}{dt} = k[A][B]$$

(See Example 4 in Section 3.3.) Thus, if the initial concentrations are [A] = a moles/L and [B] = b moles/L and we write x = [C], then we have

$$\frac{dx}{dt} = k(a - x)(b - x)$$

 (a) Assuming that a ≠ b, find x as a function of t. Use a computer algebra system to perform the integration.

CAS

- (b) Find x(t) assuming that a = b. How does this expression for x(t) simplify if it is known that [C] = a/2 after 20 seconds?
- **29.** A glucose solution is administered intravenously into the bloodstream at a constant rate *r*. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus, a model for the concentration C = C(t) of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

- (a) Suppose that the concentration at time t = 0 is  $C_0$ . Determine the concentration at any time t by solving the differential equation.
- (b) Assuming that  $C_0 < r/k$ , find  $\lim_{t \to \infty} C(t)$  and interpret your answer.
- **30.** A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let x = x(t) denote the amount of new currency in circulation at time t, with x(0) = 0.
  - (a) Formulate a mathematical model in the form of an initial-value problem that represents the "flow" of the new currency into circulation.
  - (b) Solve the initial-value problem found in part (a).
  - (c) How long will it take for the new bills to account for 90% of the currency in circulation?

CAS

- 31. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?
- **32.** A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after *t* minutes and (b) after one hour?
- 33. When a raindrop falls it increases in size, so its mass at time t is a function of t, m(t). The rate of growth of the mass is km(t) for some positive constant k. When we apply Newton's Law of Motion to the raindrop, we get (mv)' = gm, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The *terminal velocity* of the raindrop is lim<sub>t→∞</sub> v(t). Find an expression for the terminal velocity in terms of g and k.

**34.** An object of mass *m* is moving horizontally through a medium which resists the motion with a force that is a function of the velocity; that is,

$$m \frac{d^2s}{dt^2} = m \frac{dv}{dt} = f(v)$$

where v = v(t) and s = s(t) represent the velocity and position of the object at time *t*, respectively. For example, think of a boat moving through the water.

- (a) Suppose that the resisting force is proportional to the velocity, that is, f(v) = -kv, k a positive constant. Let  $v(0) = v_0$  and  $s(0) = s_0$  be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels from time t = 0?
- (b) Suppose that the resisting force is proportional to the square of the velocity, that is,  $f(v) = -kv^2$ , k > 0. Let  $v_0$  and  $s_0$  be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels in this case?
- **35.** Let A(t) be the area of a tissue culture at time t and let M be the final area of the tissue when growth is complete. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to  $\sqrt{A(t)}$ . So a reasonable model for the growth of tissue is obtained by assuming that the rate of growth of the area is jointly proportional to  $\sqrt{A(t)}$  and M A(t).
  - (a) Formulate a differential equation and use it to show that the tissue grows fastest when A(t) = M/3.
  - (b) Solve the differential equation to find an expression for A(t). Use a computer algebra system to perform the integration.
- **36.** According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass *m* that has been projected vertically upward from Earth's surface is

$$F = \frac{mgR^2}{(x+R)^2}$$

where x = x(t) is the object's distance above the surface at time t, R is Earth's radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, F = ma = m (dv/dt) and so

$$m \, \frac{dv}{dt} = -\frac{mgR^2}{(x+R)^2}$$

(a) Suppose a rocket is fired vertically upward with an initial velocity  $v_0$ . Let *h* be the maximum height above the surface reached by the object. Show that

$$v_0 = \sqrt{\frac{2gRh}{R+h}}$$

[*Hint*: By the Chain Rule, m (dv/dt) = mv (dv/dx).]

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- (b) Calculate  $v_e = \lim_{h \to \infty} v_0$ . This limit is called the *escape velocity* for Earth.
- (c) Use R = 3960 mi and g = 32 ft/s<sup>2</sup> to calculate  $v_e$  in feet per second and in miles per second.
- **37.** Let y(t) and V(t) be the height and volume of water in a tank at time *t*. If water leaks through a hole with area *a* at the bottom of the tank, then Torricelli's Law says that

$$\frac{dV}{dt} = -a\sqrt{2gy}$$

where g is the acceleration due to gravity.

(a) Suppose the tank is cylindrical with height 6 ft and radius 2 ft and the hole is circular with radius 1 in. If we take g = 32 ft/s<sup>2</sup>, show that y satisfies the differential equation

$$\frac{dy}{dt} = -\frac{1}{72}\sqrt{y}$$

- (b) Solve this equation to find the height of the water at time t, assuming the tank is full at time t = 0.
- (c) How long will it take for the water to drain completely?

Applied Proiect **38.** Suppose the tank in Exercise 37 is not cylindrical but has cross-sectional area A(y) at height y. Then the volume of water up to height y is  $V = \int_0^y A(u) du$  and so the Fundamental Theorem of Calculus gives dV/dy = A(y). It follows that

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}$$

and so Torricelli's Law becomes

$$A(y) \frac{dy}{dt} = -a \sqrt{2gy}$$

(a) Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm and we take  $g = 10 \text{ m/s}^2$ , show that y satisfies the differential equation

$$4y - y^2) \,\frac{dy}{dt} = -0.0001 \,\sqrt{20y}$$

(b) How long will it take for the water to drain completely?

# Which is Faster, Going Up or Coming Down?

Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to Earth from its maximum height? We will solve the problem in this project but, before getting started, think about that situation and make a guess based on your physical intuition.

1. A ball with mass *m* is projected vertically upward from Earth's surface with a positive initial velocity  $v_0$ . We assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude p | v(t) |, where *p* is a positive constant and v(t) is the velocity of the ball at time *t*. In both the ascent and the descent, the total force acting on the ball is -pv - mg. (During ascent, v(t) is positive and the resistance acts downward; during descent, v(t) is negative and the resistance acts upward.) So, by Newton's Second Law, the equation of motion is

$$nv' = -pv - mg$$

Solve this differential equation to show that the velocity is

$$v(t) = \left(\frac{1}{v_0} + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$$

2. Show that the height of the ball, until it hits the ground, is

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}(1 - e^{-pt/m}) - \frac{mgt}{p}$$

Experiments have shown that, for speeds up to 100 m/s, the drag force due to air resistance is approximately proportional to the speed. **3.** Let  $t_1$  be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p} \ln \left( \frac{mg + pv_0}{mg} \right)$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is  $\frac{1}{10}$  of the speed.

4. Let  $t_2$  be the time at which the ball falls back to Earth. For the particular ball in Problem 3, estimate  $t_2$  by using a graph of the height function y(t). Which is faster, going up or coming down?

5. In general, it's not easy to find  $t_2$  because it's impossible to solve the equation y(t) = 0 explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster; we determine whether  $y(2t_1)$  is positive or negative. Show that

$$y(2t_1) = \frac{m^2 g}{p^2} \left( x - \frac{1}{x} - 2 \ln x \right)$$

where  $x = e^{pt_1/m}$ . Then show that x > 1 and the function

$$f(x) = x - \frac{1}{x} - 2\ln x$$

is increasing for x > 1. Use this result to decide whether  $y(2t_1)$  is positive or negative. What can you conclude? Is ascent or descent faster?



#### Exponential Growth and Decay . . . . . . . . . . . . . . . . .

One of the models for population growth that we considered in Section 7.1 was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size P = 1000 and at a certain time it is growing at a rate of P' = 300 bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the new population was growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. In general, it seems reasonable that the growth rate should be proportional to the size.

The same assumption applies in other situations as well. In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if y(t) is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size y(t) at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the law of natural growth (if k > 0) or the **law of natural decay** (if k < 0). Because it is a separable differential equation we can solve it by the methods of Section 7.4:

$$\int \frac{dy}{y} = \int k \, dt$$
$$\ln|y| = kt + C$$
$$|y| = e^{kt+C} = e^C e^{kt}$$
$$y = A e^{kt}$$

where  $A(=\pm e^{C} \text{ or } 0)$  is an arbitrary constant. To see the significance of the constant A, we observe that

$$y(0) = Ae^{k \cdot 0} = A$$

Therefore, A is the initial value of the function.

Because Equation 1 occurs so frequently in nature, we summarize what we have just proved for future use.

The solution of the initial-value problem 2  $\frac{dy}{dt} = ky \qquad y(0) = y_0$  $y(t) = y_0 e^{kt}$ is

## **Population Growth**

What is the significance of the proportionality constant k? In the context of population growth, we can write

 $\frac{dP}{dt} = kP$  or  $\frac{1}{P}\frac{dP}{dt} = k$ 

The quantity

$$\frac{1}{P} \frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the relative growth rate. According to (3), instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant." Then (2) says that a population with constant relative growth rate must grow exponentially. Notice that the relative growth rate k appears as the coefficient of t in the exponential function  $y_0 e^{kt}$ . For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is k = 0.02 and the population grows at a rate of 2% per year. If the population at time 0 is  $P_0$ , then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

**EXAMPLE I** Assuming that the growth rate is proportional to population size, use the data in Table 1 to model the population of the world in the 20th century. What is the relative growth rate? How well does the model fit the data?

**SOLUTION** We measure the time t in years and let t = 0 in the year 1900. We measure the population P(t) in millions of people. Then the initial condition is P(0) = 1650. We are assuming that the growth rate is proportional to population size, so the initial-value problem is

$$\frac{dP}{dt} = kP \qquad P(0) = 1650$$

From (2) we know that the solution is

$$P(t) = 1650e^{kt}$$

One way to estimate the relative growth rate k is to use the fact that the population in 1910 was 1750 million. Therefore

$$P(10) = 1650e^{k(10)} = 1750$$

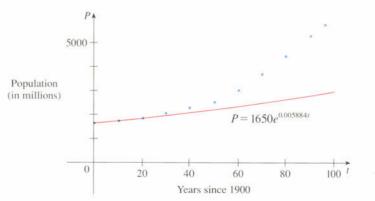
We solve this equation for k:

$$e^{10k} = \frac{1750}{1650}$$
$$k = \frac{1}{10} \ln \frac{1750}{1650} \approx 0.005884$$

Thus, the relative growth rate is about 0.6% per year and the model becomes

$$P(t) = 1650e^{0.005884t}$$

Table 2 and Figure 1 allow us to compare the predictions of this model with the actual data. You can see that the predictions become quite inaccurate after about 30 years and they underestimate by a factor of almost 2 in 1990.



Year	Populatio (millions			
1900	1650			
1910	1750			
1920	1860			
1930	2070			
194()	2300			
1950	2520			
1960	3020			
1970	3700			
1980	4450			
1990	5300			
1996	5770			

TABLE 2

Year	Model	Population		
1900	1650	1650		
1910	1750	1750		
1920	1856	1860		
1930	1969	2070		
1940	2088	2300		
1950	2214	2520		
1960	2349	3020		
1970	2491	3700		
1980	2642	4450		
1990	2802	5300		
1996	2903	5770		



e

Another possibility for estimating k would be to use the given population for 1950, for instance, instead of 1910. Then

$$P(50) = 1650e^{50k} = 2520$$
$$k = \frac{1}{50} \ln \frac{2520}{1650} \approx 0.008470$$

In Section 1.7 we modeled the same data with an exponential function, but there we used the method of least squares.

The estimate for the relative growth rate is now 0.85% per year and the model is

$$P(t) = 1650e^{0.00847t}$$

The predictions with this second model are shown in Table 3 and Figure 2. This exponential model is more accurate over a longer period of time, but it too lags behind reality in recent years.

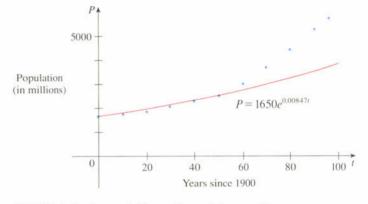


FIGURE 2 Another model for world population growth

**EXAMPLE 2** Use the data in Table 1 to model the population of the world in the second half of the 20th century. Use the model to estimate the population in 1993 and to predict the population in the year 2010.

**SOLUTION** Here we let t = 0 in the year 1950. Then the initial-value problem is

$$\frac{dP}{dt} = kP \qquad P(0) = 2520$$

and the solution is

$$P(t) = 2520e^{kt}$$

Let's estimate k by using the population in 1960:

$$P(10) = 2520e^{10k} = 3020$$

$$k = \frac{1}{10} \ln \frac{3020}{2520} \approx 0.018100$$

The relative growth rate is about 1.8% per year and the model is

$$P(t) = 2520e^{0.0181t}$$

-		
	Model	Pop

Year

TABLE 3

1900	1650	1650	
1910	1796	1750	
1920	1955	1860	1.0
1930	2127	2070	
1940	2315	2300	
1950	2520	2520	
1960	2743	3020	
1970	2985	3700	
1980	3249	4450	
1990	3536	5300	
1996	3721	5770	

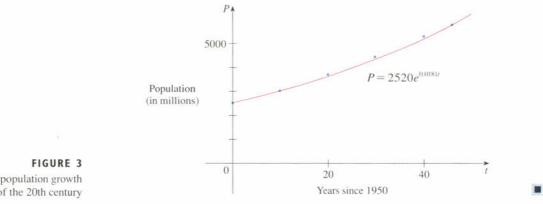
We estimate that the world population in 1993 was

$$P(43) = 2520e^{0.0181(43)} \approx 5488$$
 million

The model predicts that the population in 2010 will be

$$P(60) = 2520e^{0.0181(60)} \approx 7465$$
 million

The graph in Figure 3 shows that the model is fairly accurate to date, so the estimate for 1993 is quite reliable. But the prediction for 2010 is riskier.



A model for world population growth in the second half of the 20th century

## **Radioactive Decay**

Radioactive substances decay by spontaneously emitting radiation. If m(t) is the mass remaining from an initial mass  $m_0$  of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use (2) to show that the mass decays exponentially:

$$m(t) = m_0 e^{k_0}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

**EXAMPLE 3** The half-life of radium-226  $\binom{226}{88}$ Ra) is 1590 years. (a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of  $\binom{226}{88}$ Ra that remains after *t* years.

- (b) Find the mass after 1000 years correct to the nearest milligram.
- (c) When will the mass be reduced to 30 mg?

#### SOLUTION

(a) Let m(t) be the mass of radium-226 (in milligrams) that remains after t years. Then dm/dt = km and y(0) = 100, so (2) gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k, we use the fact that  $y(1590) = \frac{1}{2}(100)$ . Thus  $100e^{1590k} = 50$  so  $e^{1590k} = \frac{1}{2}$ 

and

$$1590k = \ln \frac{1}{2} = -\ln 2$$

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$$k = -\frac{\ln 2}{1590}$$

Therefore  $m(t) = 100e^{-(\ln 2/1590)t}$ 

We could use the fact that  $e^{\ln 2} = 2$  to write the expression for m(t) in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2/1590)1000} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that m(t) = 30, that is,

$$100e^{-(\ln 2/1590)t} = 30$$
 or  $e^{-(\ln 2/1590)t} = 0.3$ 

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590} t = \ln 0.3$$

Thus

$$t = -1590 \frac{\ln 0.3}{\ln 2} \approx 2762$$
 years

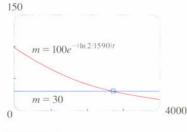
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As a check on our work in Example 3, we use a graphing device to draw the graph of m(t) in Figure 4 together with the horizontal line m = 30. These curves

intersect when  $t \approx 2800$ , and this agrees with the answer to part (c).

## **Continuously Compounded Interest**

**EXAMPLE 4** If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth \$1000(1.06) = \$1060, after 2 years it's worth \$[1000(1.06)]1.06 = \$1123.60, and after t years it's worth \$1000(1.06)'. In general, if an amount  $A_0$  is invested at an interest rate r (r = 0.06 in this example), then after t years it's worth  $A_0(1 + r)'$ . Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years,





#### SECTION 7.5 EXPONENTIAL GROWTH AND DECAY

so the value of the investment is

$$A_0\left(1+\frac{r}{n}\right)^n$$

For instance, after 3 years at 6% interest a \$1000 investment will be worth

$$\$1000(1.06)^3 = \$1191.02$$
 with annual compounding  
 $\$1000(1.03)^6 = \$1194.05$  with semiannual compounding  
 $\$1000(1.015)^{12} = \$1195.62$  with quarterly compounding  
 $\$1000(1.005)^{36} = \$1196.68$  with monthly compounding  
 $\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20$  with daily compounding

You can see that the interest paid increases as the number of compounding periods (*n*) increases. If we let  $n \to \infty$ , then we will be compounding the interest *continuously* and the value of the investment will be

$$A(t) = \lim_{n \to \infty} A_0 \left( 1 + \frac{r}{n} \right)^{nt} = \lim_{n \to \infty} A_0 \left[ \left( 1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$
$$= A_0 \left[ \lim_{n \to \infty} \left( 1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$
$$= A_0 \left[ \lim_{m \to \infty} \left( 1 + \frac{1}{m} \right)^m \right]^{rt} \quad \text{(where } m = n/r)$$

But the limit in this expression is equal to the number e (see Equation 6 in Section 3.7). So with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Returning to the example of \$1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

$$\begin{aligned} \mathbf{A}(3) &= \$1000e^{(0.06)3} \\ &= \$1000e^{0.18} = \$1197.22 \end{aligned}$$

Notice how close this is to the amount we calculated for daily compounding, \$1197.20. But the amount is easier to compute if we use continuous compounding.

Exercises · · ·



- A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.
- 2. A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 100 cells.
  - (a) Find the relative growth rate.
  - (b) Find an expression for the number of cells after *t* hours.
  - (c) Find the number of cells after 10 hours.
  - (d) When will the population reach 10,000 cells?
- **3.** A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria.
  - (a) Find an expression for the number of bacteria after *t* hours.
  - (b) Find the number of bacteria after 4 hours.
  - (c) When will the population reach 30,000?
- **4.** A bacteria culture grows with constant relative growth rate. The count was 400 after 2 hours and 25,600 after 6 hours.
  - (a) What was the initial population of the culture?
  - (b) Find an expression for the population after t hours.
  - (c) In what period of time does the population double?
  - (d) When will the population reach 100,000?
- **5.** The table gives estimates of the world population, in millions, over two centuries:

Year	1750	1800	1850	1900	1950
Population	728	906	1171	1608	2517

- (a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
- (b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
- (c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 1992. Compare with the actual 1992 population of 5.4 billion and try to explain the discrepancy.

**6.** The table gives the population of the United States, in millions, for the years 1900–1990.

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Year	Population
1900	76
1910	92
1920	106
1930	123
1940	131
1950	150
1960	179
1970	203
1980	227
1990	250

- (a) Use the exponential model and the census figures for 1900 and 1910 to predict the population in 1990. Compare with the actual figure and try to explain the discrepancy.
- (b) Use the exponential model and the census figures for 1970 and 1980 to predict the population in 1990. Compare with the actual population. Then use this model to predict the population in the years 2000 and 2010.
- (c) Draw a graph showing both of the exponential functions in parts (a) and (b) together with a plot of the actual population. Are these models reasonable ones?
- 7. Experiments show that if the chemical reaction

 $N_2O_5 \rightarrow 2NO_2 + \frac{1}{2}O_2$ 

takes place at 45 °C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

(See Example 4 in Section 3.3.)

- (a) Find an expression for the concentration  $[N_2O_5]$  after *t* seconds if the initial concentration is *C*.
- (b) How long will the reaction take to reduce the concentration of  $N_2O_5$  to 90% of its original value?
- 8. Polonium-210 has a half-life of 140 days.
  - (a) If a sample has a mass of 200 mg, find a formula for the mass that remains after *t* days.
  - (b) Find the mass after 100 days.
  - (c) When will the mass be reduced to 10 mg?
  - (d) Sketch the graph of the mass function.
- **9.** Polonium-214 has a very short half-life of  $1.4 \times 10^{-4}$  s.
  - (a) If a sample has a mass of 50 mg, find a formula for the mass that remains after *t* seconds.

- (b) Find the mass that remains after a hundredth of a second.
- (c) How long would it take for the mass to decay to 40 mg?
- **10.** After 3 days a sample of radon-222 decayed to 58% of its original amount.
  - (a) What is the half-life of radon-222?
  - (b) How long would it take the sample to decay to 10% of its original amount?
- 11. Scientists can determine the age of ancient objects by a method called *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, <sup>14</sup>C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates <sup>14</sup>C through food chains. When a plant or animal dies it stops replacing its carbon and the amount of <sup>14</sup>C begins to decrease through radioactive decay. Therefore, the level of radioactivity must also decay exponentially. A parchment fragment was discovered that had about 74% as much <sup>14</sup>C radioactivity as does plant material on Earth today. Estimate the age of the parchment.
- **12.** A curve passes through the point (0, 5) and has the property that the slope of the curve at every point *P* is twice the *y*-coordinate of *P*. What is the equation of the curve?
- 13. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. Suppose that a roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. If u(t) is the temperature of the turkey after t minutes, then Newton's Law of Cooling implies that

$$\frac{du}{dt} = k(u - 75)$$

This could be solved as a separable differential equation. Another method is to make the change of variable y = u - 75.

- (a) What initial-value problem does the new function y satisfy? What is the solution?
- (b) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 min?
- (c) When will the turkey have cooled to 100°F?
- 14. A thermometer is taken from a room where the temperature is 20 °C to the outdoors, where the temperature is 5 °C. After one minute the thermometer reads 12 °C. Use Newton's Law of Cooling to answer the following questions.
  - (a) What will the reading on the thermometer be after one more minute?

(b) When will the thermometer read  $6^{\circ}$ C?

- 15. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15 °C the pressure is
  - 101.3 kPa at sea level and 87.14 kPa at h = 1000 m.
  - (a) What is the pressure at an altitude of 3000 m?
  - (b) What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?
- 16. (a) If \$500 is borrowed at 14% interest, find the amounts due at the end of 2 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) daily, (v) hourly, and (vi) continuously.
- (b) Suppose \$500 is borrowed and the interest is compounded continuously. If A(t) is the amount due after *t* years, where  $0 \le t \le 2$ , graph A(t) for each of the interest rates 14%, 10%, and 6% on a common screen.

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- 17. (a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.
  - (b) If A(t) is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by A(t).
- **18.** How long will it take an investment to double in value if the interest rate is 6% compounded continuously?
- 19. Consider a population P = P(t) with constant relative birth and death rates  $\alpha$  and  $\beta$ , respectively, and a constant emigration rate *m*, where  $\alpha$ ,  $\beta$ , and *m* are positive constants. Assume that  $\alpha > \beta$ . Then the rate of change of the population at time *t* is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad \text{where } k = \alpha - \beta$$

- (a) Find the solution of this equation that satisfies the initial condition  $P(0) = P_0$ .
- (b) What condition on *m* will lead to an exponential expansion of the population?
- (c) What condition on *m* will result in a constant population? A population decline?
- (d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

**20.** Let *c* be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{\dagger}$$

where k is a positive constant, is called a *doomsday* equation because the exponent in the expression  $ky^{1+c}$  is larger than that for natural growth (that is, ky).

- (a) Determine the solution that satisfies the initial condition y(0) = y<sub>0</sub>.
- (b) Show that there is a finite time t = T such that  $\lim_{t \to T^{-}} y(t) = \infty$ .
- (c) An especially prolific breed of rabbits has the growth term  $ky^{1.01}$ . If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

# Applied Project

## Calculus and Baseball

In this project we explore three of the many applications of calculus to baseball. The physical interactions of the game, especially the collision of ball and bat, are quite complex and their models are discussed in detail in a book by Robert Adair, *The Physics of Baseball* (New York: Harper and Row, 1990).

1. It may surprise you to learn that the collision of baseball and bat lasts only about a thousandth of a second. Here we calculate the average force on the bat during this collision by first computing the change in the ball's momentum.

The momentum p of an object is the product of its mass m and its velocity v, that is, p = mv. Suppose an object, moving along a straight line, is acted on by a force F = F(t) that is a continuous function of time.

(a) Show that the change in momentum over a time interval [t<sub>0</sub>, t<sub>1</sub>] is equal to the integral of F from t<sub>0</sub> to t<sub>1</sub>; that is, show that

$$p(t_1) - p(t_0) = \int_{t_0}^{t_1} F(t) dt$$

This integral is called the *impulse* of the force over the time interval.

- (b) A pitcher throws a 90-mi/h fastball to a batter, who hits a line drive directly back to the pitcher. The ball is in contact with the bat for 0.001 s and leaves the bat with velocity 110 mi/h. A baseball weighs 5 oz and, in U.S. Customary units, its mass is measured in slugs: m = w/g, where g = 32 ft/s<sup>2</sup>.
  - (i) Find the change in the ball's momentum.
  - (ii) Find the average force on the bat.
- **2.** In this problem we calculate the work required for a pitcher to throw a 90-mi/h fastball by first considering kinetic energy.

The *kinetic energy K* of an object of mass *m* and velocity *v* is given by  $K = \frac{1}{2}mv^2$ . Suppose an object of mass *m*, moving in a straight line, is acted on by a force F = F(s) that depends on its position *s*. According to Newton's Second Law

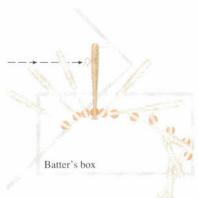
$$F(s) = ma = m \, \frac{dv}{dt}$$

where a and v denote the acceleration and velocity of the object.

(a) Show that the work done in moving the object from a position  $s_0$  to a position  $s_1$  is equal to the change in the object's kinetic energy; that is, show that

$$W = \int_{s_0}^{s_1} F(s) \, ds = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2$$

where  $v_0 = v(s_0)$  and  $v_1 = v(s_1)$  are the velocities of the object at the positions



An overhead view of the position of a baseball bat, shown every fiftieth of a second during a typical swing. (Adapted from *The Physics of Baseball*)

 $s_0$  and  $s_1$ . *Hint:* By the Chain Rule,

$$m \frac{dv}{dt} = m \frac{dv}{ds} \frac{ds}{dt} = mv \frac{dv}{ds}$$

- (b) How many foot-pounds of work does it take to throw a baseball at a speed of 90 mi/h?
- 3. (a) An outfielder fields a baseball 280 ft away from home plate and throws it directly to the catcher with an initial velocity of 100 ft/s. Assume that the velocity v(t) of the ball after t seconds satisfies the differential equation dv/dt = -v/10 because of air resistance. How long does it take for the ball to reach home plate? (Ignore any vertical motion of the ball.)
  - (b) The manager of the team wonders whether the ball will reach home plate sooner if it is relayed by an infielder. The shortstop can position himself directly between the outfielder and home plate, catch the ball thrown by the outfielder, turn, and throw the ball to the catcher with an initial velocity of 105 ft/s. The manager clocks the relay time of the shortstop (catching, turning, throwing) at half a second. How far from home plate should the shortstop position himself to minimize the total time for the ball to reach the plate? Should the manager encourage a direct throw or a relayed throw? What if the shortstop can throw at 115 ft/s?
- (c) For what throwing velocity of the shortstop does a relayed throw take the same time as a direct throw?



#### The Logistic Equation

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In this section we discuss in detail a model for population growth, the logistic model, that is more sophisticated than exponential growth. In doing so we use all the tools at our disposal—direction fields from Section 7.2, Euler's method from Section 7.3, and the explicit solution of separable differential equations from Section 7.4. In the exercises we investigate other possible models for population growth, some of which take into account harvesting and seasonal growth.

#### The Logistic Model

As we discussed in Section 7.1, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If P(t) is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** K, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incor-

porates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right)$$

Multiplying by *P*, we obtain the model for population growth known as the **logistic differential equation**:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

Notice from Equation 1 that if *P* is small compared with *K*, then *P*/*K* is close to 0 and so  $dP/dt \approx kP$ . But if  $P \rightarrow K$  (the population approaches its carrying capacity), then  $P/K \rightarrow 1$ , so  $dP/dt \rightarrow 0$ . We can deduce information about whether solutions increase or decrease directly from Equation 1. If the population *P* lies between 0 and *K*, then the right side of the equation is positive, so dP/dt > 0 and the population increases. But if the population exceeds the carrying capacity (P > K), then 1 - P/K is negative, so dP/dt < 0 and the population decreases.

#### **Direction Fields**

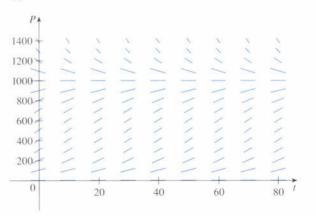
Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

**EXAMPLE 1** Draw a direction field for the logistic equation with k = 0.08 and carrying capacity K = 1000. What can you deduce about the solutions?

SOLUTION In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P \bigg( 1 - \frac{P}{1000} \bigg)$$

A direction field for this equation is shown in Figure 1. We show only the first quadrant because negative populations aren't meaningful and we are interested only in what happens after t = 0.



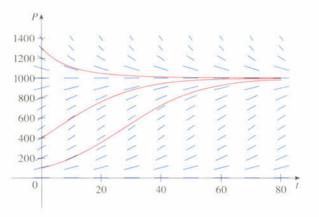
#### FIGURE I

Direction field for the logistic equation in Example 1

The logistic equation is autonomous (dP/dt depends only on *P*, not on *t*), so the slopes are the same along any horizontal line. As expected, the slopes are positive for 0 < P < 1000 and negative for P > 1000.

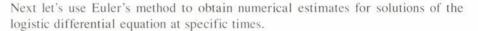
The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution P = 0 and move toward the equilibrium solution P = 1000.

In Figure 2 we use the direction field to sketch solution curves with initial populations P(0) = 100, P(0) = 400, and P(0) = 1300. Notice that solution curves that start below P = 1000 are increasing and those that start above P = 1000 are decreasing. The slopes are greatest when  $P \approx 500$  and, therefore, the solution curves that start below P = 1000 have inflection points when  $P \approx 500$ . In fact we can prove that all solution curves that start below P = 500 have an inflection point when P is exactly 500 (see Exercise 9).





**Euler's Method** 



**EXAMPLE 2** Use Euler's method with step sizes 20, 10, 5, 1, and 0.1 to estimate the population sizes P(40) and P(80), where P is the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P \left( 1 - \frac{P}{1000} \right) \qquad P(0) = 100$$

**SOLUTION** With step size h = 20,  $t_0 = 0$ ,  $P_0 = 100$ , and

$$F(t,P) = 0.08P \left(1 - \frac{P}{1000}\right)$$

we get, using the notation of Section 7.3,

$$P_{1} = 100 + 20F(0, 100) = 244$$

$$P_{2} = 244 + 20F(20, 244) \approx 539.14$$

$$P_{3} = 539.14 + 20F(40, 539.14) \approx 936.69$$

$$P_{4} = 936.69 + 20F(60, 936.69) \approx 1031.57$$

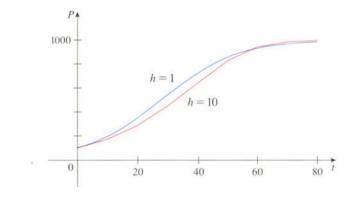
Thus, our estimates for the population sizes at times t = 40 and t = 80 are

$$P(40) \approx 539$$
  $P(80) \approx 1032$ 

For smaller step sizes we need to program a calculator or computer. The table gives the results.

Step size	Euler estimate of $P(40)$	Euler estimate of $P(80)$			
20	539	1032			
10	647	997			
5	695	991			
1	725	986			
0.1	731	985			

Figure 3 shows a graph of the Euler approximations with step sizes h = 10 and h = 1. We see that the Euler approximation with h = 1 looks very much like the lower solution curve that we drew using a direction field in Figure 2.



#### FIGURE 3

Euler approximations of the solution curve in Example 2

The Analytic Solution

The logistic equation (1) is separable and so we can solve it explicitly using the method of Section 7.4. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

we have

$$\int \frac{dP}{P(1-P/K)} = \int k \, dt$$

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/K)} = \frac{K}{P(K - P)}$$

We can further simplify this expression by writing

 $\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$ 

The method of partial fractions (Appendix F) gives a systematic approach to problems of this type.

as you can verify by combining the fractions on the right-hand side. This enables us to rewrite Equation 2:

$$\int \left(\frac{1}{P} + \frac{1}{K - P}\right) dP = \int k \, dt$$
$$\ln|P| - \ln|K - P| = kt + C$$
$$\ln\left|\frac{K - P}{P}\right| = -kt - C$$
$$\left|\frac{K - P}{P}\right| = e^{-kt - C} = e^{-C}e^{-kt}$$
$$\frac{K - P}{P} = Ae^{-kt}$$

3

SO

Thus

where  $A = \pm e^{-C}$ . Solving Equation 3 for *P*, we get

$$\frac{K}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{K} = \frac{1}{1 + Ae^{-kt}}$$
$$P = \frac{K}{1 + Ae^{-kt}}$$

We find the value of A by putting t = 0 in Equation 3. If t = 0, then  $P = P_0$  (the initial population), so

$$\frac{K - P_0}{P_0} = Ae^0 = A$$

Thus, the solution to the logistic equation is

4 
$$P(t) = \frac{K}{1 + Ae^{-kt}}$$
 where  $A = \frac{K - P_0}{P_0}$ 

**EXAMPLE 3** Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P \left( 1 - \frac{P}{1000} \right) \qquad P(0) = 100$$

and use it to find the population sizes P(40) and P(80). At what time does the population reach 900?

**SOLUTION** The differential equation is a logistic equation with k = 0.08, carrying capacity K = 1000, and initial population  $P_0 = 100$ . So Equation 4 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \quad \text{where } A = \frac{1000 - 100}{100} = 9$$
$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

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So the population sizes when t = 40 and 80 are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}} \approx 731.6 \qquad P(80) = \frac{1000}{1 + 9e^{-6.4}} \approx 985.3$$

Compare these values with the Euler estimates from Example 2:

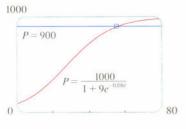
 $P(40) \approx 731 \qquad P(80) \approx 985$ 

The population reaches 900 when

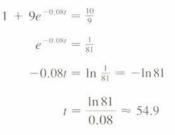
$$\frac{1000}{1+9e^{-0.08t}} = 900$$

Solving this equation for t, we get

Compare the solution curve in Figure 4 with the lowest solution curve we drew from the direction field in Figure 2.







So the population reaches 900 when t is approximately 55. As a check on our work, we graph the population curve in Figure 4 and observe where it intersects the line P = 900. The cursor indicates that  $t \approx 55$ .

#### Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G.F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

**EXAMPLE 4** Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit.

**SOLUTION** Given the relative growth rate k = 0.7944 and the initial population  $P_0 = 2$ , the exponential model is

$$P(t) = P_0 e^{kt} = 2e^{0.7944t}$$

Gause used the same value of k for his logistic model. [This is reasonable because  $P_0 = 2$  is small compared with the carrying capacity (K = 64). The equation

$$\frac{1}{P_0} \left. \frac{dP}{dt} \right|_{t=0} = k \left( 1 - \frac{2}{64} \right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.]

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Then the solution of the logistic equation in Equation 4 gives

$$P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}}$$
$$A = \frac{K - P_0}{P_0} = \frac{64 - 2}{2} = 31$$

where

So

 $P(t) = \frac{64}{1 + 31e^{-0.7944t}}$ 

t (days)	0	1	2	3	-4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	aaa										

We notice from the table and from the graph in Figure 5 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model. For  $t \ge 5$ , however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.

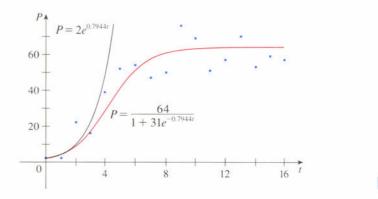


FIGURE 5 The exponential and logistic models for the *Paramecium* data



## Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth. In Exercise 14 we look at the Gompertz growth function and in Exercises 15 and 16 we investigate seasonal-growth models.

Two of the other models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) - c$$

has been used to model populations that are subject to "harvesting" of one sort or

another. (Think of a population of fish being caught at a constant rate). This equation is explored in Exercises 11 and 12.

For some species there is a minimum population level m below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, 1 - m/P, takes into account the consequences of a sparse population. (See Exercise 13.)

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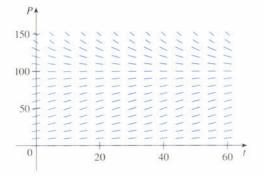
Exercises · · · · ·

1. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

- (a) What is the carrying capacity? What is the value of *k*?
- (b) A direction field for this equation is shown. Where are the slopes close to 0? Where are they largest? Which solutions are increasing? Which solutions are decreasing?



- (c) Use the direction field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At what population levels do they occur?
- (d) What are the equilibrium solutions? How are the other solutions related to these solutions?

Suppose that a population grows according to a logistic model with carrying capacity 6000 and k = 0.0015 per year.

- (a) Write the logistic differential equation for these data.
- (b) Draw a direction field (either by hand or with a computer algebra system). What does it tell you about the solution curves?

- (c) Use the direction field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of these curves? What is the significance of the inflection points?
- (d) Program a calculator or computer to use Euler's method with step size h = 1 to estimate the population after 50 years if the initial population is 1000.
- (e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).
- (f) Graph the solution in part (e) and compare with the solution curve you sketched in part (c).
- **3.** The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky \bigg( 1 - \frac{y}{K} \bigg)$$

where y(t) is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be  $K = 8 \times 10^7$  kg, and k = 0.71 per year.

- (a) If y(0) = 2 × 10<sup>7</sup> kg, find the biomass a year later.
  (b) How long will it take for the biomass to reach 4 × 10<sup>7</sup> kg?
- **4.** The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
()	18	10	509
2	39	12	597
4	-80	14	640
6	171	16	664
8	3.36	18	672

- (a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.
- (b) Use the data to estimate the initial relative growth rate.
- (c) Find both an exponential model and a logistic model for these data.
- (d) Compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.
- (e) Use your logistic model to estimate the number of yeast cells after 7 hours.
- 5. The population of the world was about 5.3 billion in 1990. Birth rates in the 1990s range from 35 to 40 million per year and death rate rates from 15 to 20 million per year. Let's assume that the carrying capacity for world population is 100 billion.
  - (a) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)
  - (b) Use the logistic model to predict the world population in the years 2000, 2100, and 2500.
  - (c) What are your predictions if the carrying capacity is 50 billion?
- 6. (a) Make a guess as to the carrying capacity for the U.S. population. Use it and the fact that the population was 228 million in 1980 to formulate a logistic model for the U.S. population.
  - (b) Determine the value of k in your model by using the fact that the population in 1990 was 250 million.
  - (c) Use your model to predict the U.S. population in the years 2100 and 2200.
  - (d) Use your model to predict the year in which the U.S. population will exceed 300 million.
- **7.** One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction *y* of the population who have heard the rumor and the fraction who have not heard the rumor.
  - (a) Write a differential equation that is satisfied by y.
  - (b) Solve the differential equation.
  - (c) A small town has 1000 inhabitants. At 8 A.M., 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?
- **8.** Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.
  - (a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.
  - (b) How long will it take for the population to increase to 5000?

**9.** (a) Show that if *P* satisfies the logistic equation (1), then

$$\frac{d^2P}{dt^2} = k^2 P \left(1 - \frac{P}{K}\right) \left(1 - \frac{2P}{K}\right)$$

- (b) Deduce that a population grows fastest when it reaches half its carrying capacity.
- 10. For a fixed value of K (say K = 10), the family of logistic functions given by Equation 4 depends on the initial value  $P_0$  and the proportionality constant k. Graph several members of this family. How does the graph change when  $P_0$  varies? How does it change when k varies?
  - **11.** Let's modify the logistic differential equation of Example 1 as follows:

$$\frac{dP}{dt} = 0.08P \left(1 - \frac{P}{1000}\right) - 15$$

- (a) Suppose P(t) represents a fish population at time *t*, where *t* is measured in weeks. Explain the meaning of the term -15.
- (b) Draw a direction field for this differential equation.
- (c) What are the equilibrium solutions?
- (d) Use the direction field to sketch several solution curves. Describe what happens to the fish population for various initial populations.
- (e) Solve this differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial populations 200 and 300. Graph the solutions and compare with your sketches in part (d).
- **CAS** 12. Consider the differential equation

CAS

$$\frac{dP}{dt} = 0.08P \left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population, where t is measured in weeks and c is a constant.

- (a) Use a CAS to draw direction fields for various values of *c*.
- (b) From your direction fields in part (a), determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?
- (c) Use the differential equation to prove what you discovered graphically in part (b).
- (d) What would you recommend for a limit to the weekly catch of this fish population?
- 13. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor (1 m/P). Thus, the modified logistic model

is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right)$$

- (a) Use the differential equation to show that any solution is increasing if m < P < K and decreasing if 0 < P < m.</li>
- (b) For the case where k = 0.08, K = 1000, and m = 200, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?
- (c) Solve the differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial population  $P_0$ .
- (d) Use the solution in part (c) to show that if  $P_0 < m$ , then the species will become extinct. [*Hint:* Show that the numerator in your expression for P(t) is 0 for some value of *t*.]
- **14.** Another model for a growth function for a limited population is given by the **Gompertz function**, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{K}{P}\right) P$$

where c is a constant and K is the carrying capacity.

- (a) Solve this differential equation.
- (b) Compute  $\lim_{t \to \infty} P(t)$ .
- (c) Graph the Gompertz growth function for K = 1000,  $P_0 = 100$ , and c = 0.05, and compare it with the

logistic function in Example 3. What are the similarities? What are the differences?

- (d) We know from Exercise 9 that the logistic function grows fastest when P = K/2. Use the Gompertz differential equation to show that the Gompertz function grows fastest when P = K/e.
- **15.** In a **seasonal-growth model**, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food.
  - (a) Find the solution of the seasonal-growth model

$$\frac{dP}{dt} = kP\cos(rt - \phi) \qquad P(0) = P_0$$

where k, r, and  $\phi$  are positive constants.

- (b) By graphing the solution for several values of k, r, and  $\phi$ , explain how the values of k, r, and  $\phi$  affect the solution. What can you say about  $\lim_{t\to\infty} P(t)$ ?
- **16.** Suppose we alter the differential equation in Exercise 15 as follows:

$$\frac{dP}{dt} = kP\cos^2(rt - \phi) \qquad P(0) = P_0$$

- (a) Solve this differential equation with the help of a table of integrals or a CAS.
- (b) Graph the solution for several values of k, r, and  $\phi$ . How do the values of k, r, and  $\phi$  affect the solution? What can you say about  $\lim_{t\to\infty} P(t)$  in this case?



# Predator-Prey Systems

earrow

We have looked at a variety of models for the growth of a single species that lives alone in an environment. In this section we consider more realistic models that take into account the interaction of two species in the same habitat. We will see that these models take the form of a pair of linked differential equations.

We first consider the situation in which one species, called the *prey*, has an ample food supply and the second species, called the *predators*, feeds on the prey. Examples of prey and predators include rabbits and wolves in an isolated forest, food fish and sharks, aphids and ladybugs, and bacteria and amoebas. Our model will have two dependent variables and both are functions of time. We let R(t) be the number of prey (using *R* for rabbits) and W(t) be the number of predators (with *W* for wolves) at time *t*.

In the absence of predators, the ample food supply would support exponential growth of the prey, that is,

$$\frac{dR}{dt} = kR$$
 where k is a positive constant

In the absence of prey, we assume that the predator population would decline at a

rate proportional to itself, that is,

$$\frac{dW}{dt} = -rW \qquad \text{where } r \text{ is a positive constant}$$

With both species present, however, we assume that the principal cause of death among the prey is being eaten by a predator, and the birth and survival rates of the predators depend on their available food supply, namely, the prey. We also assume that the two species encounter each other at a rate that is proportional to both populations and is therefore proportional to the product *RW*. (The more there are of either population, the more encounters there are likely to be.) A system of two differential equations that incorporates these assumptions is as follows:

W represents the predator. R represents the prey.

The Lotka-Volterra equations were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).

$$\frac{dR}{dt} = kR - aRW$$
$$\frac{dW}{dt} = -rW + bRW$$

10

where k, r, a, and b are positive constants. Notice that the term -aRW decreases the natural growth rate of the prey and the term bRW increases the natural growth rate of the predators.

The equations in (1) are known as the **predator-prey equations**, or the **Lotka-Volterra equations**. A **solution** of this system of equations is a pair of functions R(t) and W(t) that describe the populations of prey and predator as functions of time. Because the system is coupled (*R* and *W* occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for *R* and *W* as functions of *t*. We can, however, use graphical methods to analyze the equations.

**EXAMPLE 1** Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations (1) with k = 0.08, a = 0.001, r = 0.02, and b = 0.00002.

(a) Find the constant solutions (called the **equilibrium solutions**) and interpret the answer.

(b) Use the system of differential equations to find an expression for dW/dR.

(c) Draw a direction field for the resulting differential equation in the *RW*-plane. Then use that direction field to sketch some solution curves.

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to make sketches of R and W as functions of t.

#### SOLUTION

(a) With the given values of k, a, r, and b, the Lotka-Volterra equations become

$$\frac{dR}{dt} = 0.08R - 0.001RW$$
$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Both R and W will be constant if both derivatives are 0, that is,

$$R' = R(0.08 - 0.001W) = 0$$
$$W' = W(-0.02 + 0.00002R) = 0$$

One solution is given by R = 0 and W = 0. (This makes sense: If there are no rabbits or wolves, the populations are certainly not going to increase.) The other constant solution is

$$W = \frac{0.08}{0.001} = 80 \qquad \qquad R = \frac{0.02}{0.00002} = 1000$$

So the equilibrium populations consist of 80 wolves and 1000 rabbits. This means that 1000 rabbits are just enough to support a constant wolf population of 80. There are neither too many wolves (which would result in fewer rabbits) nor too few wolves (which would result in more rabbits).

(b) We use the Chain Rule to eliminate *t*:

$$\frac{dW}{dt} = \frac{dW}{dR} \frac{dR}{dt}$$

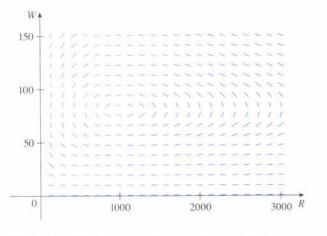
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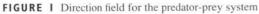
$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(c) If we think of W as a function of R, we have the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

We draw the direction field for this differential equation in Figure 1 and we use it to sketch several solution curves in Figure 2. If we move along a solution





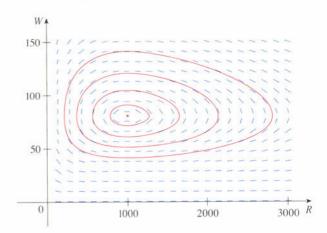


FIGURE 2 Phase portrait of the system

curve, we observe how the relationship between R and W changes as time passes. Notice that the curves appear to be closed in the sense that if we travel along a curve, we always return to the same point. Notice also that the point (1000, 80) is inside all the solution curves. That point is called an *equilibrium point* because it corresponds to the equilibrium solution R = 1000, W = 80.

When we represent solutions of a system of differential equations as in Figure 2, we refer to the *RW*-plane as the **phase plane**, and we call the solution curves **phase trajectories**. So a phase trajectory is a path traced out by solutions (R, W) as time goes by. A **phase portrait** consists of equilibrium points and typical phase trajectories, as shown in Figure 2.

(d) Starting with 1000 rabbits and 40 wolves corresponds to drawing the solution curve through the point  $P_0(1000, 40)$ . Figure 3 shows this phase trajectory with the direction field removed. Starting at the point  $P_0$  at time t = 0 and letting t increase, do we move clockwise or counterclockwise around the phase trajectory? If we put R = 1000 and W = 40 in the first differential equation, we get

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40) = 80 - 40 = 40$$

Since dR/dt > 0, we conclude that R is increasing at  $P_0$  and so we move counterclockwise around the phase trajectory.

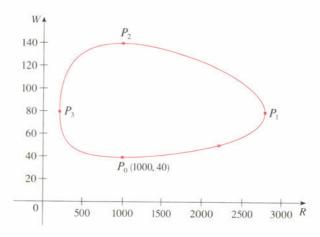
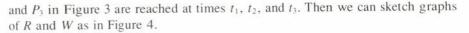
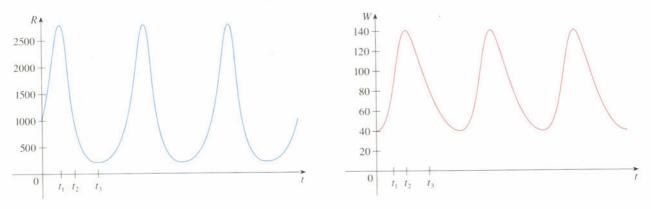


FIGURE 3 Phase trajectory through (1000, 40)

We see that at  $P_0$  there aren't enough wolves to maintain a balance between the populations, so the rabbit population increases. That results in more wolves and eventually there are so many wolves that the rabbits have a hard time avoiding them. So the number of rabbits begins to decline (at  $P_1$ , where we estimate that R reaches its maximum population of about 2800). This means that at some later time the wolf population starts to fall (at  $P_2$ , where R = 1000and  $W \approx 140$ ). But this benefits the rabbits, so their population later starts to increase (at  $P_3$ , where W = 80 and  $R \approx 210$ ). As a consequence, the wolf population eventually starts to increase as well. This happens when the populations return to their initial values of R = 1000 and W = 40, and the entire cycle begins again.

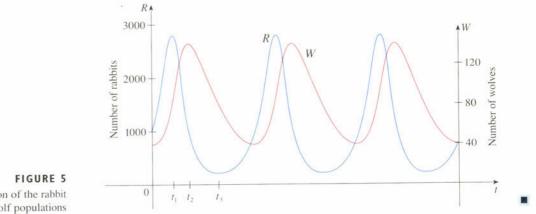
(e) From the description in part (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of R(t) and W(t). Suppose the points  $P_1$ ,  $P_2$ ,







To make the graphs easier to compare, we draw the graphs on the same axes but with different scales for R and W, as in Figure 5. Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves.



Comparison of the rabbit and wolf populations

An important part of the modeling process, as we discussed on page 76, is to interpret our mathematical conclusions as real-world predictions and to test the predictions against real data. The Hudson's Bay Company, which started trading in animal furs in Canada in 1670, has kept records that date back to the 1840s. Figure 6 shows graphs of the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded by the company over a 90-year period. You can see that the coupled oscillations in the hare and lynx populations predicted by the Lotka-Volterra model do actually occur and the period of these cycles is roughly 10 years.

Although the relatively simple Lotka-Volterra model has had some success in explaining and predicting coupled populations, more sophisticated models have also been proposed. One way to modify the Lotka-Volterra equations is to assume that, in the absence of predators, the prey grow according to a logistic model with carrying capacity K. Then the Lotka-Volterra equations (1) are replaced by the sys-

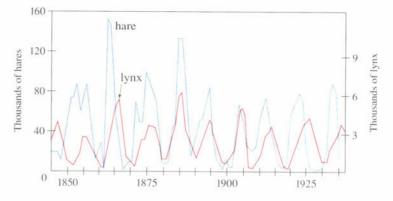


FIGURE 6

Relative abundance of hare and lynx from Hudson's Bay Company records

tem of differential equations

$$\frac{dR}{dt} = kR\left(1 - \frac{R}{K}\right) - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

This model is investigated in Exercises 9 and 10.

Models have also been proposed to describe and predict population levels of two species that compete for the same resources or cooperate for mutual benefit. Such models are explored in Exercise 2.



7 Exercises

1. For each predator-prey system, determine which of the variables, *x* or *y*, represents the prey population and which represents the predator population. Is the growth of the prey restricted just by the predators or by other factors as well? Do the predators feed only on the prey or do they have additional food sources? Explain.

(a) 
$$\frac{dx}{dt} = -0.05x + 0.0001xy$$
  
 $\frac{dy}{dt} = 0.1y - 0.005xy$   
(b)  $\frac{dx}{dt} = 0.2x - 0.0002x^2 - 0.006xy$   
 $\frac{dy}{dt} = -0.015y + 0.00008xy$ 

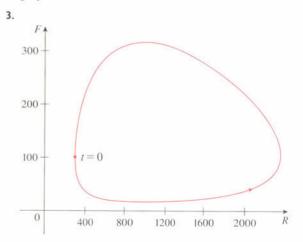
2. Each system of differential equations is a model for two species that either compete for the same resources or cooperate for mutual benefit (flowering plants and insect pollinators, for instance). Decide whether each system describes competition or cooperation and explain why it is a reasonable model. (Ask yourself what effect an increase in one species has on the growth rate of the other.)

(a) 
$$\frac{dx}{dt} = 0.12x - 0.0006x^2 + 0.00001xy$$
  
 $\frac{dy}{dt} = 0.08y + 0.00004xy$ 

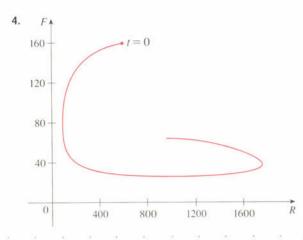
(b) 
$$\frac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$$
  
 $\frac{dy}{dt} = 0.2y - 0.00008y^2 - 0.0002xy$ 

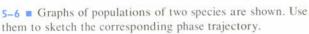
3-4 = A phase trajectory is shown for populations of rabbits (*R*) and foxes (*F*).

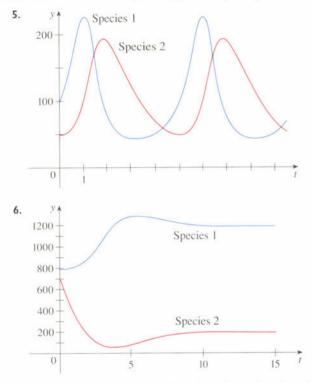
- (a) Describe how each population changes as time goes by.
- (b) Use your description to make a rough sketch of the graphs of *R* and *F* as functions of time.

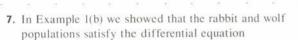


552 CHAPTER 7 DIFFERENTIAL EQUATIONS









$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

By solving this separable differential equation, show that

$$\frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$$

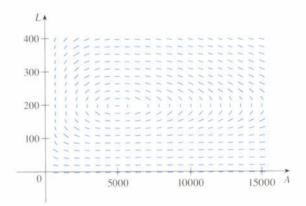
where C is a constant.

It is impossible to solve this equation for W as an explicit function of R (or vice versa). If you have a computer algebra system that graphs implicitly defined curves, use this equation and your CAS to draw the solution curve that passes through the point (1000, 40) and compare with Figure 3.

**8.** Populations of aphids and ladybugs are modeled by the equations

$$\frac{dA}{dt} = 2A - 0.01AL$$
$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

- (a) Find the equilibrium solutions and explain their significance.
- (b) Find an expression for dL/dA.
- (c) The direction field for the differential equation in part (b) is shown. Use it to sketch a phase portrait. What do the phase trajectories have in common?

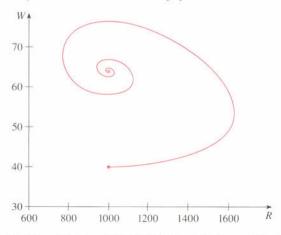


- (d) Suppose that at time t = 0 there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.
- (e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of *t*. How are the graphs related to each other?
- In Example 1 we used Lotka-Volterra equations to model populations of rabbits and wolves. Let's modify those equations as follows:

$$\frac{dR}{dt} = 0.08R(1 - 0.0002R) - 0.001RW$$
$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

- (a) According to these equations, what happens to the rabbit population in the absence of wolves?
- (b) Find all the equilibrium solutions and explain their significance.

(c) The figure shows the phase trajectory that starts at the point (1000, 40). Describe what eventually happens to the rabbit and wolf populations.



- (d) Sketch graphs of the rabbit and wolf populations as functions of time.
- **10.** In Exercise 8 we modeled populations of aphids and ladybugs with a Lotka-Volterra system. Suppose we modify

those equations as follows:

$$\frac{dA}{dt} = 2A(1 - 0.0001A) - 0.01AL$$
$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

- (a) In the absence of ladybugs, what does the model predict about the aphids?
- (b) Find the equilibrium solutions.
- (c) Find an expression for dL/dA.
- (d) Use a computer algebra system to draw a direction field for the differential equation in part (c). Then use the direction field to sketch a phase portrait. What do the phase trajectories have in common?
- (e) Suppose that at time t = 0 there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.
- (f) Use part (e) to make rough sketches of the aphid and ladybug populations as functions of *t*. How are the graphs related to each other?

# Chapter 7 Review

#### CONCEPT CHECK •

- I. (a) What is a differential equation?(b) What is the order of a differential equation?(c) What is an initial condition?
- 2. What can you say about the solutions of the equation  $y' = x^2 + y^2$  just by looking at the differential equation?
- **3.** What is a direction field for the differential equation y' = F(x, y)?
- 4. Explain how Euler's method works.
- **5.** What is a separable differential equation? How do you solve it?

- **6.** (a) Write a differential equation that expresses the law of natural growth.
  - (b) Under what circumstances is this an appropriate model for population growth?
  - (c) What are the solutions of this equation?
- 7. (a) Write the logistic equation.
  - (b) Under what circumstances is this an appropriate model for population growth?
- **8.** (a) Write Lotka-Volterra equations to model populations of food fish (*F*) and sharks (*S*).
  - (b) What do these equations say about each population in the absence of the other?

#### 🔺 TRUE-FALSE QUIZ 🔺

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

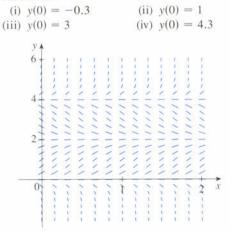
- 1. All solutions of the differential equation  $y' = -y^4$  are decreasing functions.
- 2. The function  $f(x) = (\ln x)/x$  is a solution of the differential equation  $x^2y' + xy = 1$ .
- **3.** The equation y' = x + y is separable.
- 4. The equation y' = 3y 2x + 6xy 1 is separable.
- 5. If y is the solution of the initial-value problem

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right) \qquad y(0) = 1$$

then  $\lim_{t\to\infty} y = 5$ .

#### 🕨 EXERCISES 🤞

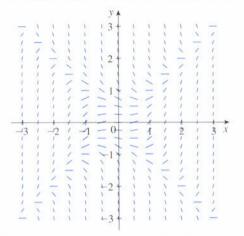
1. (a) A direction field for the differential equation y' = y(y - 2)(y - 4) is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.



- (b) If the initial condition is y(0) = c, for what values of c is  $\lim_{t\to\infty} y(t)$  finite? What are the equilibrium solutions?
- 2. (a) Sketch a direction field for the differential equation y' = x/y. Then use it to sketch the four solutions that satisfy the initial conditions y(0) = 1, y(0) = -1, y(2) = 1, and y(-2) = 1.
  - (b) Check your work in part (a) by solving the differential equation explicitly. What type of curve is each solution curve?
- (a) A direction field for the differential equation
   y' = x<sup>2</sup> y<sup>2</sup> is shown. Sketch the solution of the initial-value problem

 $y' = x^2 - y^2$  y(0) = 1

Use your graph to estimate the value of y(0.3).



(b) Use Euler's method with step size 0.1 to estimate y(0.3), where y(x) is the solution of the initial-value problem in part (a). Compare with your estimate from part (a).

- (c) On what lines are the centers of the horizontal line segments of the direction field in part (a) located? What happens when a solution curve crosses these lines?
- **4.** (a) Use Euler's method with step size 0.2 to estimate y(0.4), where y(x) is the solution of the initial-value problem

$$y' = 2xy^2 \qquad y(0) = 1$$

- (b) Repeat part (a) with step size 0.1.
- (c) Find the exact solution of the differential equation and compare the value at 0.4 with the approximations in parts (a) and (b).
- **5.** Solve the equation  $y' = 2 + 2x^2 + y + x^2y$ .

6-7 ■ Solve the initial-value problem.

**6.** 
$$1 + x = 2xyy', x > 0, y(1) = -2$$

7.  $xyy' = \ln x$ , y(1) = 2

**8.** Solve the initial-value problem  $2yy' = xe^x$ , y(0) = 1, and graph the solution.

**9–10** Find the orthogonal trajectories of the given family of curves.

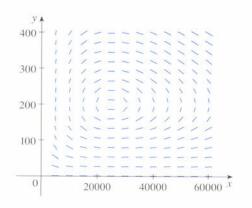
**9.** 
$$kx^2 + y^2 = 1$$
 **10.**  $y = \frac{k}{1 + x^2}$ 

- **11.** A bacteria culture starts with 1000 bacteria and the growth rate is proportional to the number of bacteria. After 2 hours the population is 9000.
  - (a) Find an expression for the number of bacteria after *t* hours.
  - (b) Find the number of bacteria after 3 h.
  - (c) How long does it take for the number of bacteria to double?
- 12. An isotope of strontium, <sup>90</sup>Sr, has a half-life of 25 years.
   (a) Find the mass of <sup>90</sup>Sr that remains from a sample of
  - 18 mg after *t* years.(b) How long would it take for the mass to decay to
  - 2 mg?
- 13. Let C(t) be the concentration of a drug in the bloodstream. As the body eliminates the drug, C(t) decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus, C'(t) = -kC(t), where k is a positive number called the *elimination constant* of the drug.
  - (a) If  $C_0$  is the concentration at time t = 0, find the concentration at time t.
  - (b) If the body eliminates half the drug in 30 h, how long does it take to eliminate 90% of the drug?

- 14. (a) The population of the world was 4.45 billion in 1980 and 5.30 billion in 1990. Find an exponential model for these data and use the model to predict the world population in the year 2020.
  - (b) According to the model in part (a), when will the world population exceed 10 billion?
  - (c) Use the data in part (a) to find a logistic model for the population. Assume a carrying capacity of 100 billion. Then use the logistic model to predict the population in 2020. Compare with your prediction from the exponential model.
  - (d) According to the logistic model, when will the world population exceed 10 billion? Compare with your prediction in part (b).
- 15. The von Bertalanffy growth model is used to predict the length L(t) of a fish over a period of time. If  $L_{\infty}$  is the largest length for a species, then the hypothesis is that the rate of growth in length is proportional to  $L_{\infty} - L$ , the length yet to be achieved.
  - (a) Formulate and solve a differential equation to find an expression for L(t).
  - (b) For the North Sea haddock it has been determined that  $L_x = 53$  cm, L(0) = 10 cm, and the constant of proportionality is 0.2. What does the expression for L(t) become with these data?
- 16. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?
- **17.** One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week. How long does it take for 80% of the population to become infected?
- **18.** Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = 0.4x - 0.002xy \qquad \frac{dy}{dt} = -0.2y + 0.000008xy$$

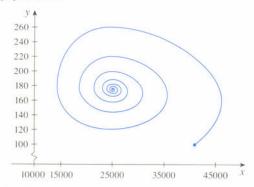
- (a) Which of the variables, x or y, represents the bird population and which represents the insect population? Explain.
- (b) Find the equilibrium solutions and explain their significance.
- (c) Find an expression for dy/dx.
- (d) The direction field for the differential equation in part (c) is shown. Use it to sketch the phase trajectory corresponding to initial populations of 100 birds and 40,000 insects. Then use the phase trajectory to describe how both populations change.



- (e) Use part (d) to make rough sketches of the bird and insect populations as functions of time. How are these graphs related to each other?
- **19.** Suppose the model of Exercise 18 is replaced by the equations

$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.002xy$$
$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

- (a) According to these equations, what happens to the insect population in the absence of birds?
- (b) Find the equilibrium solutions and explain their significance.
- (c) The figure shows the phase trajectory that starts with 100 birds and 40,000 insects. Describe what eventually happens to the bird and insect populations.



- (d) Sketch graphs of the bird and insect populations as functions of time.
- 20. Barbara weighs 60 kg and is on a diet of 1600 calories per day, of which 850 are used automatically by basal metabolism. She spends about 15 cal/kg/day times her weight doing exercise. If 1 kg of fat contains 10,000 cal and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?



I. Find all functions f such that f' is continuous and

$$[f(x)]^{2} = 100 + \int_{0}^{x} \{[f(t)]^{2} + [f'(t)]^{2}\} dt \quad \text{for all real } x$$

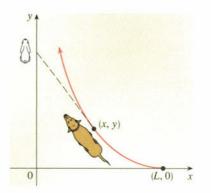
- A student forgot the Product Rule for differentiation and made the mistake of thinking that (fg)' = f'g'. However, he was lucky and got the correct answer. The function f that he used was f(x) = e<sup>x<sup>2</sup></sup> and the domain of his problem was the interval (<sup>1</sup>/<sub>2</sub>,∞). What was the function g?
- Let f be a function with the property that f(0) = 1, f'(0) = 1, and f(a + b) = f(a)f(b) for all real numbers a and b. Show that f'(x) = f(x) for all x and deduce that f(x) = e<sup>x</sup>.
- 4. Find all functions f that satisfy the equation

$$\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}\,dx\right) = -1$$

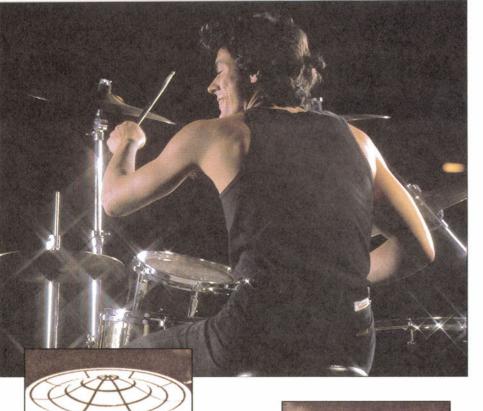
- 5. A planning engineer for a new alum plant must present some estimates to his company regarding the capacity of a silo designed to contain bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 100 ft high with a radius of 200 ft. The conveyor carries  $60,000\pi$  ft<sup>3</sup>/h and the ore maintains a conical shape whose radius is 1.5 times its height.
  - (a) If, at a certain time *t*, the pile is 60 ft high, how long will it take for the pile to reach the top of the silo?
  - (b) Management wants to know how much room will be left in the floor area of the silo when the pile is 60 ft high. How fast is the floor area of the pile growing at that height?
  - (c) Suppose a loader starts removing the ore at the rate of  $20,000\pi$  ft<sup>3</sup>/h when the height of the pile reaches 90 ft. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?
- 6. Snow began to fall during the morning of February 2 and continued steadily into the afternoon. A snowplow began to clear a street at noon, traveling at a constant rate. The plow traveled 6 km from noon to 1 p.m. but only 3 km from 1 p.m. to 2 p.m. When did the snow begin to fall? [*Hints:* To get started, let t be the time measured in hours after noon; let x(t) be the distance traveled by the plow at time t; then the speed of the plow is dx/dt. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m<sup>3</sup>/h) is constant.]
- **7.** A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:
  - (i) The rabbit is at the origin and the dog is at the point (L, 0) at the instant the dog first sees the rabbit.
  - (ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.
  - (iii) The dog runs at the same speed as the rabbit.
  - (a) Show that the dog's path is the graph of the function y = f(x), where y satisfies the differential equation

$$x \frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

- (b) Determine the solution of the equation in part (a) that satisfies the initial conditions y = y' = 0 when x = L. [*Hint*: Let z = dy/dx in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.]
- (c) Does the dog ever catch the rabbit?



- **8.** (a) Suppose that the dog in Problem 7 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.
  - (b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?



8

• • • Bessel functions, which are used to model the vibrations of drumheads and cymbals, are defined as sums of infinite series in Section 8.5.









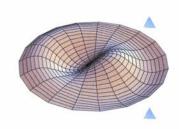


Infinite Sequences and Series

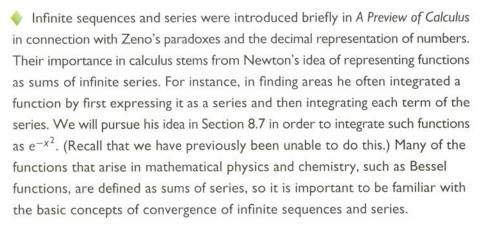
Notice how closely the computergenerated models (which involve Bessel functions and cosine functions) match the photographs of a vibrating rubber membrane.











Physicists also use series in another way, as we will see in Section 8.9. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.







A sequence can be thought of as a list of numbers written in a definite order:

 $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ 

The number  $a_1$  is called the *first term*,  $a_2$  is the *second term*, and in general  $a_n$  is the *nth term*. We will deal exclusively with infinite sequences and so each term  $a_n$  will have a successor  $a_{n+1}$ .

Notice that for every positive integer n there is a corresponding number  $a_n$  and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write  $a_n$  instead of the function notation f(n) for the value of the function at the number n.

**NOTATION** • The sequence  $\{a_1, a_2, a_3, \ldots\}$  is also denoted by

$$\{a_n\}$$
 or  $\{a_n\}_{n=1}^{\infty}$ 

**EXAMPLE 1** Some sequences can be defined by giving a formula for the *n*th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that *n* doesn't have to start at 1.

- (a)  $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$   $a_n = \frac{n}{n+1}$   $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right\}$ (b)  $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$   $a_n = \frac{(-1)^n(n+1)}{3^n}$   $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots\right\}$ (c)  $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$   $a_n = \sqrt{n-3}, n \ge 3$   $\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$
- (d)  $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$   $a_n = \cos\frac{n\pi}{6}, n \ge 0$   $\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos\frac{n\pi}{6}, \dots\right\}$

**EXAMPLE 2** Here are some sequences that don't have a simple defining equation.

(a) The sequence  $\{p_n\}$ , where  $p_n$  is the population of the world as of January 1 in the year *n*.

(b) If we let  $a_n$  be the digit in the *n*th decimal place of the number *e*, then  $\{a_n\}$  is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$$

(c) The **Fibonacci sequence**  $\{f_n\}$  is defined recursively by the conditions

$$f_1 = 1$$
  $f_2 = 1$   $f_n = f_{n-1} + f_{n-2}$   $n \ge 3$ 

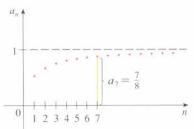
Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 35).







A more precise definition of the limit of a sequence is given in Appendix D.

FIGURE 2

A sequence such as the one in Example 1(a),  $a_n = n/(n + 1)$ , can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1)$$
  $(2, a_2)$   $(3, a_3)$  ...  $(n, a_n)$  ...

From Figure 1 or 2 it appears that the terms of the sequence  $a_n = n/(n + 1)$  are approaching 1 as *n* becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty}a_n=L$$

means that the terms of the sequence  $\{a_n\}$  approach *L* as *n* becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.5.

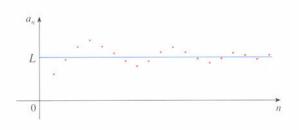
**Definition** A sequence  $\{a_n\}$  has the limit *L* and we write lim  $a_n = L$  or  $a_n \to L$  as  $n \to \infty$ 

if we can make the terms  $a_n$  as close to L as we like by taking n sufficiently large. If  $\lim_{n\to\infty} a_n$  exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

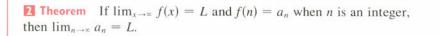
Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L.



**FIGURE 3** Graphs of two sequences with  $\lim_{n \to \infty} a_n = L$ 



If you compare Definition 1 with Definition 4 in Section 2.5 you will see that the only difference between  $\lim_{n\to\infty} a_n = L$  and  $\lim_{x\to\infty} f(x) = L$  is that *n* is required to be an integer. Thus, we have the following theorem, which is illustrated by Figure 4.



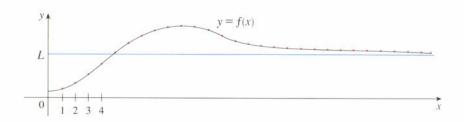


FIGURE 4

In particular, since we know from Section 2.5 that  $\lim_{x\to\infty} (1/x^r) = 0$  when r > 0, we have

$$\lim_{n \to \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If  $a_n$  becomes large as n becomes large, we use the notation

$$\lim_{n\to\infty} a_n = \infty$$

In this case the sequence  $\{a_n\}$  is divergent, but in a special way. We say that  $\{a_n\}$  diverges to  $\infty$ .

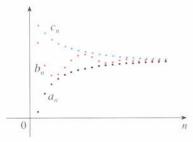
The Limit Laws given in Section 2.3 also hold for the limits of sequences and their proofs are similar.

Limit Laws for Convergent Sequences

If  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences and c is a constant, then  $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$   $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n$   $\lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n$   $\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$   $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \quad \text{if } \lim_{n \to \infty} b_n \neq 0$   $\lim_{n \to \infty} c = c$ 

The Squeeze Theorem can also be adapted for sequences as follows. (See Figure 5.)

Squeeze Theorem for Sequences



# FIGURE 5

The sequence  $\{b_n\}$  is squeezed between the sequences  $\{a_n\}$  and  $\{c_n\}$ .

This shows that the guess we made earlier from Figures I and 2 was correct. If  $a_n \leq b_n \leq c_n$  for  $n \geq n_0$  and  $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$ , then  $\lim_{n \to \infty} b_n = L$ .

Another useful fact about limits of sequences is given by the following theorem, which follows from the Squeeze Theorem because  $-|a_n| \le a_n \le |a_n|$ .

4 Theorem

If 
$$\lim |a_n| = 0$$
, then  $\lim a_n = 0$ 

**EXAMPLE 3** Find  $\lim_{n \to \infty} \frac{n}{n+1}$ 

**SOLUTION** The method is similar to the one we used in Section 2.5: Divide numerator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

Here we used Equation 3 with r = 1.

**EXAMPLE 4** Calculate 
$$\lim_{n \to \infty} \frac{\ln n}{n}$$
.

**SOLUTION** Notice that both numerator and denominator approach infinity as  $n \rightarrow \infty$ . We can't apply l'Hospital's Rule directly because it applies not to sequences but only to functions of a real variable. However, we can apply l'Hospital's Rule to the related function  $f(x) = (\ln x)/x$  and obtain

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 2 we have

$$\lim_{n \to \infty} \frac{\ln n}{n} = 0$$

**EXAMPLE 5** Determine whether the sequence  $a_n = (-1)^n$  is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

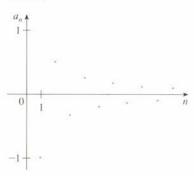
$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$$

The graph of this sequence is shown in Figure 6. Since the terms oscillate between 1 and -1 infinitely often,  $a_n$  does not approach any number. Thus,  $\lim_{n\to\infty} (-1)^n$  does not exist; that is, the sequence  $\{(-1)^n\}$  is divergent.



FIGURE 6

The graph of the sequence in Example 6 is shown in Figure 7 and supports the answer.





SOLUTION

 $\lim_{n \to \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \to \infty} \frac{1}{n} = 0$ 

Therefore, by Theorem 4,

$$\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$$

**EXAMPLE 7** Discuss the convergence of the sequence  $a_n = n!/n^n$ , where  $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n.$ 

 $n \rightarrow$ 

**SOLUTION** Both numerator and denominator approach infinity as  $n \to \infty$  but here we have no corresponding function for use with l'Hospital's Rule (x! is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to  $a_n$  as *n* gets large:

$$a_1 = 1 \qquad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \qquad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$
$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n}$$

**Creating Graphs of Sequences** 

FIGURE 7

Some computer algebra systems have special commands that enable us to create sequences and graph them directly. With most graphing calculators, however, sequences can be graphed by using parametric equations. For instance, the sequence in Example 7 can be graphed by entering the parametric equations

$$x = t \qquad y = t!/t'$$

and graphing in dot mode, starting with t = 1 and setting the *t*-step equal to 1. The result is shown in Figure 8.



FIGURE 8

It appears from these expressions and the graph in Figure 8 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 5 that

$$a_n = \frac{1}{n} \left( \frac{2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot \dots \cdot n} \right)$$
$$0 < a_n \le \frac{1}{n}$$

SO

5

We know that  $1/n \to 0$  as  $n \to \infty$ . Therefore,  $a_n \to 0$  as  $n \to \infty$  by the Squeeze Theorem.

**EXAMPLE 8** For what values of r is the sequence  $\{r^n\}$  convergent?

SOLUTION We know from Section 2.5 and the graphs of the exponential functions in Section 1.5 that  $\lim_{x \to \infty} a^x = \infty$  for a > 1 and  $\lim_{x \to \infty} a^x = 0$  for 0 < a < 1. Therefore, putting a = r and using Theorem 2, we have

$$\lim_{n \to \infty} r^n = \begin{cases} \infty & \text{if } r > 1\\ 0 & \text{if } 0 < r < 1 \end{cases}$$

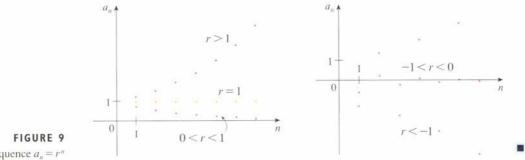
For the cases r = 1 and r = 0 we have

$$\lim_{n \to \infty} 1^n = \lim_{n \to \infty} 1 = 1 \quad \text{and} \quad \lim_{n \to \infty} 0^n = \lim_{n \to \infty} 0 = 0$$

If -1 < r < 0, then 0 < |r| < 1, so

$$\lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0$$

and, therefore,  $\lim_{n\to\infty} r^n = 0$  by Theorem 4. If  $r \leq -1$ , then  $\{r^n\}$  diverges as in Example 5. Figure 9 shows the graphs for various values of r. (The case r = -1is shown in Figure 6.)



The sequence  $a_n = r^n$ 

The results of Example 8 are summarized for future use as follows.

**6** The sequence  $\{r^n\}$  is convergent if  $-1 < r \le 1$  and divergent for all other values of r.

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

**Definition** A sequence  $\{a_n\}$  is called **increasing** if  $a_n < a_{n+1}$  for all  $n \ge 1$ , that is,  $a_1 < a_2 < a_3 < \cdots$ . It is called **decreasing** if  $a_n > a_{n+1}$  for all  $n \ge 1$ . A sequence is **monotonic** if it is either increasing or decreasing.

**EXAMPLE 9** The sequence  $\left\{\frac{3}{n+5}\right\}$  is decreasing because  $\frac{3}{n+5} > \frac{3}{n+6}$ 

for all  $n \ge 1$ . (The right side is smaller because it has a larger denominator.)

**EXAMPLE 10** Show that the sequence  $a_n = \frac{n}{n^2 + 1}$  is decreasing.

**SOLUTION** | We must show that  $a_{n+1} < a_n$ , that is,

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \qquad \Longleftrightarrow \qquad (n+1)(n^2+1) < n[(n+1)^2+1]$$
$$\iff \qquad n^3+n^2+n+1 < n^3+2n^2+2n$$
$$\iff \qquad 1 < n^2+n$$

Since  $n \ge 1$ , we know that the inequality  $n^2 + n > 1$  is true. Therefore,  $a_{n+1} < a_n$  and so  $\{a_n\}$  is decreasing.

**SOLUTION 2** Consider the function  $f(x) = \frac{x}{x^2 + 1}$ :

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0 \qquad \text{whenever } x^2 > 1$$

Thus, f is decreasing on  $(1, \infty)$  and so f(n) > f(n + 1). Therefore,  $\{a_n\}$  is decreasing.

**Definition** A sequence  $\{a_n\}$  is **bounded above** if there is a number *M* such that

 $a_n \leq M$  for all  $n \geq 1$ 

It is **bounded below** if there is a number *m* such that

$$m \leq a_n$$
 for all  $n \geq 1$ 

If it is bounded above and below, then  $\{a_n\}$  is a **bounded sequence**.

For instance, the sequence  $a_n = n$  is bounded below  $(a_n > 0)$  but not above. The sequence  $a_n = n/(n + 1)$  is bounded because  $0 < a_n < 1$  for all n.

We know that not every bounded sequence is convergent  $[a_n = (-1)^n$  satisfies  $-1 \le a_n \le 1$  but is divergent, from Example 5] and not every monotonic sequence is convergent  $(a_n = n \rightarrow \infty)$ . But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is stated without proof as Theorem 7, but intuitively you can understand why it is true by looking at Figure 10. If  $\{a_n\}$  is increasing and  $a_n \le M$  for all *n*, then the terms are forced to crowd together and approach some number *L*.

**Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

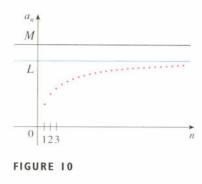
**EXAMPLE II** Investigate the sequence  $\{a_n\}$  defined by the recurrence relation

 $a_1 = 2$   $a_{n+1} = \frac{1}{2}(a_n + 6)$  for n = 1, 2, 3, ...

**SOLUTION** We begin by computing the first few terms:

$$a_1 = 2$$
  $a_2 = \frac{1}{2}(2+6) = 4$   $a_3 = \frac{1}{2}(4+6) = 5$   $a_4 = \frac{1}{2}(5+6) = 5.5$   
 $a_5 = 5.75$   $a_6 = 5.875$   $a_7 = 5.9375$   $a_8 = 5.96875$ 

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that  $a_{n+1} > a_n$  for all  $n \ge 1$ . This is true for n = 1 because  $a_2 = 4 > a_1$ . If we assume that it is true for n = k, then we have



Mathematical induction is often used in dealing with recursive sequences. See page 88 for a discussion of the Principle of Mathematical Induction.

$$a_{k+1} > a_k$$

$$a_{k+1} + 6 > a_k + 6$$

and 
$$\frac{1}{2}(a_{k+1}+6) > \frac{1}{2}(a_k+6)$$

Thus 
$$a_{k+2} > a_{k+1}$$

We have deduced that  $a_{n+1} > a_n$  is true for n = k + 1. Therefore, the inequality is true for all *n* by induction.

Next we verify that  $\{a_n\}$  is bounded by showing that  $a_n < 6$  for all n. (Since the sequence is increasing, we already know that it has a lower bound:  $a_n \ge a_1 = 2$  for all n.) We know that  $a_1 < 6$ , so the assertion is true for n = 1. Suppose it is true for n = k. Then

$$a_k < 6$$
$$a_k + 6 < 12$$

SO

SO

and  $\frac{1}{2}(a_k + 6) < \frac{1}{2}(12)$ 

Thus

This shows, by mathematical induction, that  $a_n < 6$  for all n.

Since the sequence  $\{a_n\}$  is increasing and bounded, the Monotonic Sequence Theorem guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know  $L = \lim_{n \to \infty} a_n$  exists, we can use the given recurrence relation to write

 $a_{k+1} < 6$ 

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2} (a_n + 6) = \frac{1}{2} \left( \lim_{n \to \infty} a_n + 6 \right) = \frac{1}{2} (L + 6)$$

Since  $a_n \to L$ , it follows that  $a_{n+1} \to L$ , too (as  $n \to \infty$ ,  $n + 1 \to \infty$  also). So we have

$$L = \frac{1}{2}(L + 6)$$

Solving this equation for L, we get L = 6, as we predicted.



I. (a) What is a sequence?

Exercises

- (b) What does it mean to say that  $\lim_{n \to \infty} a_n = 8$ ?
- (c) What does it mean to say that  $\lim_{n\to\infty} a_n = \infty$ ?
- (a) What is a convergent sequence? Give two examples.(b) What is a divergent sequence? Give two examples.
- 3. List the first six terms of the sequence defined by

$$a_n = \frac{n}{2n+1}$$

Does the sequence appear to have a limit? If so, find it.

4. List the first eight terms of the sequence  $\{\sin(n\pi/2)\}$ . Does this sequence appear to have a limit? If so, find it. If not, explain why. **5–8** Find a formula for the general term  $a_n$  of the sequence, assuming that the pattern of the first few terms continues.

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5.	$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$	<b>6.</b> $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right\}$
7.	$\left\{\frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, \ldots\right\}$	<b>8.</b> {0, 2, 0, 2, 0, 2,}

**9–26** Determine whether the sequence converges or diverges. If it converges, find the limit.

9. 
$$a_n = \frac{1}{5^n}$$
  
10.  $a_n = 4\sqrt{n}$   
11.  $a_n = \frac{n^2 - 1}{n^2 + 1}$   
12.  $a_n = \frac{4n - 3}{3n + 4}$ 

13. 
$$a_n = \frac{n^2}{n+1}$$
 14. { $\arctan 2n$ }

 15.  $a_n = \cos(n\pi/2)$ 
 16.  $a_n = (-1)^n \frac{n^2}{1+n^3}$ 

 17.  $\left\{\frac{\pi^n}{3^n}\right\}$ 
 18.  $\left\{\frac{3+(-1)^n}{n^2}\right\}$ 

 19.  $\left\{\frac{\ln(n^2)}{n}\right\}$ 
 20.  $\{(-1)^n \sin(1/n)\}$ 

 21.  $\{\sqrt{n+2} - \sqrt{n}\}$ 
 22.  $\left\{\frac{\ln(2+e^n)}{3n}\right\}$ 

 23.  $a_n = n2^{-n}$ 
 24.  $a_n = \ln(n+1) - \ln n$ 

 25.  $a_n = \frac{\cos^2 n}{2^n}$ 
 26.  $a_n = \frac{(-3)^n}{n!}$ 

27-32 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess. (See the margin note on page 564 for advice on graphing sequences.)

27. 
$$a_n = (-1)^n \frac{n+1}{n}$$
  
28.  $a_n = 2 + (-2/\pi)^n$   
29.  $\left\{ \arctan\left(\frac{2n}{2n+1}\right) \right\}$   
30.  $\left\{ \frac{\sin n}{\sqrt{n}} \right\}$   
31.  $a_n = \frac{n^3}{n!}$   
32.  $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{(2n)^n}$ 

**33.** (a) Determine whether the sequence defined as follows is convergent or divergent:

 $a_1 = 1$   $a_{n+1} = 4 - a_n$  for  $n \ge 1$ 

- (b) What happens if the first term is  $a_1 = 2$ ?
- 34. (a) If lim<sub>n→∞</sub> a<sub>n</sub> = L, what is the value of lim<sub>n→∞</sub> a<sub>n+1</sub>?
  (b) A sequence {a<sub>n</sub>} is defined by

 $a_1 = 1$   $a_{n+1} = 1/(1 + a_n)$  for  $n \ge 1$ 

Find the first ten terms of the sequence correct to five decimal places. Does it appear that the sequence is convergent? If so, estimate the value of the limit to three decimal places.

(c) Assuming that the sequence in part (b) has a limit, use part (a) to find its exact value. Compare with your estimate from part (b).

- **35.** (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair, which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the *n*th month? Show that the answer is  $f_n$ , where  $\{f_n\}$  is the Fibonacci sequence defined in Example 2(c).
  - (b) Let  $a_n = f_{n+1}/f_n$  and show that  $a_{n-1} = 1 + 1/a_{n-2}$ . Assuming that  $\{a_n\}$  is convergent, find its limit.

36. Find the limit of the sequence

$$\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$$

**37–40** ■ Determine whether the sequence is increasing, decreasing, or not monotonic.

**37.** 
$$a_n = \frac{1}{3n+5}$$
  
**38.**  $a_n = 3 + (-1)^n/n$   
**39.**  $a_n = \frac{n-2}{n+2}$   
**40.**  $a_n = \frac{\sqrt{n+1}}{5n+3}$ 

- 41. Suppose you know that {a<sub>n</sub>} is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?
- 42. A sequence {a<sub>n</sub>} is given by a<sub>1</sub> = √2, a<sub>n+1</sub> = √2 + a<sub>n</sub>.
  (a) By induction or otherwise, show that {a<sub>n</sub>} is increasing and bounded above by 3. Deduce that lim<sub>n→∞</sub> a<sub>n</sub> exists.
  - (b) Find  $\lim_{n\to\infty} a_n$ .
- **43.** Show that the sequence defined by

$$a_1 = 1$$
  $a_{n+1} = 3 - 1/a_n$ 

is increasing and  $a_n < 3$  for all *n*. Deduce that  $\{a_n\}$  is convergent, and find its limit.

44. Show that the sequence defined by

 $a_1 = 2$   $a_{n+1} = 1/(3 - a_n)$ 

satisfies  $0 < a_n \le 2$  and is decreasing. Deduce that the sequence is convergent, and find its limit.

- **45.** We know that  $\lim_{n\to\infty} (0.8)^n = 0$  [from (6) with r = 0.8]. Use logarithms to determine how large *n* has to be so that  $(0.8)^n < 0.000001$ .
- **46.** (a) Let  $a_1 = a$ ,  $a_2 = f(a)$ ,  $a_3 = f(a_2) = f(f(a))$ , ...,  $a_{n+1} = f(a_n)$ , where f is a continuous function. If  $\lim_{n \to \infty} a_n = L$ , show that f(L) = L.
  - (b) Illustrate part (a) by taking  $f(x) = \cos x$ , a = 1, and estimating the value of *L* to five decimal places.

**47.** Let *a* and *b* be positive numbers with a > b. Let  $a_1$  be their arithmetic mean and  $b_1$  their geometric mean:

$$a_1 = \frac{a+b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2}$$
  $b_{n+1} = \sqrt{a_n b_n}$ 

(a) Use mathematical induction to show that

 $a_n > a_{n+1} > b_{n+1} > b_n$ 

- (b) Deduce that both  $\{a_n\}$  and  $\{b_n\}$  are convergent.
- (c) Show that lim<sub>n→∞</sub> a<sub>n</sub> = lim<sub>n→∞</sub> b<sub>n</sub>. Gauss called the common value of these limits the arithmetic-geometric mean of the numbers a and b.

48. A sequence is defined recursively by

$$a_1 = 1$$
  $a_{n+1} = 1 + \frac{1}{1 + a_n}$ 

Find the first eight terms of the sequence  $\{a_n\}$ . What do you notice about the odd terms and the even terms? By considering the odd and even terms separately, show that  $\{a_n\}$  is convergent and deduce that

$$\lim_{n\to\infty}a_n=\sqrt{2}$$

This gives the continued fraction expansion

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

Laboratory Project

## Logistic Sequences

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation** 

$$p_{n+1} = k p_n (1 - p_n)$$

where  $p_n$  measures the size of the population of the *n*th generation of a single species. To keep the numbers manageable,  $p_n$  is a fraction of the maximal size of the population, so  $0 \le p_n \le 1$ . Notice that the form of this equation is similar to the logistic differential equation in Section 7.6. The discrete model—with sequences instead of continuous functions—is preferable for modeling insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks the questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program to compute the first *n* terms of this sequence starting with an initial population  $p_0$ , where  $0 < p_0 < 1$ . Use this program to do the following.

- Calculate 20 or 30 terms of the sequence for p<sub>0</sub> = <sup>1</sup>/<sub>2</sub> and for two values of k such that 1 < k < 3. Graph the sequences. Do they appear to converge? Repeat for a different value of p<sub>0</sub> between 0 and 1. Does the limit depend on the choice of p<sub>0</sub>? Does it depend on the choice of k?
- **2.** Calculate terms of the sequence for a value of *k* between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?
- **3.** Experiment with values of k between 3.4 and 3.5. What happens to the terms?
- **4.** For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change  $p_0$  by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.



If we try to add the terms of an infinite sequence  $\{a_n\}_{n=1}^{\infty}$  we get an expression of the form

1

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n$$
 or  $\sum a_n$ 

But does it make sense to talk about the sum of infinitely many terms? It would be impossible to find a finite sum for the series

$$+2+3+4+5+\dots+n+\dots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21,... and, after the *n*th term, n(n + 1)/2, which becomes very large as *n* increases.

However, if we start to add the terms of the series

1

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots + \frac{1}{2^n} + \dots$$

we get  $\frac{1}{2}$ ,  $\frac{3}{4}$ ,  $\frac{7}{8}$ ,  $\frac{15}{16}$ ,  $\frac{31}{32}$ ,  $\frac{63}{64}$ , ...,  $1 - 1/2^n$ , .... The table shows that as we add more and more terms, these partial sums become closer and closer to 1. (See also Figure 11 in *A Preview of Calculus*, page 7.) In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1. So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum. We consider the **partial sums** 

$$s_{1} = a_{1}$$

$$s_{2} = a_{1} + a_{2}$$

$$s_{3} = a_{1} + a_{2} + a_{3}$$

$$s_{4} = a_{1} + a_{2} + a_{3} + a_{4}$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{i=1}^n a_i$$

These partial sums form a new sequence  $\{s_n\}$ , which may or may not have a limit. If  $\lim_{n\to\infty} s_n = s$  exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series  $\sum a_n$ .

n	Sum of first <i>n</i> terms
1	0.5000000
1 2 3	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

**2** Definition Given a series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ , let  $s_n$  denote its *n*th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$$

If the sequence  $\{s_n\}$  is convergent and  $\lim_{n\to\infty} s_n = s$  exists as a real number, then the series  $\sum a_n$  is called **convergent** and we write

$$a_1 + a_2 + \dots + a_n + \dots = s$$
 or  $\sum_{n=1}^{\infty} a_n = s$ 

The number *s* is called the **sum** of the series. Otherwise, the series is called **divergent**.

Thus, when we write  $\sum_{n=1}^{\infty} a_n = s$  we mean that by adding sufficiently many terms of the series we can get as close as we like to the number *s*. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

**EXAMPLE 1** An important example of an infinite series is the **geometric** series

$$a + ar + ar^{2} + ar^{3} + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the common ratio r. (We have already considered the special case where  $a = \frac{1}{2}$  and  $r = \frac{1}{2}$ .)

If r = 1, then  $s_n = a + a + \dots + a = na \rightarrow \pm \infty$ . Since  $\lim_{n \to \infty} s_n$  doesn't exist, the geometric series diverges in this case.

If  $r \neq 1$ , we have

 $s_n = a + ar + ar^2 + \dots + ar^{n-1}$ 

 $rs_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$ 

and

3

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$
$$s_n = \frac{a(1 - r^n)}{1 - r}$$

If -1 < r < 1, we know from (6) in Section 8.1 that  $r^n \to 0$  as  $n \to \infty$ , so

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n \to \infty} r^n = \frac{a}{1 - r}$$

Thus, when |r| < 1 the geometric series is convergent and its sum is a/(1 - r). If  $r \le -1$  or r > 1, the sequence  $\{r^n\}$  is divergent and so, by Equation 3,

 $\lim_{n\to\infty} s_n$  does not exist. Therefore, the geometric series diverges in those cases.

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We summarize the results of Example 1 as follows.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if |r| < 1 and its sum is

$$\sum_{i=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If  $|r| \ge 1$ , the geometric series is divergent.

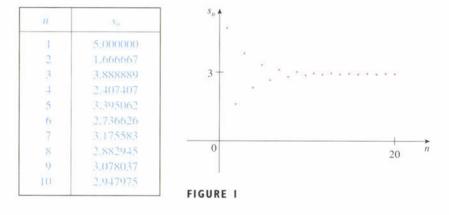
**EXAMPLE 2** Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

**SOLUTION** The first term is a = 5 and the common ratio is  $r = -\frac{2}{3}$ . Since  $|r| = \frac{2}{3} < 1$ , the series is convergent by (4) and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

What do we really mean when we say that the sum of the series in Example 2 is 3? Of course, we can't literally add an infinite number of terms, one by one. But, according to Definition 2, the total sum is the limit of the sequence of partial sums. So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. The table shows the first ten partial sums  $s_n$  and the graph in Figure 1 shows how the sequence of partial sums approaches 3.



**EXAMPLE 3** Is the series  $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$  convergent or divergent?

**SOLUTION** Let's rewrite the *n*th term of the series in the form  $ar^{n-1}$ :

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

Another way to identify a and r is to write out the first few terms:

 $4 + \frac{16}{3} + \frac{64}{9} + \cdots$ 

We recognize this series as a geometric series with a = 4 and  $r = \frac{4}{3}$ . Since r > 1, the series diverges by (4).

#### SECTION 8.2 SERIES = 573

**EXAMPLE 4** Write the number  $2.3\overline{17} = 2.3171717...$  as a ratio of integers.

SOLUTION 
$$2.3171717... = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term we have a geometric series with  $a = 17/10^3$  and  $r = 1/10^2$ . Therefore

$$2.3\overline{17} = 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$
$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$

**EXAMPLE 5** Find the sum of the series  $\sum_{n=0}^{\infty} x^n$ , where |x| < 1.

**SOLUTION** Notice that this series starts with n = 0 and so the first term is  $x^0 = 1$ . (With series, we adopt the convention that  $x^0 = 1$  even when x = 0.) Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with a = 1 and r = x. Since |r| = |x| < 1, it converges and (4) gives

5

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

**EXAMPLE 6** Show that the series  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  is convergent, and find its sum.

**SOLUTION** This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1}\right)$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$

This is similar to the method we used in Example 7 in Section 5.5. A full discussion of partial fractions appears in Appendix F.

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: Because of all the cancellations, the sum collapses (like an old-fashioned collapsing telescope) into just two terms.

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Figure 2 illustrates Example 6 by showing the graphs of the sequence of terms  $a_n = 1/[n(n + 1)]$  and the sequence  $\{s_n\}$  of partial sums. Notice that  $a_n \to 0$ and  $s_n \rightarrow 1$ . See Exercises 44 and 45 for two geometric interpretations of Example 6.

 $\{s_n\}$  $\{a_n\}$ 0

FIGURE 2

and so

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 - 0 = 1$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

## **EXAMPLE 7** Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent.

$$s_{1} = 1$$

$$s_{2} = 1 + \frac{1}{2}$$

$$s_{4} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_{8} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

Similarly,  $s_{32} > 1 + \frac{5}{2}$ ,  $s_{64} > 1 + \frac{6}{2}$ , and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that  $s_{2^n} \to \infty$  as  $n \to \infty$  and so  $\{s_n\}$  is divergent. Therefore, the harmonic series diverges.

**6** Theorem If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \to \infty} a_n = 0$ .

**Proof** Let  $s_n = a_1 + a_2 + \dots + a_n$ . Then  $a_n = s_n - s_{n-1}$ . Since  $\sum a_n$  is convergent, the sequence  $\{s_n\}$  is convergent. Let  $\lim_{n\to\infty} s_n = s$ . Since  $n-1\to\infty$  as  $n \to \infty$ , we also have  $\lim_{n\to\infty} s_{n-1} = s$ . Therefore

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (s_n - s_{n-1}) = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1}$$
$$= s - s = 0$$

The method used in Example 7 for showing that the harmonic series

diverges is due to the French scholar Nicole Oresme (1323-1382).

**NOTE I** • With any series  $\sum a_n$  we associate two sequences: the sequence  $\{s_n\}$  of its partial sums and the sequence  $\{a_n\}$  of its terms. If  $\sum a_n$  is convergent, then the limit of the sequence  $\{s_n\}$  is s (the sum of the series) and, as Theorem 6 asserts, the limit of the sequence  $\{a_n\}$  is 0.

**NOTE 2** • The converse of Theorem 6 is not true in general. If  $\lim_{n\to\infty} a_n = 0$ , we cannot conclude that  $\sum a_n$  is convergent. Observe that for the harmonic series  $\sum 1/n$  we have  $a_n = 1/n \to 0$  as  $n \to \infty$ , but we showed in Example 7 that  $\sum 1/n$  is divergent.

**The Test for Divergence** If  $\lim_{n \to \infty} a_n$  does not exist or if  $\lim_{n \to \infty} a_n \neq 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so  $\lim_{n\to\infty} a_n = 0$ .

EXAMPLE 8 Show that the series 
$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$$
 diverges.  
SOLUTION 
$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{5n^2 + 4} = \lim_{n \to \infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

**NOTE 3** • If we find that  $\lim_{n\to\infty} a_n \neq 0$ , we know that  $\sum a_n$  is divergent. If we find that  $\lim_{n\to\infty} a_n = 0$ , we know *nothing* about the convergence or divergence of  $\sum a_n$ . Remember the warning in Note 2: If  $\lim_{n\to\infty} a_n = 0$ , the series  $\sum a_n$  might converge or it might diverge.

**B** Theorem If  $\Sigma a_n$  and  $\Sigma b_n$  are convergent series, then so are the series  $\Sigma ca_n$  (where *c* is a constant),  $\Sigma (a_n + b_n)$ , and  $\Sigma (a_n - b_n)$ , and

(i)  $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$  (ii)  $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ (iii)  $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$ 

These properties of convergent series follow from the corresponding Limit Laws for Convergent Sequences in Section 8.1. For instance, here is how part (ii) of Theorem 8 is proved:

Let

 $\oslash$ 

$$s_n = \sum_{i=1}^n a_i$$
  $s = \sum_{n=1}^\infty a_n$   $t_n = \sum_{i=1}^n b_i$   $t = \sum_{n=1}^\infty b_n$ 

The *n*th partial sum for the series  $\Sigma (a_n + b_n)$  is

$$u_n = \sum_{i=1}^n (a_i + b_i)$$

and, using Equation 9 in Section 5.2, we have

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \sum_{i=1}^n (a_i + b_i) = \lim_{n \to \infty} \left( \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \right)$$
$$= \lim_{n \to \infty} \sum_{i=1}^n a_i + \lim_{n \to \infty} \sum_{i=1}^n b_i$$
$$= \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = s + t$$

Therefore,  $\Sigma (a_n + b_n)$  is convergent and its sum is

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

**EXAMPLE 9** Find the sum of the series  $\sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right)$ .

**SOLUTION** The series  $\sum 1/2^n$  is a geometric series with  $a = \frac{1}{2}$  and  $r = \frac{1}{2}$ , so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

In Example 6 we found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, by Theorem 8, the given series is convergent and

$$\sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$
$$= 3 \cdot 1 + 1 = 4$$

**NOTE 4** • A finite number of terms can't affect the convergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

it follows that the entire series  $\sum_{n=1}^{\infty} n/(n^3 + 1)$  is convergent. Similarly, if it is known that the series  $\sum_{n=N+1}^{\infty} a_n$  converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

8.2 Exercises · ·

- 1. (a) What is the difference between a sequence and a series?
  - (b) What is a convergent series? What is a divergent series?
- **2.** Explain what it means to say that  $\sum_{n=1}^{\infty} a_n = 5$ .
- 3-8 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

**3.** 
$$\sum_{n=1}^{\infty} \frac{10}{3^n}$$
  
**4.**  $\sum_{n=1}^{\infty} \sin n$   
**5.**  $\sum_{n=1}^{\infty} \frac{n}{n+1}$   
**6.**  $\sum_{n=4}^{\infty} \frac{3}{n(n-1)}$   
**7.**  $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right)$   
**8.**  $\sum_{n=1}^{\infty} \left(-\frac{2}{7}\right)^{n-1}$ 

- **9.** Let  $a_n = \frac{2n}{3n+1}$ .
  - (a) Determine whether  $\{a_n\}$  is convergent.
  - (b) Determine whether  $\sum_{n=1}^{\infty} a_n$  is convergent.
- 10. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i$$
 and  $\sum_{j=1}^{n} a_j$ 

(b) Explain the difference between

$$\sum_{i=1}^n a_i \quad \text{and} \quad \sum_{i=1}^n a_i$$

**11–28** Determine whether the series is convergent or divergent. If it is convergent, find its sum.

23. 
$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$$
 24.  $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$   
25.  $\sum_{n=1}^{\infty} \left[\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right)\right]$   
26.  $\sum_{n=1}^{\infty} \frac{1}{5+2^{-n}}$   
27.  $\sum_{n=1}^{\infty} \arctan n$  28.  $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ 

- **29–32** Express the number as a ratio of integers.
- **29.**  $0.\overline{5} = 0.5555...$  **30.**  $0.\overline{15} = 0.15151515...$
- **31.**  $0.\overline{307} = 0.307307307307...$
- **32.** 4.1570 = 4.157015701570...

**33–36** Find the values of x for which the series converges. Find the sum of the series for those values of x.

**33.** 
$$\sum_{n=0}^{\infty} (x-3)^n$$
  
**34.**  $\sum_{n=0}^{\infty} 3^n x^n$   
**35.**  $\sum_{n=0}^{\infty} \frac{1}{x^n}$   
**36.**  $\sum_{n=0}^{\infty} \tan^n x$ 

CAS 37-38 ■ Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

**37.** 
$$\sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-3)}$$
 **38.**  $\sum_{n=1}^{\infty} \frac{n^2+3n+1}{(n^2+n)^2}$ 

**39.** If the *n*th partial sum of a series  $\sum_{n=1}^{\infty} a_n$  is

$$s_n = \frac{n-1}{n+1}$$

find  $a_n$  and  $\sum_{n=1}^{\infty} a_n$ .

**40.** If the *n*th partial sum of a series  $\sum_{n=1}^{\infty} a_n$  is

$$s_n = 3 - n2^-$$

find  $a_n$  and  $\sum_{n=1}^{\infty} a_n$ .

**41.** When money is spent on goods and services, those that receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending *D* dollars. Suppose that each recipient of spent money spends 100c% and saves 100s% of the money

that he or she receives. The values c and s are called the marginal propensity to consume and the marginal propensity to save and, of course, c + s = 1.

- (a) Let  $S_n$  be the total spending that has been generated after *n* transactions. Find an equation for  $S_n$ .
- (b) Show that lim<sub>n→∞</sub> S<sub>n</sub> = kD, where k = 1/s. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

**NOTE** • The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

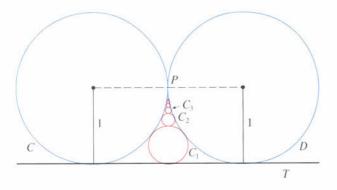
- **42.** A certain ball has the property that each time it falls from a height *h* onto a hard, level surface, it rebounds to a height *rh*, where 0 < r < 1. Suppose that the ball is dropped from an initial height of *H* meters.
  - (a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.
  - (b) Calculate the total time that the ball travels.
  - (c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity kv, where 0 < k < 1. How long will it take for the ball to come to rest?

**43.** What is the value of c if 
$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$
?

44. Graph the curves y = x<sup>n</sup>, 0 ≤ x ≤ 1, for n = 0, 1, 2, 3, 4, ... on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 6, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

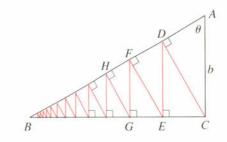
**45.** The figure shows two circles *C* and *D* of radius 1 that touch at *P*. *T* is a common tangent line;  $C_1$  is the circle that touches *C*, *D*, and *T*;  $C_2$  is the circle that touches *C*, *D*, and *C*<sub>1</sub>; *C*<sub>3</sub> is the circle that touches *C*, *D*, and *C*<sub>2</sub>. This procedure can be continued indefinitely and produces an infinite sequence of circles  $\{C_n\}$ . Find an expression for the diameter of  $C_n$  and thus provide another geometric demonstration of Example 6.



46. A right triangle ABC is given with ∠A = θ and |AC| = b. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, EF⊥AB, and this process is continued indefinitely as in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and  $\theta$ .



- 47. What is wrong with the following calculation?
  - $0 = 0 + 0 + 0 + \cdots$ = (1 - 1) + (1 - 1) + (1 - 1) + \dots = 1 - 1 + 1 - 1 + 1 - 1 + \dots = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots = 1 + 0 + 0 + 0 + \dots = 1

(Guido Ubaldus thought that this proved the existence of God because "something has been created out of nothing.")

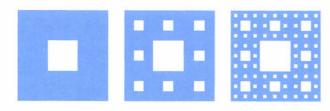
- **48.** Suppose that  $\sum_{n=1}^{\infty} a_n (a_n \neq 0)$  is known to be a convergent series. Prove that  $\sum_{n=1}^{\infty} 1/a_n$  is a divergent series.
- **49.** If  $\sum a_n$  is convergent and  $\sum b_n$  is divergent, show that the series  $\sum (a_n + b_n)$  is divergent. [*Hint:* Argue by contradiction.]
- **50.** If  $\sum a_n$  and  $\sum b_n$  are both divergent, is  $\sum (a_n + b_n)$  necessarily divergent?
- **51.** Suppose that a series  $\sum a_n$  has positive terms and its partial sums  $s_n$  satisfy the inequality  $s_n \le 1000$  for all *n*. Explain why  $\sum a_n$  must be convergent.
- **52.** The Fibonacci sequence was defined in Section 8.1 by the equations

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad n \ge 3$$

Show that each of the following statements is true.

(a) 
$$\frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n-1}}$$
  
(b)  $\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1$   
(c)  $\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2$ 

- **53.** The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval [0, 1] and remove the open interval  $(\frac{1}{3}, \frac{2}{3})$ . That leaves the two intervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$  and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in [0, 1] after all those intervals have been removed.
  - (a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.
  - (b) The Sierpinski carpet is a two-dimensional analogue of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



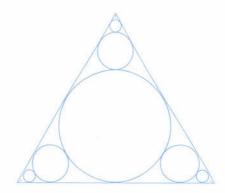
**54.** (a) A sequence  $\{a_n\}$  is defined recursively by the equation  $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$  for  $n \ge 3$ , where  $a_1$  and  $a_2$ 

can be any real numbers. Experiment with various values of  $a_1$  and  $a_2$  and use your calculator to guess the limit of the sequence.

- (b) Find  $\lim_{n\to\infty} a_n$  in terms of  $a_1$  and  $a_2$  by expressing  $a_{n+1} a_n$  in terms of  $a_2 a_1$  and summing a series.
- **55.** Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

- (a) Find the partial sums  $s_1$ ,  $s_2$ ,  $s_3$ , and  $s_4$ . Do you recognize the denominators? Use the pattern to guess a formula for  $s_n$ .
- (b) Use mathematical induction to prove your guess.
- (c) Show that the given infinite series is convergent, and find its sum.
- **56.** In the figure there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.





# The Integral and Comparison Tests; Estimating Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and for the series  $\sum 1/[n(n + 1)]$  because in each of those cases we could find a simple formula for the *n*th partial sum  $s_n$ . But usually it is not easy to compute  $\lim_{n\to\infty} s_n$ . Therefore, in this section and the next we develop tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. In some cases, however, our methods will enable us to find good estimates of the sum.

In this section we deal only with series with positive terms, so the partial sums are increasing. In view of the Monotonic Sequence Theorem, to decide whether a series is convergent or divergent, we need to determine whether the partial sums are bounded or not.

### Testing with an Integral

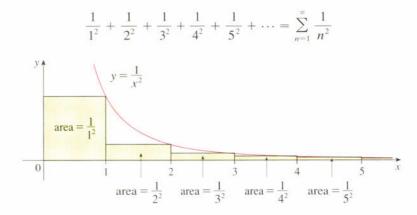
Let's investigate the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

П	$s_n = \sum_{i=1}^n \frac{1}{i^2}$
5	1.4636
1()	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.64.39
5000	1.6447

There's no simple formula for the sum  $s_n$  of the first *n* terms, but the computergenerated table of values given in the margin suggests that the partial sums are approaching a number near 1.64 as  $n \rightarrow \infty$  and so it looks as if the series is convergent.

We can confirm this impression with a geometric argument. Figure 1 shows the curve  $y = 1/x^2$  and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function  $y = 1/x^2$  at the right endpoint of the interval. So the sum of the areas of the rectangles is



#### FIGURE I

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve  $y = 1/x^2$  for  $x \ge 1$ , which is the value of the integral  $\int_{1}^{\infty} (1/x^2) dx$ . In Section 5.9 we discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2} \, dx = 2$$

Thus, the partial sums are bounded and the series converges. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 2$$

[The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be  $\pi^2/6$ , but the proof of this fact is beyond the scope of this book.]

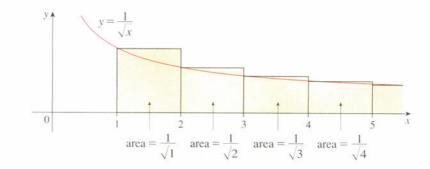
Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

П	$s_n = \sum_{i=1}^n \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

FIGURE 2

The table of values of  $s_n$  suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent. Again we use a picture for confirmation. Figure 2 shows the curve  $y = 1/\sqrt{x}$ , but this time we use rectangles whose tops lie *above* the curve.



The base of each rectangle is an interval of length 1. The height is equal to the value of the function  $y = 1/\sqrt{x}$  at the *left* endpoint of the interval. So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve  $y = 1/\sqrt{x}$  for  $x \ge 1$ , which is equal to the integral  $\int_{1}^{\infty} (1/\sqrt{x}) dx$ . But we know from Section 5.9 that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite, that is, the series is divergent.

The same sort of geometric reasoning that we used for these two series can be used to prove the following test.

**The Integral Test** Suppose *f* is a continuous, positive, decreasing function on  $[1, \infty)$  and let  $a_n = f(n)$ . Then the series  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if the improper integral  $\int_{1}^{\infty} f(x) dx$  is convergent. In other words:

(a) If ∫<sub>1</sub><sup>∞</sup> f(x) dx is convergent, then ∑<sub>n=1</sub><sup>∞</sup> a<sub>n</sub> is convergent.
(b) If ∫<sub>1</sub><sup>∞</sup> f(x) dx is divergent, then ∑<sub>n=1</sub><sup>∞</sup> a<sub>n</sub> is divergent.

**NOTE** • When we use the Integral Test it is not necessary to start the series or the integral at n = 1. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \quad \text{we use} \quad \int_4^{\infty} \frac{1}{(x-3)^2} \, dx$$

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then  $\sum_{n=N}^{\infty} a_n$  is convergent, so  $\sum_{n=1}^{\infty} a_n$  is convergent by Note 4 of Section 8.2.

**EXAMPLE I** Determine whether the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  converges or diverges.

**SOLUTION** The function  $f(x) = (\ln x)/x$  is positive and continuous for x > 1 because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus, f'(x) < 0 when  $\ln x > 1$ , that is, x > e. It follows that f is decreasing when x > e and so we can apply the Integral Test:

$$\int_{1}^{\infty} \frac{\ln x}{x} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{\ln x}{x} dx = \lim_{t \to \infty} \frac{(\ln x)^2}{2} \bigg]_{1}^{t}$$
$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series  $\sum (\ln n)/n$  is also divergent by the Integral Test.

**EXAMPLE 2** For what values of p is the series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  convergent?

**SOLUTION** If p < 0, then  $\lim_{n\to\infty} (1/n^p) = \infty$ . If p = 0, then  $\lim_{n\to\infty} (1/n^p) = 1$ . In either case  $\lim_{n\to\infty} (1/n^p) \neq 0$ , so the given series diverges by the Test for Divergence [see (7) in Section 8.2].

If p > 0, then the function  $f(x) = 1/x^p$  is clearly continuous, positive, and decreasing on  $[1, \infty)$ . We found in Chapter 5 [see (2) in Section 5.9] that

$$\int_{1}^{\infty} \frac{1}{x^{p}} dx \text{ converges if } p > 1 \text{ and diverges if } p \le 1$$

It follows from the Integral Test that the series  $\sum 1/n^p$  converges if p > 1 and diverges if 0 . (For <math>p = 1, this series is the harmonic series discussed in Example 7 in Section 8.2.)

The series in Example 2 is called the p-series. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

The *p*-series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent if p > 1 and divergent if  $p \le 1$ .

Testing by Comparing

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series  $\sum_{n=1}^{\infty} 1/2^n$ , which is a geometric series with a = 1 and  $r = \frac{1}{2}$  and is therefore convergent. Because the series (2) is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

shows that our given series (2) has smaller terms that those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series). This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < 2$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

**The Comparison Test** Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms.

- (a) If  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all *n*, then  $\sum a_n$  is also convergent.
- (b) If  $\Sigma b_n$  is divergent and  $a_n \ge b_n$  for all n, then  $\Sigma a_n$  is also divergent.

In using the Comparison Test we must, of course, have some known series  $\sum b_n$  for the purpose of comparison. Most of the time we use either a *p*-series  $[\sum 1/n^n \text{ converges} \text{ if } p > 1 \text{ and diverges} \text{ if } p \leq 1; \text{ see (1)}] \text{ or a geometric series } [\sum ar^{n-1} \text{ converges if } |r| < 1 \text{ and diverges} \text{ if } |r| \geq 1; \text{ see (4) in Section 8.2]}.$ 

**EXAMPLE 3** Determine whether the series  $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$  converges or diverges.

**SOLUTION** For large *n* the dominant term in the denominator is  $2n^2$ , so we compare the given series with the series  $\sum 5/(2n^2)$ . Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test,  $a_n$  is the left side and  $b_n$  is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Standard series for use with the Comparison Test

is convergent (*p*-series with p = 2 > 1). Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (a) of the Comparison Test.

Although the condition  $a_n \leq b_n$  or  $a_n \geq b_n$  in the Comparison Test is given for all *n*, we need verify only that it holds for  $n \geq N$ , where *N* is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

**EXAMPLE 4** Test the series 
$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$
 for convergence or divergence.

**SOLUTION** We used the Integral Test to test this series in Example 1, but we can also test it by comparing it with the harmonic series. Observe that  $\ln n > 1$  for  $n \ge 3$  and so

$$\frac{\ln n}{n} > \frac{1}{n} \qquad n \ge 3$$

We know that  $\sum 1/n$  is divergent (*p*-series with p = 1). Thus, the given series is divergent by the Comparison Test.

**NOTE** • The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply. Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because  $\sum b_n = \sum \left(\frac{1}{2}\right)^n$  is convergent and  $a_n > b_n$ . Nonetheless, we have the feeling that  $\sum 1/(2^n - 1)$  ought to be convergent because it is very similar to the convergent geometric series  $\sum \left(\frac{1}{2}\right)^n$ . In such cases the following test can be used.

**The Limit Comparison Test** Suppose that  $\Sigma a_n$  and  $\Sigma b_n$  are series with positive terms. If

$$\lim_{n\to\infty}\frac{a_n}{b_n}=c$$

where c is a finite number and c > 0, then either both series converge or both diverge.

Although we won't prove the Limit Comparison Test, it seems reasonable because for large  $n, a_n \approx cb_n$ .

**EXAMPLE 5** Test the series  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}$$
  $b_n = \frac{1}{2^n}$ 

and obtain

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2^n}{2^n - 1} = \lim_{n \to \infty} \frac{1}{1 - 1/2^n} = 1$$

Since this limit exists and  $\sum 1/2^n$  is a convergent geometric series, the given series converges by the Limit Comparison Test.



#### Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series  $\sum a_n$  is convergent and we now want to find an approximation to the sum *s* of the series. Of course, any partial sum  $s_n$  is an approximation to *s* because  $\lim_{n\to\infty} s_n = s$ . But how good is such an approximation? To find out, we need to estimate the size of the **remainder** 

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder  $R_n$  is the error made when  $s_n$ , the sum of the first *n* terms, is used as an approximation to the total sum.

We use the same notation and ideas as in the Integral Test. Comparing the areas of the rectangles with the area under y = f(x) for x > n in Figure 3, we see that

$$R_n = a_{n+1} + a_{n+2} + \dots \leq \int_n^\infty f(x) \, dx$$

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \dots \ge \int_{n+1}^{\infty} f(x) \, dx$$

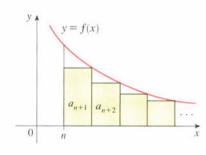
So we have proved the following error estimate.

**B** Remainder Estimate for the Integral Test If  $\sum a_n$  converges by the Integral Test and  $R_n = s - s_n$ , then

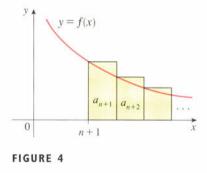
$$\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_n^{\infty} f(x) \, dx$$

#### EXAMPLE 6

(a) Approximate the sum of the series  $\sum 1/n^3$  by using the sum of the first 10 terms. Estimate the error involved in this approximation.







(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

**SOLUTION** In both parts (a) and (b) we need to know  $\int_{a}^{\infty} f(x) dx$ . With  $f(x) = 1/x^{3}$ , we have

(a) 
$$\int_{n}^{\infty} \frac{1}{x^{3}} dx = \lim_{t \to \infty} \left[ -\frac{1}{2x^{2}} \right]_{n}^{t} = \lim_{t \to \infty} \left( -\frac{1}{2t^{2}} + \frac{1}{2n^{2}} \right) = \frac{1}{2n^{2}}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^{3}} \approx s_{10} = \frac{1}{1^{3}} + \frac{1}{2^{3}} + \frac{1}{3^{3}} + \dots + \frac{1}{10^{3}} \approx 1.1975$$

According to the remainder estimate in (3), we have

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^3} dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of *n* such that  $R_n \leq 0.0005$ . Since

$$R_n \le \int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$
  
 $\frac{1}{2n^2} < 0.0005$ 

we want

$$n^2 > \frac{1}{0.001} = 1000$$
 or  $n > \sqrt{1000} \approx 31.6$ 

We need 32 terms to ensure accuracy to within 0.0005.

If we add  $s_n$  to each side of the inequalities in (3), we get

$$s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_n^{\infty} f(x) \, dx$$

because  $s_n + R_n = s$ . The inequalities in (4) give a lower bound and an upper bound for s. They provide a more accurate approximation to the sum of the series than the partial sum  $s_n$  does.

**EXAMPLE 7** Use (4) with n = 10 to estimate the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ .

**SOLUTION** The inequalities in (4) become

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3} dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^3} dx$$

1

From Example 6 we know that

$$\int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$

 $s_{10} + \frac{1}{2(11)^2} \le s \le s_{10} + \frac{1}{2(10)^2}$ 

SO

Using 
$$s_{10} \approx 1.197532$$
, we get

$$1.201664 \le s \le 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

If we compare Example 7 with Example 6, we see that the improved estimate in (4) can be much better than the estimate  $s \approx s_n$ . To make the error smaller than 0.0005 we had to use 32 terms in Example 6 but only 10 terms in Example 7.

If we have used the Comparison Test to show that a series  $\sum a_n$  converges by comparison with a series  $\sum b_n$ , then we may be able to estimate the sum  $\sum a_n$  by comparing remainders, as the following example shows.

**EXAMPLE 8** Use the sum of the first 100 terms to approximate the sum of the series  $\sum 1/(n^3 + 1)$ . Estimate the error involved in this approximation.

**SOLUTION** Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. The remainder  $T_n$  for the comparison series  $\sum 1/n^3$  was estimated in Example 6. There we found that

$$T_n \leqslant \int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$

Therefore, the remainder  $R_n$  for the given series satisfies

$$R_n \le T_n \le \frac{1}{2n^2}$$

With n = 100 we have

$$R_{100} \le \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005.



- Exercises · · · · · · · · · · · ·
- I. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_{1}^{\infty} \frac{1}{x^{1.3}} \, dx$$

What can you conclude about the series?

 Suppose f is a continuous positive decreasing function for x ≥ 1 and a<sub>n</sub> = f(n). By drawing a picture, rank the following three quantities in increasing order:

$$\int_{1}^{n} f(x) \, dx \qquad \sum_{i=1}^{n-1} a_i \qquad \sum_{i=2}^{n} a_i$$

- Suppose Σ a<sub>n</sub> and Σ b<sub>n</sub> are series with positive terms and Σ b<sub>n</sub> is known to be convergent.
  - (a) If  $a_n > b_n$  for all *n*, what can you say about  $\sum a_n$ ? Why?
  - (b) If  $a_n < b_n$  for all *n*, what can you say about  $\sum a_n$ ? Why?
- Suppose Σ a<sub>n</sub> and Σ b<sub>n</sub> are series with positive terms and Σ b<sub>n</sub> is known to be divergent.
  - (a) If  $a_n > b_n$  for all *n*, what can you say about  $\sum a_n$ ? Why?
  - (b) If  $a_n < b_n$  for all *n*, what can you say about  $\sum a_n$ ? Why?
- 5. It's important to distinguish between

$$\sum_{n=1}^{\infty} n^b \quad \text{and} \quad \sum_{n=1}^{\infty} b^n$$

What name is given to the first series? To the second? For what values of b does the first series converge? For what values of b does the second series converge?

Determine whether the series is convergent or divergent.

6. 
$$\sum_{n=1}^{\infty} \left( \frac{2}{n\sqrt{n}} + \frac{3}{n^3} \right)$$
  
7. 
$$\sum_{n=1}^{\infty} ne^{-n^2}$$
8. 
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$
  
9. 
$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$
10. 
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$
  
11. 
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$
12. 
$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$
  
13. 
$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^2}$$
14. 
$$\sum_{n=1}^{\infty} \frac{3}{4^n + 5}$$
  
15. 
$$\sum_{n=0}^{\infty} \frac{1 + 5^n}{4^n}$$
16. 
$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$$

**17.** 
$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$
  
**18.**  $\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$   
**19.**  $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$   
**20.**  $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$   
**21.**  $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$   
**22.**  $\sum_{n=1}^{\infty} \frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}$ 

**23.** Find the values of *p* for which the following series is convergent:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

- 24. (a) Find the partial sum s<sub>10</sub> of the series ∑<sup>∞</sup><sub>n=1</sub> 1/n<sup>4</sup>. Estimate the error in using s<sub>10</sub> as an approximation to the sum of the series.
  - (b) Use (4) with n = 10 to give an improved estimate of the sum.
  - (c) Find a value of n so that  $s_n$  is within 0.00001 of the sum.
- **25.** (a) Use the sum of the first 10 terms to estimate the sum of the series  $\sum_{n=1}^{\infty} 1/n^2$ . How good is this estimate?
  - (b) Improve this estimate using (4) with n = 10.
  - (c) Find a value of *n* that will ensure that the error in the approximation  $s \approx s_n$  is less than 0.001.
- **26.** Find the sum of the series  $\sum_{n=1}^{\infty} 1/n^5$  correct to three decimal places.
- **27.** Estimate  $\sum_{n=1}^{\infty} n^{-3/2}$  to within 0.01.
- **28.** How many terms of the series  $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$  would you need to add to find its sum to within 0.01?

29-30 • Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

**29.** 
$$\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2}$$
 **30.**  $\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$ 

**31.** (a) Use a graph of y = 1/x to show that if  $s_n$  is the *n*th partial sum of the harmonic series, then

$$s_n \leq 1 + \ln n$$

- (b) The harmonic series diverges but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.
- (a) Show that the series ∑<sup>∞</sup><sub>n=1</sub> (ln n)<sup>2</sup>/n<sup>2</sup> is convergent.
   (b) Find an upper bound for the error in the approximation s ≈ s<sub>n</sub>.

- (c) What is the smallest value of *n* such that this upper bound is less than 0.05?
- (d) Find  $s_n$  for this value of n.
- **33.** The meaning of the decimal representation of a number  $0.d_1d_2d_3...$  (where the digit  $d_i$  is one of the numbers 0, 1, 2, ..., 9) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series always converges.

- **34.** Find all positive values of *b* for which the series  $\sum_{n=1}^{\infty} b^{\ln n}$  converges.
- **35.** If  $\sum a_n$  is a convergent series with positive terms, is it true that  $\sum \sin(a_n)$  is also convergent?
- **36.** Show that if  $a_n > 0$  and  $\sum a_n$  is convergent, then  $\sum \ln(1 + a_n)$  is convergent.

and a second second



# Other Convergence Tests

The convergence tests that we have looked at so far apply only to series with positive terms. In this section we learn how to deal with series whose terms are not necessarily positive.

#### Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$
$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n$$
 or  $a_n = (-1)^n b_n$ 

where  $b_n$  is a positive number. (In fact,  $b_n = |a_n|$ .)

The following test says that if the terms of an alternating series decrease to 0 in absolute value, then the series converges.

The Alternating Series Test If the alternating series

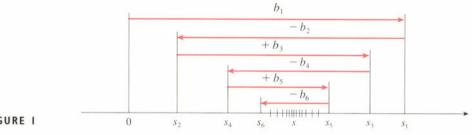
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(a) 
$$b_{n+1} \le b_n$$
 for all  $n$   
(b)  $\lim_{n \to \infty} b_n = 0$ 

then the series is convergent.

We won't present a formal proof of this test, but Figure 1 gives a picture of the idea behind the proof. We first plot  $s_1 = b_1$  on a number line. To find  $s_2$  we subtract  $b_2$ , so  $s_2$  is to the left of  $s_1$ . Then to find  $s_3$  we add  $b_3$ , so  $s_3$  is to the right of  $s_2$ . But, since  $b_3 < b_2$ ,  $s_3$  is to the left of  $s_1$ . Continuing in this manner, we see that the partial sums oscillate back and forth. Since  $b_n \rightarrow 0$ , the successive steps are becoming smaller and smaller. The even partial sums  $s_2, s_4, s_6, \ldots$  are increasing and the odd partial sums  $s_1, s_3, s_5, \ldots$  are decreasing. Thus, it seems plausible that both are converging to some number s, which is the sum of the series.





the graphs of the terms  $a_n = (-1)^{n-1}/n$ and the partial sums  $s_n$ . Notice how the values of  $s_n$  zigzag across the limiting value, which appears to be about 0.7. In fact, it can be proved that the exact sum of the series is  $\ln 2 \approx 0.693$ .

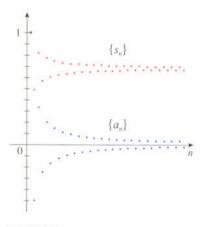


Figure 2 illustrates Example 1 by showing **EXAMPLE 1** The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(a) 
$$b_{n+1} < b_n$$
 because  $\frac{1}{n+1} < \frac{1}{n}$   
(b)  $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$ 

so the series is convergent by the Alternating Series Test.

**EXAMPLE 2** The series  $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$  is alternating, but  $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{3n}{4n - 1} = \lim_{n \to \infty} \frac{3}{4 - \frac{1}{4}} = \frac{3}{4}$ 

FIGURE 2

so condition (b) is not satisfied. Instead, we look at the limit of the nth term of the series:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{(-1)^n 3n}{4n - 1}$$

This limit does not exist, so the series diverges by the Test for Divergence.

**EXAMPLE 3** Test the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}$  for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (a) and (b) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by  $b_n = n^2/(n^3 + 1)$  is decreasing. However, if we consider the related function  $f(x) = x^2/(x^3 + 1)$ , we find that

$$f'(x) = \frac{x(2 - x^3)}{(x^3 + 1)^2}$$

Since we are considering only positive x, we see that f'(x) < 0 if  $2 - x^3 < 0$ , that is,  $x > \sqrt[3]{2}$ . Thus, f is decreasing on the interval  $(\sqrt[3]{2}, \infty)$ . This means that f(n + 1) < f(n) and therefore  $b_{n+1} < b_n$  when  $n \ge 2$ . (The inequality  $b_2 < b_1$  can be verified directly but all that really matters is that the sequence  $\{b_n\}$  is eventually decreasing.)

Condition (b) is readily verified:

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n^2}{n^3 + 1} = \lim_{n \to \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$$

Thus, the given series is convergent by the Alternating Series Test.

The error involved in using the partial sum  $s_n$  as an approximation to the total sum s is the remainder  $R_n = s - s_n$ . The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than  $b_{n+1}$ , which is the absolute value of the first neglected term.

Alternating Series Estimation Theorem If  $s = \sum (-1)^{n-1} b_n$  is the sum of an alternating series that satisfies (a)  $0 < b_{n+1} \le b_n$  and (b)  $\lim_{n \to \infty} b_n = 0$ then  $|R_n| = |s - s_n| \le b_{n+1}$ 

You can see geometrically why this is true by looking at Figure 1. Notice that  $s - s_4 < b_5$ ,  $|s - s_5| < b_6$ , and so on.

**EXAMPLE 4** Find the sum of the series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  correct to three decimal places. (By definition, 0! = 1.)

**SOLUTION** We first observe that the series is convergent by the Alternating Series Test because

(a) 
$$\frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!}$$
  
(b)  $0 < \frac{1}{n!} < \frac{1}{n} \to 0$  so  $\frac{1}{n!} \to 0$  as  $n \to \infty$ 

Instead of verifying condition (a) of the Alternating Series Test by computing a derivative, we could verify that  $b_{n+1} < b_n$  directly by using the technique of Solution 1 of Example 10 in Section 8.1.

To give us a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots$$

Notice that

 $b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$ 

and

 $\oslash$ 

 $s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$ 

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \le b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have

$$s \approx 0.368$$

correct to three decimal places.

In Section 8.7 we will prove that  $e^x = \sum_{n=0}^{\infty} x^n/n!$  for all *x*, so what we have obtained in this example is actually an approximation to the number  $e^{-1}$ .

**NOTE** • The rule that the error (in using  $s_n$  to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

#### Absolute Convergence

Given any series  $\sum a_n$ , we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

**Definition** A series  $\Sigma a_n$  is called **absolutely convergent** if the series of absolute values  $\Sigma |a_n|$  is convergent.

Notice that if  $\sum a_n$  is a series with positive terms, then  $|a_n| = a_n$  and so absolute convergence is the same as convergence.

**EXAMPLE 5** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

We have convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 7 that the idea of absolute convergence sometimes helps in such cases. is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent *p*-series (p = 2).

**EXAMPLE 6** We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent (see Example 1), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (*p*-series with p = 1) and is therefore divergent.

Example 6 shows it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

**Theorem** If a series  $\sum a_n$  is absolutely convergent, then it is convergent.

To see why Theorem 1 is true, observe that the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

is true because  $|a_n|$  is either  $a_n$  or  $-a_n$ . If  $\sum a_n$  is absolutely convergent, then  $\sum |a_n|$  is convergent, so  $\sum 2 |a_n|$  is convergent. Therefore, by the Comparison Test,  $\sum (a_n + |a_n|)$  is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

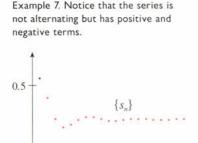
is the difference of two convergent series and is therefore convergent.

**EXAMPLE 7** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

**SOLUTION** This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive. The signs change irregularly.) We can apply the Comparison



 $\{a_n\}$ 

Figure 3 shows the graphs of the terms

 $a_n$  and partial sums  $s_n$  of the series in



0

Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since  $|\cos n| \le 1$  for all *n*, we have

$$\frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$

We know that  $\sum 1/n^2$  is convergent (*p*-series with p = 2) and, therefore,

 $\sum |\cos n|/n^2$  is convergent by the Comparison Test. Thus, the given series

 $\sum (\cos n)/n^2$  is absolutely convergent and therefore convergent by Theorem 1.

## The Ratio Test

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio To	st					
(a) If $\lim_{n\to\infty}$	(a) If $\lim_{n \to \infty} \left  \frac{a_{n+1}}{a_n} \right  = L < 1$ , then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent					
	(and therefore convergent).					
(b) If $\lim_{n \to \infty}$	$\frac{a_{n+1}}{a_n} = L > 1 \text{ or } \lim_{n \to \infty} \left  \frac{a_{n+1}}{a_n} \right  = \infty$ , then the series	$\sum_{n=1}^{\infty} a_n$ is				
diverge	t.					

The Ratio Test can be proved by comparing the given series to a geometric series. It's understandable that geometric series are involved because, for those series, the ratio r of consecutive terms is constant and the series converges if |r| < 1. In part (a) of the Ratio Test, the ratio of consecutive terms isn't constant but  $|a_{n+1}/a_n| \rightarrow L$  so, for large n,  $|a_{n+1}/a_n|$  is almost constant and the series converges if L < 1.

**NOTE** • If  $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$ , the Ratio Test gives no information. For instance, for the convergent series  $\sum 1/n^2$  we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series  $\sum 1/n$  we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \quad \text{as } n \to \infty$$

Therefore, if  $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$ , the series  $\sum a_n$  might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

**EXAMPLE 8** Test the series  $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$  for absolute convergence.

**SOLUTION** We use the Ratio Test with  $a_n = (-1)^n n^3 / 3^n$ :

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3} \left( \frac{n+1}{n} \right)^3 = \frac{1}{3} \left( 1 + \frac{1}{n} \right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

**EXAMPLE 9** Test the convergence of the series  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ .

**SOLUTION** Since the terms  $a_n = n^n/n!$  are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$
$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e \quad \text{as } n \to \infty$$

(see Equation 6 in Section 3.7). Since e > 1, the given series is divergent by the Ratio Test.

**NOTE** • Although the Ratio Test works in Example 9, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \ge n$$

it follows that  $a_n$  does not approach 0 as  $n \to \infty$ . Therefore, the given series is divergent by the Test for Divergence.



I. (a) What is an alternating series?

Exercises ·

- (b) Under what conditions does an alternating series converge?
- (c) If these conditions are satisfied, what can you say about the remainder after *n* terms?
- **2.** What can you say about the series  $\sum a_n$  in each of the following cases?

(a) 
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$$

We have used various methods for estimating the sum of a series—the method depended on which test was used to prove convergence. What about series for which the Ratio Test works? There are two possibilities: If the series happens to be an alternating series, as in Example 8, then it is best to use the Alternating Series Estimation Theorem. If the terms are all positive, then use the special methods explained in Exercise 34.

**Estimating Sums** 

(b) 
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8$$
  
(c)  $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ 

**3−8** ■ Test the series for convergence or divergence.

**3.** 
$$\frac{3}{5} - \frac{3}{6} + \frac{3}{7} - \frac{3}{8} + \frac{3}{9} - \cdots$$
  
**4.**  $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \frac{1}{\ln 6} - \cdots$   
**5.**  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$   
**6.**  $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2 + 1}$   
**7.**  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$   
**8.**  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$ 

- **9.** Is the 50th partial sum  $s_{50}$  of the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1}/n$  an overestimate or an underestimate of the total sum? Explain.
- 🚰 10. Calculate the first 10 partial sums of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

and graph both the sequence of terms and the sequence of partial sums on the same screen. Estimate the error in using the 10th partial sum to approximate the total sum.

**11.** For what values of *p* is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

**12–14** How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

12. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$
 (error < 0.001)  
13.  $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$  (error < 0.01)  
14.  $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$  (error < 0.002)

15-16 Graph both the sequence of terms and the sequence of partial sums on the same screen. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.

**15.** 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$$
 **16.**  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ 

17–18 Approximate the sum of the series to the indicated accuracy.

17. 
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$
 (four decimal places)  
18. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6}$$
 (five decimal places)

**19–28** Determine whether the series is absolutely convergent.

$$19. \sum_{n=1}^{\infty} \frac{(-3)^n}{n^3} \qquad 20. \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} \\
21. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} \qquad 22. \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1} \\
23. \sum_{n=1}^{\infty} \frac{\sin 2n}{n^2} \qquad 24. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}5^{n-1}}{(n+1)^24^{n+2}} \\
25. \sum_{n=1}^{\infty} \frac{(n+1)5^n}{n3^{2n}} \qquad 26. \sum_{n=1}^{\infty} \frac{\cos(n\pi/6)}{n\sqrt{n}} \\
27. \sum_{n=1}^{\infty} \frac{(n+2)!}{n! 10^n} \\
28. \frac{1}{3} + \frac{1 \cdot 4}{3 \cdot 5} + \frac{1 \cdot 4 \cdot 7}{2 \cdot 5 \cdot 7} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{2 \cdot 5 \cdot 7} + \dots$$

$$\frac{1}{3} + \frac{1}{3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots + \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} + \cdots$$

**29.** The terms of a series are defined recursively by the equations

$$a_1 = 2$$
  $a_{n+1} = \frac{5n+1}{4n+3} a_n$ 

Determine whether  $\sum a_n$  converges or diverges.

**30.** A series  $\sum a_n$  is defined by the equations

$$a_1 = 1$$
  $a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$ 

Determine whether  $\sum a_n$  converges or diverges.

**31.** For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a) 
$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$
 (b)  $\sum_{n=1}^{\infty} \frac{n}{2^n}$   
(c)  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$  (d)  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ 

**32.** For which positive integers k is the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

convergent?

- (a) Show that ∑<sub>n=0</sub><sup>∞</sup> x<sup>n</sup>/n! converges for all x.
  (b) Deduce that lim<sub>n→∞</sub> x<sup>n</sup>/n! = 0 for all x.
- 34. Let Σ a<sub>n</sub> be a series with positive terms and let r<sub>n</sub> = a<sub>n+1</sub>/a<sub>n</sub>. Suppose that lim<sub>n→∞</sub> r<sub>n</sub> = L < 1, so Σ a<sub>n</sub> converges by the Ratio Test. As usual, we let R<sub>n</sub> be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) If  $\{r_n\}$  is a decreasing sequence and  $r_{n+1} < 1$ , show, by summing a geometric series, that

$$R_n \le \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) If  $\{r_n\}$  is an increasing sequence, show that

$$R_n \leqslant \frac{a_{n+1}}{1-L}$$

**35.** (a) Find the partial sum  $s_5$  of the series

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

Use Exercise 34 to estimate the error in using  $s_5$  as an approximation to the sum of the series.

- (b) Find a value of n so that  $s_n$  is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.
- **36.** Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 34 to estimate the error.



Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the  $c_n$ 's are constants called the **coefficients** of the series. For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take  $c_n = 1$  for all *n*, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}$$

which converges when -1 < x < 1 and diverges when  $|x| \ge 1$  (see Equation 5 in Section 8.2).

More generally, a series of the form

2 
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

is called a **power series in** (x - a) or a **power series centered at** a or a **power series about** a. Notice that in writing out the term corresponding to n = 0 in Equations 1 and 2 we have adopted the convention that  $(x - a)^0 = 1$  even when

x = a. Notice also that when x = a all of the terms are 0 for  $n \ge 1$  and so the power series (2) always converges when x = a.

**EXAMPLE I** For what values of x is the series  $\sum_{n=0}^{\infty} n! x^n$  convergent?

**SOLUTION** We use the Ratio Test. If we let  $a_n$ , as usual, denote the *n*th term of the series, then  $a_n = n!x^n$ . If  $x \neq 0$ , we have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right|$$
$$= \lim_{n \to \infty} (n+1) |x| = \infty$$

By the Ratio Test, the series diverges when  $x \neq 0$ . Thus, the given series converges only when x = 0.

**EXAMPLE 2** For what values of x does the series  $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$  converge?

**SOLUTION** Let  $a_n = (x - 3)^n/n$ . Then

$$\frac{a_{n+1}}{a_n} = \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right|$$
$$= \frac{1}{1+\frac{1}{n}} |x-3| \to |x-3| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when |x - 3| < 1 and divergent when |x - 3| > 1. Now

 $|x-3| < 1 \iff -1 < x - 3 < 1 \iff 2 < x < 4$ 

so the series converges when 2 < x < 4 and diverges when x < 2 or x > 4.

The Ratio Test gives no information when |x - 3| = 1 so we must consider x = 2 and x = 4 separately. If we put x = 4 in the series, it becomes  $\sum 1/n$ , the harmonic series, which is divergent. If x = 2, the series is  $\sum (-1)^n/n$ , which converges by the Alternating Series Test. Thus, the given power series converges for  $2 \le x < 4$ .

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, the sum of the power series in the next example is called a **Bessel function**, after the German astronomer Friedrich Bessel (1784–1846), and the function given in Exercise 21 is another example of a Bessel function. In fact, these functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead (see the photographs on pages 558–559).

**EXAMPLE 3** Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

**SOLUTION** Let  $a_n = (-1)^n x^{2n} / [2^{2n} (n!)^2]$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} [(n+1)!]^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right|$$
$$= \frac{x^{2n+2}}{2^{2n+2} (n+1)^2 (n!)^2} \cdot \frac{2^{2n} (n!)^2}{x^{2n}}$$
$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \quad \text{for all } x$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function  $J_0$  is  $(-\infty, \infty) = \mathbb{R}$ .

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 3 as the sum of a series we mean that, for every real number x,

$$J_0(x) = \lim_{n \to \infty} s_n(x) \quad \text{where} \quad s_{2n}(x) = \sum_{i=0}^n \frac{(-1)^i x^{2i}}{2^{2i} (i!)^2}$$

The first few partial sums are

56(x

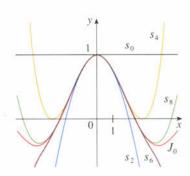
$$s_0(x) = 1 \qquad s_2(x) = 1 - \frac{x^2}{4} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$
$$y = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_8(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147,456}$$

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function  $J_0$ , but notice that the approximations become better when more terms are included. Figure 2 shows a more complete graph of the Bessel function.

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval for the geometric series and the series in Example 2, the infinite interval  $(-\infty, \infty)$  in Example 3, and a collapsed interval  $[0, 0] = \{0\}$  in Example 1]. The following theorem, which we won't prove, says that this is true in general.

**3 Theorem** For a given power series  $\sum_{n=0}^{\infty} c_n (x - a)^n$  there are only three possibilities:

- (i) The series converges only when x = a.
- (ii) The series converges for all x.
- (iii) There is a positive number R such that the series converges if |x a| < R and diverges if |x a| > R.



**FIGURE 1** Partial sums of the Bessel function  $J_0$ 

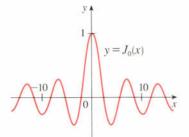
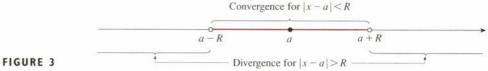


FIGURE 2

The number R in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is R = 0 in case (i) and  $R = \infty$ in case (ii). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is  $(-\infty, \infty)$ . In case (iii) note that the inequality |x - a| < R can be rewritten as a - R < x < a + R. When x is an *endpoint* of the interval, that is,  $x = a \pm R$ , anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus, in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R)$$
  $(a - R, a + R]$   $[a - R, a + R]$   $[a - R, a + R]$ 

The situation is illustrated in Figure 3.



We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	R = 1	(-1,1)
Example 1	$\sum_{n=0}^{\infty} n! x^n$	R = 0	{0}
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	R = 1	[2,4)
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$	$R = \infty$	$(-\infty,\infty)$

The Ratio Test can be used to determine the radius of convergence R in most cases. The Ratio Test always fails when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

**EXAMPLE 4** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

**SOLUTION** Let  $a_n = (-3)^n x^n / \sqrt{n+1}$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$
$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}} |x| \to 3|x| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if 3|x| < 1 and diverges if 3|x| > 1. Thus, it converges if  $|x| < \frac{1}{3}$  and diverges if  $|x| > \frac{1}{3}$ . This means that the radius of convergence is  $R = \frac{1}{3}$ .

We know the series converges in the interval  $\left(-\frac{1}{3}, \frac{1}{3}\right)$ , but we must now test for convergence at the endpoints of this interval. If  $x = -\frac{1}{3}$ , the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which diverges. (Use the Integral Test or simply observe that it is a *p*-series with  $p = \frac{1}{2} < 1$ .) If  $x = \frac{1}{3}$ , the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore, the given power series converges when  $-\frac{1}{3} < x \leq \frac{1}{3}$ , so the interval of convergence is  $\left(-\frac{1}{3}, \frac{1}{3}\right]$ .

**EXAMPLE 5** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

**SOLUTION** If  $a_n = n(x + 2)^n / 3^{n+1}$ , then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right|$$
$$= \left( 1 + \frac{1}{n} \right) \frac{|x+2|}{3} \to \frac{|x+2|}{3} \quad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if |x + 2|/3 < 1 and it diverges if |x + 2|/3 > 1. So it converges if |x + 2| < 3 and diverges if |x + 2| > 3. Thus, the radius of convergence is R = 3.

The inequality |x + 2| < 3 can be written as -5 < x < 1, so we test the series at the endpoints -5 and 1. When x = -5, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence  $[(-1)^n n$  doesn't converge to 0]. When x = 1, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus, the series converges only when -5 < x < 1, so the interval of convergence is (-5, 1).



I. What is a power series?

Exercises

- (a) What is the radius of convergence of a power series? How do you find it?
  - (b) What is the interval of convergence of a power series? How do you find it?
- **3.** If  $\sum_{n=0}^{\infty} c_n 4^n$  is convergent, does it follow that the following series are convergent?

(a) 
$$\sum_{n=0}^{\infty} c_n (-2)^n$$
 (b)  $\sum_{n=0}^{\infty} c_n (-4)^n$ 

**4.** Suppose that  $\sum_{n=0}^{\infty} c_n x^n$  converges when x = -4 and diverges when x = 6. What can be said about the convergence or divergence of the following series?

(a) 
$$\sum_{n=0}^{\infty} c_n$$
 (b)  $\sum_{n=0}^{\infty} c_n 8^n$   
(c)  $\sum_{n=0}^{\infty} c_n (-3)^n$  (d)  $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$ 

5–18 ■ Find the radius of convergence and interval of convergence of the series.

5. 
$$\sum_{n=0}^{\infty} \frac{x^n}{n+2}$$
  
6.  $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$   
7.  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$   
8.  $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$   
9.  $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$   
10.  $\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$   
11.  $\sum_{n=0}^{\infty} \frac{n}{4^n} (2x-1)^n$   
12.  $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$   
13.  $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{\sqrt{n}}$   
14.  $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n}$   
15.  $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$   
16.  $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n(n+1)}$   
17.  $\sum_{n=1}^{\infty} n! (2x-1)^n$   
18.  $\sum_{n=1}^{\infty} \frac{nx^n}{1\cdot 3\cdot 5\cdot \cdots (2n-1)}$ 

**19.** If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x$$

- **20.** Graph the first several partial sums  $s_n(x)$  of the series  $\sum_{n=0}^{\infty} x^n$ , together with the sum function f(x) = 1/(1 x), on a common screen. On what interval do these partial sums appear to be converging to f(x)?
  - **21.** The function  $J_1$  defined by

. . . . . . . . . . . . . . . . . .

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}$$

is called the Bessel function of order 1.

- (a) Find its domain.
- (b) Graph the first several partial sums on a common screen.
- (c) If your CAS has built-in Bessel functions, graph  $J_1$ on the same screen as the partial sums in part (b) and observe how the partial sums approximate  $J_1$ .
  - **22.** The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \dots$$

is called the *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).(a) Find the domain of the Airy function.

- (b) Graph the first several partial sums  $s_n(x)$  on a common screen.
- (c) If your CAS has built-in Airy functions, graph A on the same screen as the partial sums in part (b) and observe how the partial sums approximate A.
  - **23.** A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

that is, its coefficients are  $c_{2n} = 1$  and  $c_{2n+1} = 2$  for all  $n \ge 0$ . Find the interval of convergence of the series and find an explicit formula for f(x).

- **24.** If  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ , where  $c_{n+4} = c_n$  for all  $n \ge 0$ , find the interval of convergence of the series and a formula for f(x).
- **25.** Suppose the series  $\sum c_n x^n$  has radius of convergence 2 and the series  $\sum d_n x^n$  has radius of convergence 3. What is the radius of convergence of the series  $\sum (c_n + d_n)x^n$ ? Explain.
- **26.** Suppose that the radius of convergence of the power series  $\sum c_n x^n$  is *R*. What is the radius of convergence of the power series  $\sum c_n x^{2n}$ ?



## Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. (Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.)

We start with an equation that we have seen before:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

We first encountered this equation in Example 5 in Section 8.2, where we obtained it by observing that it is a geometric series with a = 1 and r = x. But here our point of view is different. We now regard Equation 1 as expressing the function f(x) = 1/(1 - x) as a sum of a power series.

**EXAMPLE 1** Express  $1/(1 + x^2)$  as the sum of a power series and find the interval of convergence.

**SOLUTION** Replacing x by  $-x^2$  in Equation 1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Because this is a geometric series, it converges when  $|-x^2| < 1$ , that is,  $x^2 < 1$ , or |x| < 1. Therefore, the interval of convergence is (-1, 1). (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.)

**EXAMPLE 2** Find a power series representation for 1/(x + 2).

**SOLUTION** In order to put this function in the form of the left side of Equation 1 we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

This series converges when |-x/2| < 1, that is, |x| < 2. So the interval of convergence is (-2, 2).

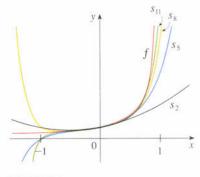
A geometric illustration of Equation I is shown in Figure I. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n \to \infty} s_n(x)$$

1

where

 $s_n(x) = 1 + x + x^2 + \dots + x^n$ is the *n*th partial sum. Notice that as *n* increases,  $s_n(x)$  becomes a better approximation to f(x) for -1 < x < 1.



**FIGURE I**  $f(x) = \frac{1}{1-x}$  and some partial sums

**EXAMPLE 3** Find a power series representation of  $x^3/(x + 2)$ .

**SOLUTION** Since this function is just  $x^3$  times the function in Example 2, all we have to do is to multiply that series by  $x^3$ :

It's legitimate to move  $x^3$  across the sigma sign because it doesn't depend on *n*. [Use Theorem 8(i) in Section 8.2 with  $c = x^3$ .]

$$\frac{x^3}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$
$$= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \dots$$

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

As in Example 2, the interval of convergence is (-2, 2).

## Differentiation and Integration of Power Series

The sum of a power series is a function  $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$  whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

**2** Theorem If the power series  $\sum c_n(x-a)^n$  has radius of convergence R > 0, then the function *f* defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval (a - R, a + R)and

(a) 
$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{n} nc_n(x-a)^{n-1}$$

(b) 
$$\int f(x) dx = C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \dots$$
  
=  $C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n+1}$ 

The radii of convergence of the power series in Equations (a) and (b) are both R.

NOTE I • Equations (a) and (b) in Theorem 2 can be rewritten in the form

(c) 
$$\frac{d}{dx}\left[\sum_{n=0}^{\infty}c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n]$$
  
(d)  $\int\left[\sum_{n=0}^{\infty}c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^ndx$ 

In part (b),  $\int c_0 dx = c_0 x + C_1$  is written as  $c_0(x - a) + C$ , where  $C = C_1 + ac_0$ , so all the terms of the series have the same form. We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (c) and (d) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple; see Exercise 32.)

**NOTE 2** • Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there. (See Exercise 33.)

**NOTE 3** • The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. We will discuss this method in Section 8.10.

**EXAMPLE 4** In Example 3 in Section 8.5 we saw that the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

is defined for all x. Thus, by Theorem 2,  $J_0$  is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n} (n!)^2}$$

**EXAMPLE 5** Express  $1/(1 - x)^2$  as a power series by differentiating Equation 1. What is the radius of convergence?

**SOLUTION** Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$$

we get

If we wish, we can replace n by n + 1 and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, R = 1.

**EXAMPLE 6** Find a power series representation for  $\ln(1 - x)$  and its radius of convergence.

**SOLUTION** We notice that, except for a factor of -1, the derivative of this function is 1/(1 - x). So we integrate both sides of Equation 1:

$$-\ln(1-x) = \int \frac{1}{1-x} dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$
$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

To determine the value of C we put x = 0 in this equation and obtain  $-\ln(1 - 0) = C$ . Thus, C = 0 and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as for the original series: R = 1.

Notice what happens if we put  $x = \frac{1}{2}$  in the result of Example 6. Since  $\ln \frac{1}{2} = -\ln 2$ , we see that

$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \dots = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

**EXAMPLE 7** Find a power series representation for  $f(x) = \tan^{-1}x$ .

**SOLUTION** We observe that  $f'(x) = 1/(1 + x^2)$  and find the required series by integrating the power series for  $1/(1 + x^2)$  found in Example 1.

$$\tan^{-1}x = \int \frac{1}{1+x^2} dx = \int (1-x^2+x^4-x^6+\cdots) dx$$
$$= C+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

To find C we put x = 0 and obtain  $C = \tan^{-1} 0 = 0$ . Therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of the series for  $1/(1 + x^2)$  is 1, the radius of convergence of this series for  $\tan^{-1}x$  is also 1.

#### EXAMPLE 8

- (a) Evaluate  $\int [1/(1 + x^7)] dx$  as a power series.
- (b) Use part (a) to approximate  $\int_{0}^{0.5} [1/(1 + x^7)] dx$  correct to within  $10^{-7}$ .

#### SOLUTION

(a) The first step is to express the integrand,  $1/(1 + x^7)$ , as the sum of a power series. As in Example 1, we start with Equation 1 and replace x by  $-x^7$ :

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \dots$$

Now we integrate term by term:

$$\int \frac{1}{1+x^7} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$
$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots$$

This series converges for  $|-x^7| < 1$ , that is, for |x| < 1.

The power series for  $\tan^{-1}x$  obtained in Example 7 is called *Gregory's series* after the Scottish mathematician James Gregory (1638–1675), who had anticipated some of Newton's discoveries. We have shown that Gregory's series is valid when  $-1 \le x \le 1$ , but it turns out (although it isn't easy to prove) that it is also valid when  $x = \pm 1$ . Notice that when x = 1 the series becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This beautiful result is known as the Leibniz formula for  $\pi$ .

This example demonstrates one way in which power series representations are useful. Integrating  $1/(1 + x^7)$  by hand is incredibly difficult. Different computer algebra systems return different forms of the answer, but they are all extremely complicated. (If you have a CAS, try it yourself.) The infinite series answer that we obtain in Example 8(a) is actually much easier to deal with than the finite answer provided by a CAS. 1

(b) In applying the Evaluation Theorem it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with C = 0:

$$\int_{0}^{0.5} \frac{1}{1+x^{7}} dx = \left[ x - \frac{x^{8}}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \right]_{0}^{1/2}$$
$$= \frac{1}{2} - \frac{1}{8 \cdot 2^{8}} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots + \frac{(-1)^{n}}{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with n = 3, the error is smaller than the term with n = 4:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_{0}^{0.5} \frac{1}{1+x^{7}} dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^{8}} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$

and a second second second second second



Exercises

1. If the radius of convergence of the power series  $\sum_{n=0}^{\infty} c_n x^n$  is 10, what is the radius of convergence of the series  $\sum_{n=1}^{\infty} nc_n x^{n-1}$ ? Why?

**2.** Suppose you know that the series  $\sum_{n=0}^{\infty} b_n x^n$  converges for |x| < 2. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

**3–8** Find a power series representation for the function and determine the interval of convergence.

**3.** 
$$f(x) = \frac{1}{1+x}$$
  
**4.**  $f(x) = \frac{x}{1-x}$   
**5.**  $f(x) = \frac{1}{1+4x^2}$   
**6.**  $f(x) = \frac{1}{x^4+16}$   
**7.**  $f(x) = \frac{x}{x-3}$   
**8.**  $f(x) = \frac{1+x^2}{1-x^2}$ 

**9–14** ■ Find a power series representation for the function and determine the radius of convergence.

9. 
$$f(x) = \frac{1}{(1+x)^2}$$
  
10.  $f(x) = \ln(1+x)$   
11.  $f(x) = \frac{1}{(1+x)^3}$   
12.  $f(x) = x \ln(1+x)$ 

**15–18** Find a power series representation for 
$$f$$
, and graph  $f$  and several partial sums  $s_n(x)$  on the same screen. What happens as  $n$  increases?

**13.**  $f(x) = \ln(5 - x)$  **14.**  $f(x) = \frac{x^2}{(1 - 2x)^2}$ 

**15.** 
$$f(x) = \ln(3 + x)$$
  
**16.**  $f(x) = \frac{1}{x^2 + 25}$   
**17.**  $f(x) = \ln\left(\frac{1+x}{1-x}\right)$   
**18.**  $f(x) = \tan^{-1}(2x)$ 

**19–22** Evaluate the indefinite integral as a power series.

**19.** 
$$\int \frac{1}{1+x^4} dx$$
  
**20.**  $\int \frac{x}{1+x^5} dx$   
**21.**  $\int \frac{\arctan x}{x} dx$   
**22.**  $\int \tan^{-1}(x^2) dx$ 

**23–26** Use a power series to approximate the definite integral to six decimal places.

- **23.**  $\int_{0}^{0.2} \frac{1}{1+x^4} dx$  **24.**  $\int_{0}^{1/2} \tan^{-1}(x^2) dx$ **25.**  $\int_{0}^{1/3} x^2 \tan^{-1}(x^4) dx$  **26.**  $\int_{0}^{0.5} \frac{dx}{1+x^6}$
- **27.** Use the result of Example 6 to compute ln 1.1 correct to five decimal places.

**28.** Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

**29.** (a) Show that  $J_0$  (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^{2}J_{0}''(x) + xJ_{0}'(x) + x^{2}J_{0}(x) = 0$$

- (b) Evaluate  $\int_0^1 J_0(x) dx$  correct to three decimal places.
- 30. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}$$

(a) Show that  $J_1$  satisfies the differential equation

$$x^{2}J_{1}''(x) + xJ_{1}'(x) + (x^{2} - 1)J_{1}(x) = 0$$

- (b) Show that  $J'_0(x) = -J_1(x)$ .
- **31.** (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x)$$



## (b) Show that $f(x) = e^x$ .

**32.** Let  $f_n(x) = (\sin nx)/n^2$ . Show that the series  $\sum f_n(x)$  converges for all values of x but the series of derivatives  $\sum f'_n(x)$  diverges when  $x = 2n\pi$ , n an integer. For what values of x does the series  $\sum f''_n(x)$  converge?

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f, f', and f''.

**34.** (a) Starting with the geometric series  $\sum_{n=0}^{\infty} x^n$ , find the sum of the series

$$\sum_{n=1}^{\infty} nx^{n-1} \qquad |x| < 1$$

(b) Find the sum of each of the following series.

(i) 
$$\sum_{n=1}^{\infty} nx^n$$
,  $|x| < 1$  (ii)  $\sum_{n=1}^{\infty} \frac{n}{2^n}$ 

(c) Find the sum of each of the following series.

(i) 
$$\sum_{n=2}^{\infty} n(n-1)x^n$$
,  $|x| < 1$   
(ii)  $\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$   
(iii)  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ 

## Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \dots$$
$$|x - a| < R$$

Let's try to determine what the coefficients  $c_n$  must be in terms of f. To begin, notice that if we put x = a in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

By Theorem 2 in Section 8.6, we can differentiate the series in Equation 1 term by term:

2 
$$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots$$
  
 $|x - a| < R$ 

R

and substitution of x = a in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots$$

Again we put x = a in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

4

3

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \dots$$
$$|x - a| < R$$

and substitution of x = a in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute x = a, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Solving this equation for the *n*th coefficient  $c_n$ , we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for n = 0 if we adopt the conventions that 0! = 1 and  $f^{(0)} = f$ . Thus, we have proved the following theorem.

**5** Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for  $c_n$  back into the series, we see that *if f* has a power series expansion at *a*, then it must be of the following form:

6 
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$
$$= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \cdots$$

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The Taylor series is named after the English mathematician Brook Taylor (1685-1731) and the Maclaurin series is named in honor of the Scottish mathematician Colin Maclaurin (1698-1746) despite the fact that the Maclaurin series is really just a special case of the Taylor series. But the idea of representing particular functions as sums of power series goes back to Newton, and the general Taylor series was known to the Scottish mathematician James Gregory in 1668 and to the Swiss mathematician John Bernoulli in the 1690s. Taylor was apparently unaware of the work of Gregory and Bernoulli when he published his discoveries on series in 1715 in his book Methodus incrementorum directa et inversa. Maclaurin series are named after Colin Maclaurin because he popularized them in his calculus textbook Treatise of Fluxions published in 1742.

The series in Equation 6 is called the **Taylor series of the function** f at a (or **about** a or **centered at** a). For the special case a = 0 the Taylor series becomes

7 
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name Maclaurin series.

**NOTE** • We have shown that *if* f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series. An example of such a function is given in Exercise 50.

**EXAMPLE 1** Find the Maclaurin series of the function  $f(x) = e^x$  and its radius of convergence.

**SOLUTION** If  $f(x) = e^x$ , then  $f^{(n)}(x) = e^x$ , so  $f^{(n)}(0) = e^0 = 1$  for all *n*. Therefore, the Taylor series for *f* at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence we let  $a_n = x^n/n!$ . Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is  $R = \infty$ .

The conclusion we can draw from Theorem 5 and Example 1 is that *if*  $e^x$  has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

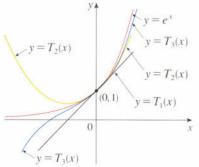
So how can we determine whether  $e^x$  does have a power series representation?

Let's investigate the more general question: Under what circumstances is a function f(x) equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

As with any convergent series, this means that f(x) is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i$$
  
=  $f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n$ 





As *n* increases,  $T_n(x)$  appears to approach  $e^x$  in Figure I. This suggests that  $e^x$  is equal to the sum of its Taylor series. Notice that  $T_n$  is a polynomial of degree *n* called the *n*th-degree Taylor polynomial of *f* at *a*. For instance, for the exponential function  $f(x) = e^x$ , the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with n = 1, 2, and 3 are

$$T_1(x) = 1 + x$$
  $T_2(x) = 1 + x + \frac{x^2}{2!}$   $T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ 

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

In general, f(x) is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

If we let  $R_n(x)$  be the remainder of the series, then

$$R_n(x) = f(x) - T_n(x)$$
 so  $f(x) = T_n(x) + R_n(x)$ 

If we can somehow show that  $\lim_{n\to\infty} R_n(x) = 0$ , then it follows that

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} \left[ f(x) - R_n(x) \right] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)$$

We have therefore proved the following theorem.

**B** Theorem If  $f(x) = T_n(x) + R_n(x)$ , where  $T_n$  is the *n*th-degree Taylor polynomial of f at a and

$$\lim R_n(x) = 0$$

for |x - a| < R, then *f* is equal to the sum of its Taylor series on the interval |x - a| < R.

When trying to show that  $\lim_{n\to\infty} R_n(x) = 0$  for a specific function *f*, we usually use the following fact.

**2** Taylor's Inequality If  $|f^{(n+1)}(x)| \le M$  for |x - a| < R, then the remainder  $R_n(x)$  of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$$
 for  $|x-a| < R$ 

To see why this is true for n = 1, we assume that  $|f''(x)| \le M$ . In particular, we have  $f''(x) \le M$ , so

$$\int_{a}^{x} f''(t) dt \leq \int_{a}^{x} M dt$$

An antiderivative of f'' is f', so by the Evaluation Theorem, we have

$$f'(x) - f'(a) \le M(x - a)$$
 or  $f'(x) \le f'(a) + M(x - a)$ 

 $\int_{-\infty}^{\infty} f'(t) dt < \int_{-\infty}^{\infty} \left[ f'(t) + M(t - t) \right] dt$ 

Thus

$$\int_{a}^{a} f(t) dt \leq \int_{a}^{a} [f(a) + M(t - a)] dt$$
$$f(x) - f(a) \leq f'(a) (x - a) + M \frac{(x - a)^{2}}{2}$$
$$f(x) - f(a) - f'(a) (x - a) \leq \frac{M}{2} (x - a)^{2}$$

But  $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$ . So

$$R_1(x) \le \frac{M}{2} \left(x - a\right)^2$$

A similar argument, using  $f''(x) \ge -M$ , shows that

$$R_1(x) \ge -\frac{M}{2}(x-a)^2$$

So

$$|R_1(x)| \le \frac{M}{2}|x-a|^2$$

This proves Taylor's Inequality for the case where n = 1. The result for any n is proved in a similar way by integrating n + 1 times.

**NOTE** • In Section 8.9 we will explore the use of Taylor's Inequality in approximating functions. Our immediate use of it is in conjunction with Theorem 8.

In applying Theorems 8 and 9 it is often helpful to make use of the following fact:

10

 $\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$ 

This is true because we know from Example 1 that the series  $\sum x^n/n!$  converges for all x and so its *n*th term approaches 0.

**EXAMPLE 2** Prove that  $e^x$  is equal to the sum of its Taylor series.

**SOLUTION** If  $f(x) = e^x$ , then  $f^{(n+1)}(x) = e^x$  for all *n*. So, for any fixed number *x*, we can take  $M = e^x$  in Taylor's Inequality (with a = 0) for every value of *n*:

$$|R_n(x)| \le \frac{e^x}{(n+1)!} |x|^{n+1}$$

But, from Equation 10, we have

$$\lim_{n \to \infty} \frac{e^x}{(n+1)!} |x|^{n+1} = e^x \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that  $\lim_{n\to\infty} |R_n(x)| = 0$  and therefore  $\lim_{n\to\infty} R_n(x) = 0$ . By Theorem 8,  $e^x$  is equal to the sum of its Taylor series, that is,

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} \quad \text{for all } x$$

In particular, if we put x = 1 in Equation 11, we obtain the following expression for the number e as a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

**EXAMPLE 3** Find the Taylor series for  $f(x) = e^x$  at a = 2.

**SOLUTION** We have  $f^{(n)}(2) = e^2$  and so, putting a = 2 in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$$

Again it can be verified, as in Example 1, that the radius of convergence is  $R = \infty$ . As in Example 2 we can verify that  $\lim_{n \to \infty} R_n(x) = 0$ , so

13 
$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$$
 for all x

We have two power series expansions for  $e^x$ , the Maclaurin series in Equation 11 and the Taylor series in Equation 13. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

**EXAMPLE 4** Find the Maclaurin series for  $\sin x$  and prove that it represents  $\sin x$  for all x.

**SOLUTION** We arrange our computation in two columns as follows:

$$f(x) = \sin x \qquad f(0) = 0$$
  

$$f'(x) = \cos x \qquad f'(0) = 1$$
  

$$f''(x) = -\sin x \qquad f''(0) = 0$$
  

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$
  

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Figure 2 shows the graph of  $\sin x$  together with its Taylor (or Maclaurin) polynomials

$$T_{1}(x) = x$$

$$T_{3}(x) = x - \frac{x^{3}}{3!}$$

$$T_{5}(x) = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!}$$

Notice that, as *n* increases,  $T_n(x)$  becomes a better approximation to  $\sin x$ .

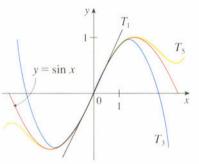


FIGURE 2

lynomials $T_1(x) = x$ 

Since  $f^{(n+1)}(x)$  is  $\pm \sin x$  or  $\pm \cos x$ , we know that  $|f^{(n+1)}(x)| \le 1$  for all x. So we can take M = 1 in Taylor's Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!} |x^{n+1}| \le \frac{|x|^{n+1}}{(n+1)!}$$

By Equation 10 the right side of this inequality approaches 0 as  $n \to \infty$ , so  $|R_n(x)| \to 0$  by the Squeeze Theorem. It follows that  $R_n(x) \to 0$  as  $n \to \infty$ , so sin x is equal to the sum of its Maclaurin series by Theorem 8.

We state the result of Example 4 for future reference:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

**EXAMPLE 5** Find the Maclaurin series for  $\cos x$ .

**SOLUTION** We could proceed directly as in Example 4 but it is easier to differentiate the Maclaurin series for  $\sin x$  given by Equation 15:

$$\cos x = \frac{d}{dx} (\sin x) = \frac{d}{dx} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$
$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Since the Maclaurin series for  $\sin x$  converges for all x, Theorem 2 in Section 8.6 tells us that the differentiated series for  $\cos x$  also converges for all x. Thus

The Maclaurin series for  $e^x$ , sin x, and  $\cos x$  that we found in Examples 2, 4, and 5 were first discovered, using different methods, by Newton. These equations are remarkable because they say we know everything about each of these functions if we know all its derivatives at the single number 0.

16

14

15

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x$$

**EXAMPLE 6** Find the Maclaurin series for the function  $f(x) = x \cos x$ .

**SOLUTION** Instead of computing derivatives and substituting in Equation 7, it is easier to multiply the series for  $\cos x$  (Equation 16) by *x*:

$$x\cos x = x\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

**EXAMPLE 7** Represent  $f(x) = \sin x$  as the sum of its Taylor series centered at  $\pi/3$ .

SOLUTION Arranging our work in columns, we have

$$f(x) = \sin x \qquad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

and this pattern repeats indefinitely. Therefore, the Taylor series at  $\pi/3$  is

$$\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \dots$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \dots$$

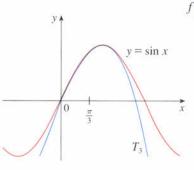
The proof that this series represents sin x for all x is very similar to that in Example 4. [Just replace x by  $x - \pi/3$  in (14).] We can write the series in sigma notation if we separate the terms that contain  $\sqrt{3}$ :

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!} \left( x - \frac{\pi}{3} \right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!} \left( x - \frac{\pi}{3} \right)^{2n+1}$$

The power series that we obtained by indirect methods in Examples 5 and 6 and in Section 8.6 are indeed the Taylor or Maclaurin series of the given functions because Theorem 5 asserts that, no matter how a power series representation  $f(x) = \sum c_n(x - a)^n$  is obtained, it is always true that  $c_n = f^{(n)}(a)/n!$ . In other words, the coefficients are uniquely determined.

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

We have obtained two different series representations for sin x, the Maclaurin series in Example 4 and the Taylor series in Example 7. It is best to use the Maclaurin series for values of x near 0 and the Taylor series for x near  $\pi/3$ . Notice that the third Taylor polynomial  $T_3$  in Figure 3 is a good approximation to sin x near  $\pi/3$  but not as good near 0. Compare it with the third Maclaurin polynomial  $T_3$  in Figure 2, where the opposite is true.





Important Maclaurin series and their intervals of convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$
 (-1,1)

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots$$
 (-\infty,\infty)

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (-\infty, \infty)$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad [-1,1]$$

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle. In fact, in the introduction to this chapter we mentioned that Newton often integrated functions by first expressing them as power series and then integrating the series term by term. The function  $f(x) = e^{-x^2}$  can't be integrated by techniques discussed so far because its anti-derivative is not an elementary function (see Section 5.7). In the following example we use Newton's idea to integrate this function.

## EXAMPLE 8

- (a) Evaluate  $\int e^{-x^2} dx$  as an infinite series.
- (b) Evaluate  $\int_{0}^{1} e^{-x^{2}} dx$  correct to within an error of 0.001.

### SOLUTION

(a) First we find the Maclaurin series for  $f(x) = e^{-x^2}$ . Although it's possible to use the direct method, let's find it simply by replacing x with  $-x^2$  in the series for  $e^x$  given in the table of Maclaurin series. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Now we integrate term by term:

$$\int e^{-x^2} dx = \int \left( 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + (-1)^n \frac{x^{2n}}{n!} + \dots \right) dx$$
$$= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \dots$$

This series converges for all x because the original series for  $e^{-x^2}$  converges for all x.

(b) The Evaluation Theorem gives

$$\int_{0}^{1} e^{-x^{2}} dx = \left[ x - \frac{x^{3}}{3 \cdot 1!} + \frac{x^{5}}{5 \cdot 2!} - \frac{x^{7}}{7 \cdot 3!} + \frac{x^{9}}{9 \cdot 4!} - \cdots \right]_{0}^{1}$$
$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$
$$\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$$

We can take C = 0 in the antiderivative in part (a).

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11\cdot 5!} = \frac{1}{1320} < 0.001$$

Another use of Taylor series is illustrated in the next example. The limit could be found with l'Hospital's Rule, but instead we use a series.

**EXAMPLE 9** Evaluate 
$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$
.

**SOLUTION** Using the Maclaurin series for  $e^x$ , we have

1

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - x}{x^2}$$
$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2}$$
$$= \lim_{x \to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right)$$
$$= \frac{1}{2}$$

because power series are continuous functions.

## Multiplication and Division of Power Series

If power series are added or subtracted, they behave like polynomials (Theorem 8 in Section 8.2 shows this). In fact, as the following example illustrates, they can also be multiplied and divided like polynomials. We find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

**EXAMPLE 10** Find the first three nonzero terms in the Maclaurin series for (a)  $e^x \sin x$  and (b)  $\tan x$ .

#### SOLUTION

(a) Using the Maclaurin series for  $e^x$  and  $\sin x$  in the table, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\frac{1 + x + \frac{1}{2}x^{2} + \frac{1}{6}x^{3} + \cdots}{x - \frac{1}{6}x^{3} + \cdots} \\
\frac{x - \frac{1}{6}x^{3} + \cdots}{-\frac{1}{6}x^{3} - \frac{1}{6}x^{4} - \cdots} \\
\frac{-\frac{1}{6}x^{3} - \frac{1}{6}x^{4} - \cdots}{x + x^{2} + \frac{1}{3}x^{3} + \cdots}$$

Some computer algebra systems compute limits in this way.

í.

Thus

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in the table, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}$$

We use a procedure like long division:

$$1 - \frac{1}{2}x^{2} + \frac{1}{24}x^{4} - \dots )\overline{x - \frac{1}{6}x^{3} + \frac{1}{120}x^{5} - \dots} \\ \frac{x - \frac{1}{2}x^{3} + \frac{1}{24}x^{5} - \dots}{\frac{1}{3}x^{3} - \frac{1}{30}x^{5} + \dots} \\ \frac{\frac{1}{3}x^{3} - \frac{1}{30}x^{5} + \dots}{\frac{1}{3}x^{3} - \frac{1}{6}x^{5} + \dots} \\ \frac{2}{15}x^{5} + \dots \\ \frac{1}{3}x^{3} + \frac{2}{15}x^{5} + \dots \\ \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \dots \\ \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \dots \\ \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \dots \\ \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \frac{1}{3}x^{5} + \dots \\ \frac{1}{3}x^{5} + \frac{1}{3}x^{5} +$$

Thus

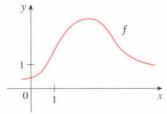
Although we have not attempted to justify the formal manipulations used in Example 10, they are legitimate. There is a theorem which states that if both 
$$f(x) = \sum c_n x^n$$
 and  $g(x) = \sum b_n x^n$  converge for  $|x| < R$  and the series are multiplied as if they were polynomials, then the resulting series also converges for  $|x| < R$  and represents  $f(x)g(x)$ . For division we require  $b_0 \neq 0$ ; the resulting series converges for sufficiently small  $|x|$ .

8.7

Exercises

- 1. If  $f(x) = \sum_{n=0}^{\infty} b_n (x 5)^n$  for all x, write a formula for  $b_8$ .
- **2.** The graph of f is shown. Explain why the series
  - $2 0.8(x 1) + 0.4(x 1)^2 0.1(x 1)^3 + \cdots$

is not the Taylor series of f centered at 1.



**3-6** Find the Maclaurin series for f(x) using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that  $R_n(x) \rightarrow 0$ .] Also find the associated radius of convergence.

**3.** 
$$f(x) = \cos x$$
 **4.**  $f(x) = \sin 2x$ 

**5.** 
$$f(x) = \frac{1}{(1+x)^2}$$
 **6.**  $f(x) = \frac{x}{1-x}$ 

**7–12** Find the Taylor series for f(x) at the given value of *a*. [Assume that *f* has a power series expansion. Do not show that  $R_n(x) \rightarrow 0$ .]

- 7.  $f(x) = e^x$ , a = 3 8.  $f(x) = \ln x$ , a = 2 

   9. f(x) = 1/x, a = 1 10.  $f(x) = \sqrt{x}$ , a = 4 

   11.  $f(x) = \sin x$ ,  $a = \pi/4$  12.  $f(x) = \cos x$ ,  $a = -\pi/4$
- **13.** Prove that the series obtained in Exercise 3 represents cos *x* for all *x*.
- **14.** Prove that the series obtained in Exercise 11 represents sin *x* for all *x*.

**15–22** Use a Maclaurin series derived in this section to obtain the Maclaurin series for the given function.

**15.** 
$$f(x) = e^{3x}$$
 **16.**  $f(x) = \sin 2x$ 

**17.** 
$$f(x) = x^2 \cos x$$
  
**18.**  $f(x) = \cos(x^3)$   
**19.**  $f(x) = x \sin(x/2)$   
**20.**  $f(x) = xe^{-x}$   
**21.**  $f(x) = \sin^2 x$  [*Hint*: Use  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ .]  
**22.**  $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0\\ 1 & \text{if } x = 0 \end{cases}$ 

23-26 Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen.

**23.** 
$$f(x) = \sqrt{1 + x}$$
  
**24.**  $f(x) = 1/\sqrt{1 + 2x}$   
**25.**  $f(x) = (1 + x)^{-3}$   
**26.**  $f(x) = 2^x$ 

- 27. Find the Maclaurin series for ln(1 + x) and use it to calculate ln 1.1 correct to five decimal places.
- Use the Maclaurin series for sin x to compute sin 3° correct to five decimal places.

**29–32** Evaluate the indefinite integral as an infinite series.

**29.** 
$$\int \sin(x^2) dx$$
  
**30.**  $\int \frac{\sin x}{x} dx$   
**31.**  $\int \sqrt{x^3 + 1} dx$   
**32.**  $\int e^{x^3} dx$ 

**33–36** Use series to approximate the definite integral to within the indicated accuracy.

- **33.**  $\int_0^1 \sin(x^2) dx$  (three decimal places)
- **34.**  $\int_0^{0.5} \cos(x^2) dx$  (three decimal places)

**35.** 
$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}}$$
 (error < 10<sup>-8</sup>)

**36.** 
$$\int_0^{0.5} x^2 e^{-x^2} dx$$
 (error < 0.001)

## The Binomial Series

 $\mathcal{A}$ 

You may be acquainted with the Binomial Theorem, which states that if a and b are any real numbers and k is a positive integer, then

$$(a + b)^{k} = a^{k} + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^{2} + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^{2} + \dots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}a^{k-n}b^{n} + \dots + kab^{k-1} + b^{k}$$

**37–39** Use series to evaluate the limit.

**37.** 
$$\lim_{x \to 0} \frac{x - \tan^{-1}x}{x^3}$$
**38.** 
$$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}$$
**39.** 
$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

40. Use the series in Example 10(b) to evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

We found this limit in Example 4 in Section 4.5 using l'Hospital's Rule three times. Which method do you prefer?

**41–44** Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

**41.** 
$$y = e^{-x^2} \cos x$$
  
**42.**  $y = \sec x$   
**43.**  $y = \frac{\ln(1-x)}{e^x}$   
**44.**  $y = e^x \ln(1-x)$ 

**45–49** ■ Find the sum of the series.

**45.** 
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$$
**46.** 
$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$$
**47.** 
$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$$
**48.** 
$$\sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!}$$
**49.** 
$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$$

50. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0\\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series.

(b) Graph the function in part (a) and comment on its behavior near the origin.

The traditional notation for the binomial coefficients is

$$\binom{k}{0} = 1 \qquad \binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad n = 1, 2, \dots, k$$

which enables us to write the Binomial Theorem in the abbreviated form

$$(a+b)^{k} = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^{n}$$

In particular, if we put a = 1 and b = x, we get

$$(1+x)^k = \sum_{n=0}^k \binom{k}{n} x^n$$

One of Newton's accomplishments was to extend the Binomial Theorem (Equation 1) to the case in which k is no longer a positive integer. (See the Writing Project on page 623). In this case the expression for  $(1 + x)^k$  is no longer a finite sum; it becomes an infinite series. To find this series we compute the Maclaurin series of  $(1 + x)^k$  in the usual way:

$$f(x) = (1 + x)^{k}$$

$$f(0) = 1$$

$$f'(x) = k(1 + x)^{k-1}$$

$$f''(0) = k$$

$$f''(x) = k(k - 1)(1 + x)^{k-2}$$

$$f''(0) = k(k - 1)$$

$$f'''(0) = k(k - 1)$$

$$f'''(0) = k(k - 1)(k - 2)$$

$$\vdots$$

$$f'''(0) = k(k - 1)(k - 2)$$

$$\vdots$$

$$f'''(0) = k(k - 1)(k - 2)$$

$$\vdots$$

$$f^{(n)}(x) = k(k - 1)\cdots(k - n + 1)(1 + x)^{k-n}$$

$$f^{(n)}(0) = k(k - 1)\cdots(k - n + 1)$$

Therefore, the Maclaurin series of  $f(x) = (1 + x)^k$  is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called the **binomial series**. If its *n*th term is  $a_n$ , then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$
$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \quad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if |x| < 1 and diverges if |x| > 1.

The following theorem states that  $(1 + x)^k$  is equal to the sum of its Maclaurin series. It is possible to prove this by showing that the remainder term  $R_n(x)$  approaches 0, but that turns out to be quite difficult. The proof outlined in Exercise 15 is much easier.

The Binomial Series If k is any real number and 
$$|x| < 1$$
, then  

$$(1 + x)^{k} = 1 + kx + \frac{k(k-1)}{2!}x^{2} + \frac{k(k-1)(k-2)}{3!}x^{3} + \cdots$$

$$= \sum_{n=0}^{\infty} {\binom{k}{n}}x^{n}$$
where  ${\binom{k}{n}} = \frac{k(k-1)\cdots(k-n+1)}{n!}$   $(n \ge 1)$  and  ${\binom{k}{0}} = 1$ 

Although the binomial series always converges when |x| < 1, the question of whether or not it converges at the endpoints,  $\pm 1$ , depends on the value of k. It turns out that the series converges at 1 if  $-1 < k \le 0$  and at both endpoints if  $k \ge 0$ . Notice that if k is a positive integer and n > k, then the expression for  $\binom{k}{n}$  contains a factor (k - k), so  $\binom{k}{n} = 0$  for n > k. This means that the series terminates and reduces to the ordinary Binomial Theorem (Equation 1) when k is a positive integer.

As we have seen, the binomial series is just a special case of the Maclaurin series; it occurs so frequently that it is worth remembering.

**EXAMPLE I** Expand 
$$\frac{1}{(1 + x)^2}$$
 as a power series.

**SOLUTION** We use the binomial series with k = -2. The binomial coefficient is

$$\binom{-2}{n} = \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!}$$
$$= \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdots n(n+1)}{n!} = (-1)^n (n+1)$$

and so, when |x| < 1,

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{n=0}^{\infty} {\binom{-2}{n}} x^n$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

**EXAMPLE 2** Find the Maclaurin series for the function  $f(x) = \frac{1}{\sqrt{4-x}}$  and its radius of convergence.

**SOLUTION** As given, f(x) is not quite of the form  $(1 + x)^k$  so we rewrite it as follows:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Using the binomial series with  $k = -\frac{1}{2}$  and with x replaced by -x/4, we have

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2} \left( 1 - \frac{x}{4} \right)^{-1/2} = \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \left( -\frac{x}{4} \right)^n$$

$$= \frac{1}{2} \left[ 1 + \left( -\frac{1}{2} \right) \left( -\frac{x}{4} \right) + \frac{\left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right)}{2!} \left( -\frac{x}{4} \right)^2 + \frac{\left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{5}{2} \right)}{3!} \left( -\frac{x}{4} \right)^3$$

$$+ \dots + \frac{\left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{5}{2} \right) \dots \left( -\frac{1}{2} - n + 1 \right)}{n!} \left( -\frac{x}{4} \right)^n + \dots \right]$$

$$= \frac{1}{2} \left[ 1 + \frac{1}{8} x + \frac{1 \cdot 3}{2! 8^2} x^2 + \frac{1 \cdot 3 \cdot 5}{3! 8^3} x^3 + \dots + \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n! 8^n} x^n + \dots \right]$$

We know from (2) that this series converges when |-x/4| < 1, that is, |x| < 4, so the radius of convergence is R = 4.

A binomial series is a special case of a Taylor series. Figure 1 shows the graphs of the first three Taylor polynomials computed from the answer to Example 2.

Exercises

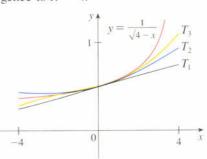


FIGURE I

**1–6** ■ Use the binomial series to expand the function as a power series. State the radius of convergence.

1. 
$$\sqrt{1 + x}$$
  
3.  $\frac{1}{(1 + 2x)^4}$   
5.  $\sqrt[4]{1 - x^4}$   
2.  $\frac{1}{(1 + x)^3}$   
4.  $\sqrt[3]{1 + x^2}$   
6.  $\frac{x^2}{\sqrt{2 + x}}$ 

7-8 Use the binomial series to expand the function as a Maclaurin series and to find the first three Taylor polynomials T<sub>1</sub>, T<sub>2</sub>, and T<sub>3</sub>. Graph the function and these Taylor polynomials in the interval of convergence.

**7.** 
$$\frac{1}{\sqrt[3]{8+x}}$$
 **8.**  $(4+x)^{3/2}$ 

- 9. (a) Use the binomial series to expand 1/√1 x<sup>2</sup>.
  (b) Use part (a) to find the Maclaurin series for sin<sup>-1</sup>x.
- 10. (a) Expand  $\sqrt[3]{8 + x}$  as a power series.

- (b) Use part (a) to estimate  $\sqrt[3]{8.2}$  correct to four decimal places.
- (a) Expand f(x) = x/(1 x)<sup>2</sup> as a power series.
  (b) Use part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

(a) Expand f(x) = (x + x<sup>2</sup>)/(1 - x)<sup>3</sup> as a power series.
(b) Use part (a) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

- 13. (a) Use the binomial series to find the Maclaurin series of  $f(x) = \sqrt{1 + x^2}$ .
  - (b) Use part (a) to evaluate  $f^{(10)}(0)$ .
- 14. (a) Use the binomial series to find the Maclaurin series of  $f(x) = 1/\sqrt{1 + x^3}$ .
  - (b) Use part (a) to evaluate  $f^{(9)}(0)$ .
- **15.** Use the following steps to prove (2).

(a) Let  $g(x) = \sum_{n=0}^{\infty} {k \choose n} x^n$ . Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1+x}$$
  $-1 < x < 1$ 

- (b) Let h(x) = (1 + x)<sup>-k</sup>g(x) and show that h'(x) = 0.
  (c) Deduce that g(x) = (1 + x)<sup>k</sup>.
- 16. The period of a pendulum with length L that makes a maximum angle  $\theta_0$  with the vertical is

$$T = 4 \ \sqrt{\frac{L}{g}} \ \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where  $k = \sin(\frac{1}{2}\theta_0)$  and g is the acceleration due to gravity. (In Exercise 28 in Section 5.8 we approximated this integral using Simpson's Rule.)

- (a) Expand the integrand as a binomial series and use
  - the result of Exercise 36 in Section 5.6 to show that

$$T = 2\pi \sqrt{\frac{L}{g}} \left[ 1 + \frac{1^2}{2^2} k^2 + \frac{1^2 3^2}{2^2 4^2} k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2} k^6 + \cdots \right]$$

Writing Project If  $\theta_0$  is not too large, the approximation  $T \approx 2\pi \sqrt{L/g}$ , obtained by using only the first term in the series, is often used. A better approximation is obtained by using two terms:

$$T\approx 2\pi \sqrt{\frac{L}{g}} \left(1+\frac{1}{4}k^2\right)$$

(b) Notice that all the terms in the series after the first one have coefficients that are at most <sup>1</sup>/<sub>4</sub>. Use this fact to compare this series with a geometric series and show that

$$2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{1}{4}k^2\right) \le T \le 2\pi \sqrt{\frac{L}{g}} \frac{4 - 3k^2}{4 - 4k^2}$$

(c) Use the inequalities in part (b) to estimate the period of a pendulum with L = 1 meter and  $\theta_0 = 10^\circ$ . How does it compare with the estimate  $T \approx 2\pi \sqrt{L/g}$ ? What if  $\theta_0 = 42^\circ$ ?

## How Newton Discovered the Binomial Series

The Binomial Theorem, which gives the expansion of  $(a + b)^k$ , was known to Chinese mathematicians many centuries before the time of Newton for the case where the exponent k is a positive integer. In 1665, when he was 22, Newton was the first to discover the infinite series expansion of  $(a + b)^k$  when k is a fractional exponent (positive or negative). He didn't publish his discovery, but he stated it and gave examples of how to use it in a letter (now called the *epistola prior*) dated June 13, 1676, that he sent to Henry Oldenburg, secretary of the Royal Society of London, to transmit to Leibniz. When Leibniz replied, he asked how Newton had discovered the binomial series. Newton wrote a second letter, the *epistola posterior* of October 24, 1676, in which he explained in great detail how he arrived at his discovery by a very indirect route. He was investigating the areas under the curves  $y = (1 - x^2)^{n/2}$  from 0 to x for  $n = 0, 1, 2, 3, 4, \ldots$ . These are easy to calculate if n is even. By observing patterns and interpolating, Newton was able to guess the answers for odd values of n. Then he realized he could get the same answers by expressing  $(1 - x^2)^{n/2}$  as an infinite series.

Write a report on Newton's discovery of the binomial series. Start by giving the statement of the binomial series in Newton's notation (see the *epistola prior* on page 285 of [4] or page 402 of [2]). Explain why Newton's version is equivalent to Theorem 2 on page 621. Then read Newton's *epistola posterior* (page 287 in [4] or page 404 in [2]) and explain the patterns that Newton discovered in the areas under the curves  $y = (1 - x^2)^{n/2}$ . Show how he was able to guess the areas under the remaining curves and how he verified his answers. Finally, explain how these discoveries led to the binomial series. The books by Edwards [1] and Katz [3] contain commentaries on Newton's letters.

 C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 178–187.

- **2.** John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillian Press, 1987).
- 3. Victor Katz, A History of Mathematics: An Introduction (New York: Harper-Collins, 1993), pp. 463-466.
- **4.** D. J. Struik, ed., A Sourcebook in Mathematics, 1200–1800 (Princeton, N.J.: Princeton University Press, 1969).

# 8.9

## Applications of Taylor Polynomials

Suppose that f(x) is equal to the sum of its Taylor series at *a*:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

In Section 8.7 we introduced the notation  $T_n(x)$  for the *n*th partial sum of this series and called it the *n*th-degree Taylor polynomial of *f* at *a*. Thus

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i$$
  
=  $f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n$ 

Since *f* is the sum of its Taylor series, we know that  $T_n(x) \to f(x)$  as  $n \to \infty$  and so  $T_n$  can be used as an approximation to  $f: f(x) \approx T_n(x)$ . It is useful to be able to approximate a function by a polynomial because polynomials are the simplest of functions. In this section we explore the use of such approximations by physical scientists and computer scientists.

Notice that the first-degree Taylor polynomial

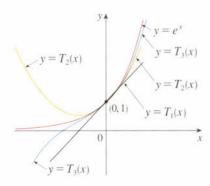
$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as the linearization of f at a that we discussed in Sections 2.9 and 3.8. Notice also that  $T_1$  and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of  $T_n$  at a agree with those of f up to and including derivatives of order n.

To illustrate these ideas let's take another look at the graphs of  $y = e^x$  and its first few Taylor polynomials, as shown in Figure 1. The graph of  $T_1$  is the tangent line to  $y = e^x$  at (0, 1); this tangent line is the best linear approximation to  $e^x$  near (0, 1). The graph of  $T_2$  is the parabola  $y = 1 + x + x^2/2$ , and the graph of  $T_3$  is the cubic curve  $y = 1 + x + x^2/2 + x^3/6$ , which is a closer fit to the exponential curve  $y = e^x$  than  $T_2$ . The next Taylor polynomial  $T_4$  would be an even better approximation, and so on.

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials  $T_n(x)$  to the function  $y = e^x$ . We see that when x = 0.2 the convergence is very rapid, but when x = 3 it is somewhat slower. In fact, the farther x is from 0, the slower  $T_n$  converges to x.

When using a Taylor polynomial  $T_n$  to approximate a function f, we have to ask the questions: How good an approximation is it? How large should we take n to be





	x = 0.2	$\chi = 3.0$
$T_2(x)$	1.220000	8,500000
$T_{A}(x)$	1.221400	16.375000
$T_{n}(x)$	1.221403	19,412500
$T_{\gamma}(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
$\mathbf{C}^{N}$	1.221403	20.085537

in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

- 1. If a graphing device is available, we can use it to graph  $|R_n(x)|$  and thereby estimate the error.
- 2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
- 3. In all cases we can use Taylor's Inequality (Theorem 9 in Section 8.7), which says that if  $|f^{(n+1)}(x)| \leq M$ , then

$$R_n(x) | \le \frac{M}{(n+1)!} |x - a|^{n+1}$$

## EXAMPLE I

(a) Approximate the function  $f(x) = \sqrt[3]{x}$  by a Taylor polynomial of degree 2 at a = 8.

(b) How accurate is this approximation when  $7 \le x \le 9$ ?

## SOLUTION

(a)

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(8) = 2$$
$$f'(x) = \frac{1}{3}x^{-2/3} \qquad f'(8) = \frac{1}{12}$$
$$f''(x) = -\frac{2}{9}x^{-5/3} \qquad f''(8) = -\frac{1}{144}$$
$$f'''(x) = \frac{10}{27}x^{-8/3}$$

Thus, the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$
$$= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(b) The Taylor series is not alternating when x < 8, so we can't use the Alternating Series Estimation Theorem in this example. But we can use Taylor's Inequality with n = 2 and a = 8:

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3$$

where  $|f'''(x)| \leq M$ . Because  $x \geq 7$ , we have  $x^{8/3} \geq 7^{8/3}$  and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \le \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Therefore, we can take M = 0.0021. Also  $7 \le x \le 9$ , so  $-1 \le x - 8 \le 1$  and

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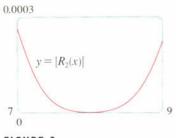
 $|x - 8| \le 1$ . Then Taylor's Inequality gives

$$|R_2(x)| \le \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if  $7 \le x \le 9$ , the approximation in part (a) is accurate to within 0.0004.

2.5  $T_2$   $y = \sqrt[3]{x}$ 0
15





Let's use a graphing device to check the calculation in Example 1. Figure 2 shows that the graphs of  $y = \sqrt[3]{x}$  and  $y = T_2(x)$  are very close to each other when x is near 8. Figure 3 shows the graph of  $|R_2(x)|$  computed from the expression

$$R_2(x) = |\sqrt[3]{x} - T_2(x)|$$

We see from the graph that

$$|R_2(x)| < 0.0003$$

when  $7 \le x \le 9$ . Thus, the error estimate from graphical methods is slightly better than the error estimate from Taylor's Inequality in this case.

## EXAMPLE 2

(a) What is the maximum error possible in using the approximation

S

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when  $-0.3 \le x \le 0.3$ ? Use this approximation to find sin 12° correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

## SOLUTION

(a) Notice that the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is alternating for all nonzero values of x, so we can use the Alternating Series Estimation Theorem. The error in approximating  $\sin x$  by the first three terms of its Maclaurin series is at most

$$\left| \frac{x^7}{7!} \right| = \frac{|x|^7}{5040}$$

If  $-0.3 \le x \le 0.3$ , then  $|x| \le 0.3$ , so the error is smaller than

$$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$$

To find  $\sin 12^\circ$  we first convert to radian measure.

$$\sin 12^\circ = \sin\left(\frac{12\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right)$$
$$\approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!}$$
$$\approx 0.20791169$$



Thus, correct to six decimal places,  $\sin 12^{\circ} \approx 0.207912$ . (b) The error will be smaller than 0.00005 if

$$\frac{\|x\|^7}{5040} < 0.00005$$

Solving this inequality for x, we get

 $|x|^7 < 0.252$  or  $|x| < (0.252)^{1/7} \approx 0.821$ 

So the given approximation is accurate to within 0.00005 when |x| < 0.82.

What if we use Taylor's Inequality to solve Example 2? Since  $f^{(7)}(x) = -\cos x$ , we have  $|f^{(7)}(x)| \le 1$  and so

$$|R_6(x)| \le \frac{1}{7!} |x|^7$$

So we get the same estimates as with the Alternating Series Estimation Theorem. What about graphical methods? Figure 4 shows the graph of

$$|R_6(x)| = \left|\sin x - \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\right|$$

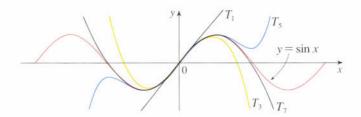
and we see from it that  $|R_6(x)| < 4.3 \times 10^{-8}$  when  $|x| \le 0.3$ . This is the same estimate that we obtained in Example 2. For part (b) we want  $|R_6(x)| < 0.00005$ , so we graph both  $y = |R_6(x)|$  and y = 0.00005 in Figure 5. By placing the cursor on the right intersection point we find that the inequality is satisfied when |x| < 0.82. Again this is the same estimate that we obtained in the solution to Example 2.

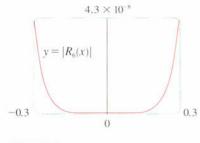
If we had been asked to approximate  $\sin 72^{\circ}$  instead of  $\sin 12^{\circ}$  in Example 2, it would have been wise to use the Taylor polynomials at  $a = \pi/3$  (instead of a = 0) because they are better approximations to  $\sin x$  for values of x close to  $\pi/3$ . Notice that  $72^{\circ}$  is close to  $60^{\circ}$  (or  $\pi/3$  radians) and the derivatives of  $\sin x$  are easy to compute at  $\pi/3$ .

Figure 6 shows the graphs of the Taylor polynomial approximations

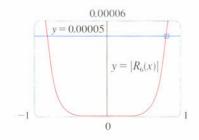
$$T_{1}(x) = x T_{3}(x) = x - \frac{x^{3}}{3!}$$
$$T_{5}(x) = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} T_{7}(x) = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \frac{x^{7}}{7!}$$

to the sine curve. You can see that as *n* increases,  $T_n(x)$  is a good approximation to sin *x* on a larger and larger interval.













One use of the type of calculation done in Examples 1 and 2 occurs in calculators and computers. For instance, when you press the sin or  $e^x$  key on your calculator, or when a computer programmer uses a subroutine for a trigonometric or exponential or Bessel function, in many machines a polynomial approximation is calculated. The polynomial is often a Taylor polynomial that has been modified so that the error is spread more evenly throughout an interval.

## Applications to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series. In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's Inequality can then be used to gauge the accuracy of the approximation. The following example shows one way in which this idea is used in special relativity.

**EXAMPLE 3** In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where  $m_0$  is the mass of the object when at rest and *c* is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0 c^2$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics:  $K = \frac{1}{2}m_0v^2$ .

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when  $|v| \le 100$  m/s.

## SOLUTION

(a) Using the expressions given for K and m, we get

$$K = mc^{2} - m_{0}c^{2} = \frac{m_{0}c^{2}}{\sqrt{1 - v^{2}/c^{2}}} - m_{0}c^{2}$$
$$= m_{0}c^{2} \left[ \left( 1 - \frac{v^{2}}{c^{2}} \right)^{-1/2} - 1 \right]$$

With  $x = -v^2/c^2$ , the Maclaurin series for  $(1 + x)^{-1/2}$  is most easily computed as a binomial series with  $k = -\frac{1}{2}$ . (Notice that |x| < 1 because v < c.) Therefore, we have

$$(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^{2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^{3} + \cdots$$
$$= 1 - \frac{1}{2}x + \frac{3}{8}x^{2} - \frac{5}{16}x^{3} + \cdots$$
$$K = m_{0}c^{2}\left[\left(1 + \frac{1}{2}\frac{v^{2}}{c^{2}} + \frac{3}{8}\frac{v^{4}}{c^{4}} + \frac{5}{16}\frac{v^{6}}{c^{6}} + \cdots\right) - 1\right]$$
$$= m_{0}c^{2}\left(\frac{1}{2}\frac{v^{2}}{c^{2}} + \frac{3}{8}\frac{v^{4}}{c^{4}} + \frac{5}{16}\frac{v^{6}}{c^{6}} + \cdots\right)$$

and

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0 c^2 \left( \frac{1}{2} \frac{v^2}{c^2} \right) = \frac{1}{2} m_0 v^2$$

(b) If  $x = -v^2/c^2$ ,  $f(x) = m_0 c^2 [(1 + x)^{-1/2} - 1]$ , and *M* is a number such that  $|f''(x)| \leq M$ , then we can use Taylor's Inequality to write

$$\left|R_{1}(x)\right| \leq \frac{M}{2!} x^{2}$$

We have  $f''(x) = \frac{3}{4}m_0c^2(1 + x)^{-5/2}$  and we are given that  $|v| \le 100$  m/s, so

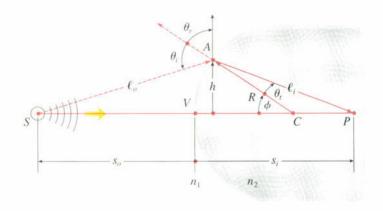
$$|f''(x)| = \frac{3m_0c^2}{4(1-v^2/c^2)^{5/2}} \le \frac{3m_0c^2}{4(1-100^2/c^2)^{5/2}} \quad (=M)$$

Thus, with  $c = 3 \times 10^8$  m/s,

$$|R_1(x)| \le \frac{1}{2} \cdot \frac{3m_0c^2}{4(1-100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

So when  $|v| \le 100$  m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most  $(4.2 \times 10^{-10})m_0$ .

Another application to physics occurs in optics. Figure 7 is adapted from a book by Eugene Hecht: *Optics*, 2d ed. (Reading, MA: Addison-Wesley, 1987), page 133. It depicts a wave from the point source *S* meeting a spherical interface of radius *R* centered at *C*. The ray *SA* is refracted toward *P*.



**FIGURE 7** Refraction at a spherical interface

I.

Using Fermat's principle that light travels so as to minimize the time taken, Hecht derives the equation

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R} \left( \frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o} \right)$$

where  $n_1$  and  $n_2$  are indexes of refraction and  $\ell_o$ ,  $\ell_i$ ,  $s_o$ , and  $s_i$  are the distances

indicated in Figure 7. By the Law of Cosines, applied to triangles ACS and ACP, we have

2

$$\ell_o = \sqrt{R^2 + (s_o + R)^2} - 2R(s_o + R)\cos\phi$$
$$\ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Because Equation 1 is cumbersome to work with, Gauss, in 1841, simplified it by using the linear approximation  $\cos \phi \approx 1$  for small values of  $\phi$ . (This amounts to using the Taylor polynomial of degree 1.) Then Equation 1 becomes the following simpler equation [as you are asked to show in Exercise 24(a)]:

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n}{R}$$

The resulting optical theory is known as *Gaussian optics*, or *first-order optics*, and has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by approximating  $\cos \phi$  by its Taylor polynomial of degree 3 (which is the same as the Taylor polynomial of degree 2). This takes into account rays for which  $\phi$  is not so small, that is, rays that strike the surface at greater distances *h* above the axis. In Exercise 24(b) you are asked to use this approximation to derive the more accurate equation

$$\boxed{1} \quad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2 \left[ \frac{n_1}{2s_o} \left( \frac{1}{s_o} + \frac{1}{R} \right)^2 + \frac{n_2}{2s_i} \left( \frac{1}{R} - \frac{1}{s_i} \right)^2 \right]$$

The resulting optical theory is known as third-order optics.

Other applications of Taylor polynomials to physics are explored in Exercises 25 and 26 and in the Applied Project on page 632.

8.9

Exercises

- **1.** (a) Find the Taylor polynomials up to degree 6 for  $f(x) = \cos x$  centered at a = 0. Graph f and these polynomials on a common screen.
  - (b) Evaluate f and these polynomials at x = π/4, π/2, and π.
  - (c) Comment on how the Taylor polynomials converge to f(x).
  - **2.** (a) Find the Taylor polynomials up to degree 3 for f(x) = 1/x centered at a = 1. Graph f and these polynomials on a common screen.
    - (b) Evaluate f and these polynomials at x = 0.9 and 1.3.
    - (c) Comment on how the Taylor polynomials converge to f(x).
- **3**-8 Find the Taylor polynomial  $T_n(x)$  for the function f at the number a. Graph f and  $T_n$  on the same screen.

**3.**  $f(x) = \sin x$ ,  $a = \pi/6$ , n = 3

4. 
$$f(x) = \cos x$$
,  $a = 2\pi/3$ ,  $n = 4$ 

**5.**  $f(x) = \tan x$ , a = 0, n = 4

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- 6.  $f(x) = \tan x$ ,  $a = \pi/4$ , n = 4
- 7.  $f(x) = \sec x$ ,  $a = \pi/3$ , n = 3
- **8.**  $f(x) = \sqrt{x}, a = 9, n = 3$

- **CAS** 9–10 Use a computer algebra system to find the Taylor polynomials  $T_n$  at a = 0 for the given values of n. Then graph these polynomials and f on the same screen.
  - 9.  $f(x) = \sec x$ , n = 2, 4, 6, 8

**10.**  $f(x) = \tan x$ , n = 1, 3, 5, 7, 9

#### 11-16 =

- (a) Approximate *f* by a Taylor polynomial with degree *n* at the number *a*.
- (b) Use Taylor's Inequality to estimate the accuracy of the approximation  $f(x) \approx T_n(x)$  when x lies in the given interval.

Here we use the identity  $\cos(\pi - \phi) = -\cos\phi$ 

- (c) Check your result in part (b) by graphing  $|R_n(x)|$ . 11.  $f(x) = \sqrt{1 + x}$ , a = 0, n = 1,  $0 \le x \le 0.1$ 12. f(x) = 1/x, a = 1, n = 3,  $0.8 \le x \le 1.2$ 13.  $f(x) = \sin x$ ,  $a = \pi/4$ , n = 5,  $0 \le x \le \pi/2$ 14.  $f(x) = \cos x$ ,  $a = \pi/3$ , n = 4,  $0 \le x \le 2\pi/3$ 15.  $f(x) = e^{x^2}$ , a = 0, n = 3,  $0 \le x \le 0.1$ 16.  $f(x) = \ln x$ , a = 4, n = 3,  $3 \le x \le 5$ 
  - Use the information from Exercise 3 to estimate sin 35° correct to five decimal places.
  - **18.** Use the information from Exercise 14 to estimate  $\cos 69^\circ$  correct to five decimal places.
  - **19.** Use Taylor's Inequality to determine the number of terms of the Maclaurin series for  $e^x$  that should be used to estimate  $e^{0.1}$  to within 0.00001.
  - **20.** How many terms of the Maclaurin series for  $\ln(1 + x)$  do you need to use to estimate  $\ln 1.4$  to within 0.001?
- 21-22 Use the Alternating Series Estimation Theorem or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error. Check your answer graphically.

**21.** 
$$\sin x \approx x - \frac{x^3}{6}$$
, error < 0.01  
**22.**  $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$ , error < 0.005

- 23. A car is moving with speed 20 m/s and acceleration 2 m/s<sup>2</sup> at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute?
- 24. (a) Derive Equation 3 for Gaussian optics from Equation 1 by approximating cos φ in Equation 2 by its first-degree Taylor polynomial.
  - (b) Show that if cos φ is replaced by its third-degree Taylor polynomial in Equation 2, then Equation 1 becomes Equation 4 for third-order optics. [*Hint:* Use the first two terms in the binomial series for ℓ<sub>0</sub><sup>-1</sup> and ℓ<sub>i</sub><sup>-1</sup>. Also, use φ ≈ sin φ.]
- **25.** An electric dipole consists of two electric charges of equal magnitude and opposite signs. If the charges are q and -q and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for *E* as a series in powers of d/D, show that *E* is approximately proportional to  $1/D^3$  when *P* is far away from the dipole.

**26.** The resistivity  $\rho$  of a conducting wire is the reciprocal of the conductivity and is measured in units of ohmmeters ( $\Omega$ -m). The resistivity of a given metal depends on the temperature according to the equation

$$\rho(t) = \rho_{20} e^{\alpha(t-20)}$$

where t is the temperature in °C. There are tables that list the values of  $\alpha$  (called the temperature coefficient) and  $\rho_{20}$  (the resistivity at 20 °C) for various metals. Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for  $\rho(t)$  by its first- or second-degree Taylor polynomial at t = 20.

(a) Find expressions for these linear and quadratic approximations.

Æ

 $\wedge$ 

- (b) For copper, the tables give α = 0.0039/°C and ρ<sub>20</sub> = 1.7 × 10<sup>-8</sup> Ω-m. Graph the resistivity of copper and the linear and quadratic approximations for -250 °C ≤ t ≤ 1000 °C.
- (c) For what values of *t* does the linear approximation agree with the exponential expression to within one percent?
- **27.** In Section 4.8 we considered Newton's method for approximating a root r of the equation f(x) = 0, and from an initial approximation  $x_1$  we obtained successive approximations  $x_2, x_3, \ldots$ , where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use Taylor's Inequality with n = 1,  $a = x_n$ , and x = r to show that if f''(x) exists on an interval *I* containing *r*,  $x_n$ , and  $x_{n+1}$ , and  $|f''(x)| \le M$ ,  $|f'(x)| \ge K$  for all  $x \in I$ , then

$$|x_{n+1} - r| \le \frac{M}{2K} |x_n - r|^2$$

[This means that if  $x_n$  is accurate to *d* decimal places, then  $x_{n+1}$  is accurate to about 2*d* decimal places. More precisely, if the error at stage *n* is at most  $10^{-m}$ , then the error at stage n + 1 is at most  $(M/2K)10^{-2m}$ .]



# Radiation from the Stars

Any object emits radiation when heated. A *blackbody* is a system that absorbs all the radiation that falls on it. For instance, a mat black surface or a large cavity with a small hole in its wall (like a blastfurnace) is a blackbody and emits blackbody radiation. Even the radiation from the Sun is close to being blackbody radiation.

Proposed in the late 19th century, the Rayleigh-Jeans Law expresses the energy density of blackbody radiation of wavelength  $\lambda$  as

$$f(\lambda) = \frac{8\pi kT}{\lambda^4}$$

where  $\lambda$  is measured in meters, T is the temperature in kelvins, and k is Boltzmann's constant. The Rayleigh-Jeans Law agrees with experimental measurements for long wavelengths but disagrees drastically for short wavelengths. [The law predicts that  $f(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow 0^+$  but experiments have shown that  $f(\lambda) \rightarrow 0$ .] This fact is known as the *ultraviolet catastrophe*.

In 1900 Max Planck found a better model (known now as Planck's Law) for blackbody radiation:

$$f(\lambda) = \frac{8\pi h c \lambda^{-5}}{e^{h c / (\lambda kT)} - 1}$$

where  $\lambda$  is measured in meters, T is the temperature in kelvins, and

 $h = \text{Planck's constant} = 6.6262 \times 10^{-34} \text{ J} \cdot \text{s}$ 

c = speed of light = 2.997925  $\times$  10<sup>8</sup> m/s

 $k = \text{Boltzmann's constant} = 1.3807 \times 10^{-23} \text{ J/K}$ 

I. Use l'Hospital's Rule to show that

 $\lim_{\lambda \to 0^+} f(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} f(\lambda) = 0$ 

for Planck's Law. So this law models blackbody radiation better than the Rayleigh-Jeans Law for short wavelengths.

- **2.** Use a Taylor polynomial to show that, for large wavelengths, Planck's Law gives approximately the same values as the Rayleigh-Jeans Law.
- 3. Graph f as given by both laws on the same screen and comment on the similarities and differences. Use T = 5700 K (the temperature of the Sun). (You may want to change from meters to the more convenient unit of micrometers:  $1 \ \mu m = 10^{-6} \ m$ .)
  - **4.** Use your graph in Problem 3 to estimate the value of  $\lambda$  for which  $f(\lambda)$  is a maximum under Planck's Law.
- **5.** Investigate how the graph of f changes as T varies. (Use Planck's Law.) In particular, graph f for the stars Betelgeuse (T = 3400 K), Procyon (T = 6400 K), and Sirius (T = 9200 K) as well as the Sun. How does the total radiation emitted (the area under the curve) vary with T? Use the graph to comment on why Sirius is known as a blue star and Betelgeuse as a red star.





# Using Series to Solve Differential Equations

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions. This is true even for a simple-looking equation like

y'' - 2xy' + y = 0

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics. In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients  $c_0, c_1, c_2, \ldots$ 

Before using power series to solve Equation 1, we illustrate the method on the simpler equation y'' + y = 0 in Example 1.

**EXAMPLE I** Use power series to solve the equation y'' + y = 0.

SOLUTION We assume there is a solution of the form

2 
$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots = \sum_{n=0}^{\infty} c_n x^n$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2x + 3c_3x^2 + \dots = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$y'' = 2c_2 + 2 \cdot 3c_3x + \dots = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

$$y'' = \sum_{n=0}^{\infty} (n + 2) (n + 1)c_{n+2}x'$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

ог

4

П

5 
$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1)c_{n+2} + c_n \right] x^n = 0$$

If two power series are equal, then the corresponding coefficients must be equal.

By writing out the first few terms of (4), you can see that it is the same as (3). To obtain (4) we replaced n by n + 2 and began the summation at 0 instead of 2.

Therefore, the coefficients of  $x^n$  in Equation 5 must be 0:

$$(n+2)(n+1)c_{n+2} + c_n = 0$$

$$c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$$
  $n = 0, 1, 2, 3, ...$ 

Equation 6 is called a recursion relation. If  $c_0$  and  $c_1$  are known, this equation allows us to determine the remaining coefficients recursively by putting n = 0, 1, 2, 3, ... in succession.

Put 
$$n = 0$$
:  $c_2 = -\frac{c_0}{1 \cdot 2}$   
Put  $n = 1$ :  $c_3 = -\frac{c_1}{2 \cdot 3}$   
Put  $n = 2$ :  $c_4 = -\frac{c_2}{3 \cdot 4} = \frac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{c_0}{4!}$   
Put  $n = 3$ :  $c_5 = -\frac{c_3}{4 \cdot 5} = \frac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{c_1}{5!}$   
Put  $n = 4$ :  $c_6 = -\frac{c_4}{5 \cdot 6} = -\frac{c_0}{4!5 \cdot 6} = -\frac{c_0}{6!}$   
Put  $n = 5$ :  $c_7 = -\frac{c_5}{6 \cdot 7} = -\frac{c_1}{5!6 \cdot 7} = -\frac{c_1}{7!}$ 

By now we see the pattern:

For the even coefficients, 
$$c_{2n} = (-1)^n \frac{c_0}{(2n)!}$$

For the odd coefficients, 
$$c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}$$

Putting these values back into Equation 2, we write the solution as

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$
  
=  $c_0 \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \right)$   
+  $c_1 \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \right)$   
=  $c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ 

Notice that there are two arbitrary constants,  $c_0$  and  $c_1$ .

**NOTE 1** • We recognize the series obtained in Example 1 as being the Maclaurin series for  $\cos x$  and  $\sin x$ . Therefore, we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions. **EXAMPLE 2** Solve y'' - 2xy' + y = 0.

SOLUTION We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and

7

 $\sum_{n=1}^{\infty} 2nc_n x^n = \sum_{n=0}^{\infty} 2nc_n x^n$ 

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

as in Example 1. Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (2n-1)c_n]x^n = 0$$

This equation is true if the coefficient of  $x^n$  is 0:

$$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$$
$$c_{n+2} = \frac{2n-1}{(n+1)(n+2)}c_n \qquad n = 0, 1, 2, 3, \dots$$

We solve this recursion relation by putting n = 0, 1, 2, 3, ... successively in Equation 7:

Put $n = 0$ :	$c_2 = \frac{-1}{1 \cdot 2} c_0$	
Put $n = 1$ :	$c_3 = \frac{1}{2 \cdot 3} c_1$	
Put $n = 2$ :	$c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0$	$= -\frac{3}{4!} c_0$
Put $n = 3$ :	$c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 =$	$\frac{1\cdot 5}{5!}c_1$
Put $n = 4$ :	$c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4! 5 \cdot 6} c_0 = -$	$\frac{3\cdot 7}{6!}c_0$
Put $n = 5$ :	$c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5! 6 \cdot 7} c_1 = \frac{1 \cdot 5}{5! 6! 5! 5! 5! 5! 5! 5! 5! 5! 5! 5! 5! 5! 5!$	$\frac{5\cdot 9}{7!} c_1$

Put 
$$n = 6$$
:  $c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0$   
Put  $n = 7$ :  $c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$ 

In general, the even coefficients are given by

$$c_{2n} = -\frac{3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-5)}{(2n)!} c_0$$

and the odd coefficients are given by

$$c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4n-3)}{(2n+1)!} c_1$$

The solution is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$
  
=  $c_0 \left( 1 - \frac{1}{2!} x^2 - \frac{3}{4!} x^4 - \frac{3 \cdot 7}{6!} x^6 - \frac{3 \cdot 7 \cdot 11}{8!} x^8 - \cdots \right)$   
+  $c_1 \left( x + \frac{1}{3!} x^3 + \frac{1 \cdot 5}{5!} x^5 + \frac{1 \cdot 5 \cdot 9}{7!} x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} x^9 + \cdots \right)$ 

or

B  

$$y = c_0 \left( 1 - \frac{1}{2!} x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \dots \cdot (4n-5)}{(2n)!} x^{2n} \right) + c_1 \left( x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4n-3)}{(2n+1)!} x^{2n+1} \right)$$

**NOTE 2** • In Example 2 we had to assume that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

**NOTE 3** • Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions. The functions

$$y_1(x) = 1 - \frac{1}{2!} x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \dots \cdot (4n-5)}{(2n)!} x^{2n}$$
$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

and

are perfectly good functions but they can't be expressed in terms of familiar functions. We can use these power series expressions for  $y_1$  and  $y_2$  to compute approximate values of the functions and even to graph them. Figure 1 shows the first few partial sums  $T_0, T_2, T_4, \ldots$  (Taylor polynomials) for  $y_1(x)$ , and we see how they converge to  $y_1$ . In this way we can graph both  $y_1$  and  $y_2$  in Figure 2.

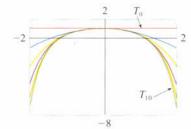
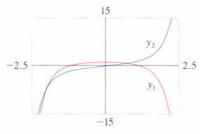


FIGURE I





**NOTE 4** • If we were asked to solve the initial-value problem

$$y'' - 2xy' + y = 0$$
  $y(0) = 0$   $y'(0) = 1$ 

we would observe that

$$c_0 = y(0) = 0$$
  $c_1 = y'(0) = 1$ 

This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is

$$y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$



Exercises · · ·

- 1-9 Use power series to solve the differential equation.
- **1.** y' = 6y **2.** y' = xy
- **3.**  $y' = x^2 y$  **4.** y'' = y
- **5.** y'' + 3xy' + 3y = 0 **6.** y'' = xy
- 7. y'' xy' y = 0, y(0) = 1, y'(0) = 0

**8.** 
$$y'' + x^2 y = 0$$
,  $y(0) = 1$ ,  $y'(0) = 0$ 

**9.**  $y'' + x^2y' + xy = 0$ , y(0) = 0, y'(0) = 1



# **Chapter 8 Review**

**10.** The solution of the initial-value problem

 $x^{2}y'' + xy' + x^{2}y = 0$  y(0) = 1 y'(0) = 0

is called a Bessel function of order 0.

- (a) Solve the initial-value problem to find a power series expansion for the Bessel function.
- (b) Graph several Taylor polynomials until you reach one that looks like a good approximation to the Bessel function on the interval [-5, 5].

• CONCEPT CHECK •

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- I. (a) What is a convergent sequence?
  - (b) What is a convergent series?
  - (c) What does  $\lim_{n\to\infty} a_n = 3$  mean?
  - (d) What does  $\sum_{n=1}^{\infty} a_n = 3$  mean?
- 2. (a) What is a bounded sequence?
  - (b) What is a monotonic sequence?
  - (c) What can you say about a bounded monotonic sequence?
- **3.** (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?
  - (b) What is a p-series? Under what circumstances is it convergent?
- Suppose Σ a<sub>n</sub> = 3 and s<sub>n</sub> is the nth partial sum of the series. What is lim<sub>n→∞</sub> a<sub>n</sub>? What is lim<sub>n→∞</sub> s<sub>n</sub>?
- 5. State the following.
  - (a) The Test for Divergence
  - (b) The Integral Test
  - (c) The Comparison Test
  - (d) The Limit Comparison Test

- (e) The Alternating Series Test
- (f) The Ratio Test
- 6. (a) What is an absolutely convergent series?
  - (b) What can you say about such a series?
- **7.** (a) If a series is convergent by the Integral Test, how do you estimate its sum?
  - (b) If a series is convergent by the Comparison Test, how do you estimate its sum?
  - (c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?
- 8. (a) Write the general form of a power series.
  - (b) What is the radius of convergence of a power series?
  - (c) What is the interval of convergence of a power series?
- **9.** Suppose f(x) is the sum of a power series with radius of convergence *R*.
  - (a) How do you differentiate f? What is the radius of convergence of the series for f'?

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- (b) How do you integrate f? What is the radius of convergence of the series for  $\int f(x) dx$ ?
- **10.** (a) Write an expression for the *n*th-degree Taylor polynomial of *f* centered at *a*.
  - (b) Write an expression for the Taylor series of *f* centered at *a*.
  - (c) Write an expression for the Maclaurin series of f.
  - (d) How do you show that f(x) is equal to the sum of its Taylor series?

- (e) State Taylor's Inequality.
- **II.** Write the Maclaurin series and the interval of convergence for each of the following functions.
  - (a) 1/(1 x) (b)  $e^x$ (c)  $\sin x$  (d)  $\cos x$ 
    - (e)  $\tan^{-1}x$
- 12. Write the binomial series expansion of  $(1 + x)^k$ . What is the radius of convergence of this series?

# 🔺 TRUE-FALSE QUIZ 🔺

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

- **1.** If  $\lim_{n\to\infty} a_n = 0$ , then  $\sum a_n$  is convergent.
- **2.** If  $\sum c_n 6^n$  is convergent, then  $\sum c_n (-2)^n$  is convergent.
- **3.** If  $\sum c_n 6^n$  is convergent, then  $\sum c_n (-6)^n$  is convergent.
- **4.** If  $\sum c_n x^n$  diverges when x = 6, then it diverges when x = 10.
- 5. The Ratio Test can be used to determine whether  $\sum 1/n^3$  converges.
- The Ratio Test can be used to determine whether Σ 1/n! converges.
- **7.** If  $0 \le a_n \le b_n$  and  $\Sigma b_n$  diverges, then  $\Sigma a_n$  diverges.

- **8.**  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$
- 9. If  $-1 < \alpha < 1$ , then  $\lim_{n \to \infty} \alpha^n = 0$ .
- **10.** If  $\Sigma a_n$  is divergent, then  $\Sigma |a_n|$  is divergent.
- **II.** If  $f(x) = 2x x^2 + \frac{1}{3}x^3 \cdots$  converges for all *x*, then f'''(0) = 2.
- 12. If  $\{a_n\}$  and  $\{b_n\}$  are divergent, then  $\{a_n + b_n\}$  is divergent.
- **13.** If  $\{a_n\}$  and  $\{b_n\}$  are divergent, then  $\{a_nb_n\}$  is divergent.
- **14.** If  $\{a_n\}$  is decreasing and  $a_n > 0$  for all *n*, then  $\{a_n\}$  is convergent.
- **15.** If  $a_n > 0$  and  $\sum a_n$  converges, then  $\sum (-1)^n a_n$  converges.
- **16.** If  $a_n > 0$  and  $\lim_{n \to \infty} (a_{n+1}/a_n) < 1$ , then  $\lim_{n \to \infty} a_n = 0$ .

# EXERCISES 🧄

**1–7** Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

1. $a_n = \frac{n}{2n+5}$	<b>2.</b> $a_n = 5 - (0.9)^n$
<b>3.</b> $a_n = 2n + 5$	<b>4.</b> $a_n = n/\ln n$
<b>5.</b> $a_n = \sin n$	<b>6.</b> $a_n = (\sin n)/n$
<b>7.</b> $\{(1 + 3/n)^{4n}\}$	

8. A sequence is defined recursively by the equations  $a_1 = 1, a_{n+1} = \frac{1}{3}(a_n + 4)$ . Show that  $\{a_n\}$  is increasing and  $a_n < 2$  for all *n*. Deduce that  $\{a_n\}$  is convergent and find its limit.

**9–18** ■ Determine whether the series is convergent or divergent.

9. 
$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$
 II.  $\sum_{n=1}^{\infty} \frac{n + n^2}{n + n^4}$ 

11. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$$
12. 
$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}$$
13. 
$$\sum_{n=1}^{\infty} \frac{4^n}{n3^n}$$
14. 
$$\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{n}}$$
15. 
$$\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$$
16. 
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$
17. 
$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{5^n n!}$$
18. 
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$$

**19–22** Find the sum of the series.

**19.** 
$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n}$$
 **20.**  $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ 

**21.**  $\sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1}n]$ 

**22.** 
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{2n} n!}$$

- **23.** Express the repeating decimal 1.2345345345... as a
- fraction.
- **24.** For what values of x does the series  $\sum_{n=1}^{\infty} (\ln x)^n$  converge?
- **25.** Find the sum of the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$  correct to four decimal places.
- **26.** (a) Find the partial sum  $s_5$  of the series  $\sum_{n=1}^{\infty} 1/n^6$  and estimate the error in using it as an approximation to the sum of the series.
  - (b) Find the sum of this series correct to five decimal places.
- **27.** Use the sum of the first eight terms to approximate the sum of the series  $\sum_{n=1}^{\infty} (2 + 5^n)^{-1}$ . Estimate the error involved in this approximation.
- **28.** (a) Show that the series  $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$  is convergent. (b) Deduce that  $\lim_{n \to \infty} \frac{n^n}{(2n)!} = 0.$
- **29.** Prove that if the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then the series

$$\sum_{n=1}^{\infty} \left( \frac{n+1}{n} \right) a_n$$

is also absolutely convergent.

**30–33** Find the radius of convergence and interval of convergence of the series.

**30.** 
$$\sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n+1}$$
**31.** 
$$\sum_{n=1}^{\infty} \frac{x^n}{3^n n^3}$$
**32.** 
$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{(n+1)!}$$
**33.** 
$$\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$$

34. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

- **35.** Find the Taylor series of  $f(x) = \sin x$  at  $a = \pi/6$ .
- **36.** Find the Taylor series of  $f(x) = \cos x$  at  $a = \pi/3$ .

**37–44** Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (defini-

tion of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for  $e^x$ and sin x.

- **37.**  $f(x) = \frac{x^2}{1+x}$  **38.**  $f(x) = \sqrt{1-x^2}$  **39.**  $f(x) = \ln(1-x)$  **40.**  $f(x) = xe^{2x}$  **41.**  $f(x) = \sin(x^4)$  **42.**  $f(x) = 10^x$  **43.**  $f(x) = 1/\sqrt[4]{16-x}$ **44.**  $f(x) = (1-3x)^{-5}$
- **45.** Evaluate  $\int \frac{e^x}{x} dx$  as an infinite series.
- **46.** Use series to approximate  $\int_0^1 \sqrt{1 + x^4} dx$  correct to two decimal places.

## 47-48

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- (a) Approximate f by a Taylor polynomial with degree n at the number a.
- (b) Graph f and  $T_n$  on a common screen.
  - (c) Use Taylor's Inequality to estimate the accuracy of the approximation  $f(x) \approx T_n(x)$  when x lies in the given interval.
- (d) Check your result in part (c) by graphing  $|R_n(x)|$ .

**47.** 
$$f(x) = \sqrt{x}$$
,  $a = 1$ ,  $n = 3$ ,  $0.9 \le x \le 1$ .

**48.**  $f(x) = \sec x$ , a = 0, n = 2,  $0 \le x \le \pi/6$ 

- **49.** Use series to evaluate  $\lim_{x\to\infty} x^2(1 e^{-1/x^2})$ .
- **50.** The force due to gravity on an object with mass *m* at a height *h* above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity.

- (a) Express F as a series in powers of h/R.
- (b) Observe that if we approximate F by the first term in the series, we get the expression  $F \approx mg$  that is usually used when h is much smaller than R. Use the Alternating Series Estimation Theorem to estimate the range of values of h for which the approximation  $F \approx mg$  is accurate to within 1%. (Use R = 6400 km.)
- 51. Use power series to solve the initial-value problem

y'' + xy' + y = 0 y(0) = 0 y'(0) = 1

52. Use power series to solve the equation

$$y'' - xy' - 2y = 0$$



Before you look at the solution of the following example, cover it up and first try to solve the problem yourself.

**Example** Find the sum of the series 
$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$$
.

**Solution** The problem-solving principle that is relevant here is *recognizing some-thing familiar*. Does the given series look anything like a series that we already know? Well, it does have some ingredients in common with the Maclaurin series for the exponential function:

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$$

We can make this series look more like our given series by replacing x by x + 2:

$$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = 1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \cdots$$

But here the exponent in the numerator matches the number in the denominator whose factorial is taken. To make that happen in the given series, let's multiply and divide by  $(x + 2)^3$ :

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \frac{1}{(x+2)^3} \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}$$
$$= (x+2)^{-3} \left[ \frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \cdots \right]$$

We see that the series between brackets is just the series for  $e^{x+2}$  with the first three terms missing. So

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = (x+2)^{-3} \left[ e^{x+2} - 1 - (x+2) - \frac{(x+2)^2}{2!} \right]$$

1. If  $f(x) = \sin(x^3)$ , find  $f^{(15)}(0)$ .

- **2.** Let  $\{P_n\}$  be a sequence of points determined as in the figure. Thus  $|AP_1| = 1$ ,  $|P_nP_{n+1}| = 2^{n-1}$ , and angle  $AP_nP_{n+1}$  is a right angle. Find  $\lim_{n\to\infty} \angle P_nAP_{n+1}$ .
- 3. (a) Show that  $\tan \frac{1}{2}x = \cot \frac{1}{2}x 2 \cot x$ . (b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

4. A function *f* is defined by

$$f(x) = \lim_{n \to \infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is *f* continuous?

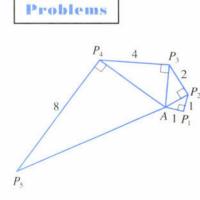
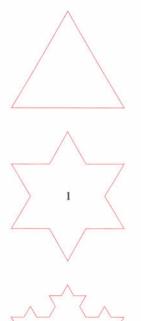
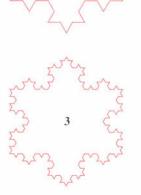


Figure for Problem 2

\_





2

Figure for Problem 5

- 5. To construct the snowflake curve, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part (see the figure). Step 2 is to repeat Step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.
  - (a) Let  $s_n$ ,  $l_n$ , and  $p_n$  represent the number of sides, the length of a side, and the total length of the *n*th approximating curve (the curve obtained after Step *n* of the construction), respectively. Find formulas for  $s_n$ ,  $l_n$ , and  $p_n$ .
  - (b) Show that  $p_n \to \infty$  as  $n \to \infty$ .
  - (c) Sum an infinite series to find the area enclosed by the snowflake curve. Parts (b) and (c) show that the snowflake curve is infinitely long but encloses only a finite area.
- 6. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \dots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

**7.** (a) Show that for  $xy \neq -1$ ,

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

if the left side lies between  $-\pi/2$  and  $\pi/2$ .

(b) Show that

$$\arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

(c) Deduce the following formula of John Machin (1680-1751):

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

(d) Use the Maclaurin series for arctan to show that

 $0.197395560 < \arctan \frac{1}{5} < 0.197395562$ 

(e) Show that

 $0.004184075 < \arctan \frac{1}{239} < 0.004184077$ 

(f) Deduce that, correct to seven decimal places,

 $\pi \approx 3.1415927$ 

Machin used this method in 1706 to find  $\pi$  correct to 100 decimal places. In this century, with the aid of computers, the value of  $\pi$  has been computed to increasingly greater accuracy. In 1995 Jonathan and Peter Borwein of Simon Fraser University and Yasumasa Kanada of the University of Tokyo calculated the value of  $\pi$  to 4,294,967,286 decimal places!

**8.** If 
$$a_0 + a_1 + a_2 + \dots + a_k = 0$$
, show that

$$\lim_{n \to \infty} (a_0 \sqrt{n} + a_1 \sqrt{n+1} + a_2 \sqrt{n+2} + \dots + a_k \sqrt{n+k}) = 0$$

If you don't see how to prove this, try the problem-solving strategy of using analogy (see page 87). Try the special cases k = 1 and k = 2 first. If you can see how to prove the assertion for these cases, then you will probably see how to prove it in general.

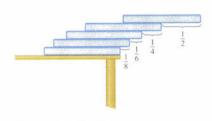


Figure for Problem 10

- 9. Find the interval of convergence of  $\sum_{n=1}^{\infty} n^3 x^n$  and find its sum.
- **10.** Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Use the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (Try it yourself with a deck of cards.) Consider centers of mass.



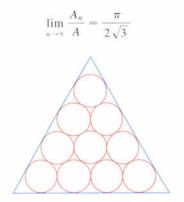
$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \dots$$
$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \dots$$
$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \dots$$

Show that  $u^3 + v^3 + w^3 - 3uvw = 1$ .

**12.** If p > 1, evaluate the expression



13. Suppose that circles of equal diameter are packed tightly in *n* rows inside an equilateral triangle. (The figure illustrates the case n = 4.) If *A* is the area of the triangle and  $A_n$  is the total area occupied by the *n* rows of circles, show that



14. A sequence  $\{a_n\}$  is defined recursively by the equations

 $a_0 = a_1 = 1$   $n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$ 

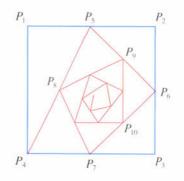
Find the sum of the series  $\sum_{n=0}^{\infty} a_n$ .

**15.** Consider the series whose terms are the reciprocals of the positive integers that can be written in base 10 notation without using the digit 0. Show that this series is convergent and the sum is less than 90.

16. Starting with the vertices P<sub>1</sub>(0, 1), P<sub>2</sub>(1, 1), P<sub>3</sub>(1, 0), P<sub>4</sub>(0, 0) of a square, we construct further points as shown in the figure: P<sub>5</sub> is the midpoint of P<sub>1</sub>P<sub>2</sub>, P<sub>6</sub> is the midpoint of P<sub>2</sub>P<sub>3</sub>, P<sub>7</sub> is the midpoint of P<sub>3</sub>P<sub>4</sub>, and so on. The polygonal spiral path P<sub>1</sub>P<sub>2</sub>P<sub>3</sub>P<sub>4</sub>P<sub>5</sub>P<sub>6</sub>P<sub>7</sub>... approaches a point P inside the square.
(a) If the coordinates of P<sub>n</sub> are (x<sub>n</sub>, y<sub>n</sub>), show that

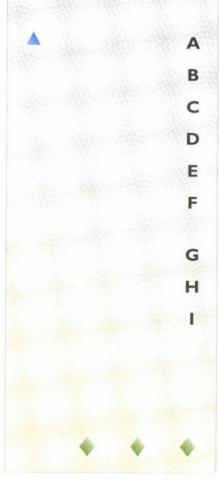
$$\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$$

and find a similar equation for the *y*-coordinates. (b) Find the coordinates of *P*.





# **APPENDIXES**



Intervals, Inequalities, and Absolute Values A2 Coordinate Geometry A7 Trigonometry A19 Precise Definitions of Limits A32 A Few Proofs A40 Integration of Rational Functions by Partial Fractions A42 Polar Coordinates A51 Complex Numbers A67

Answers to Odd-Numbered Exercises A76

b

h



a

a

Closed interval [a, b]

FIGURE 2

FIGURE I Open interval (a, b)

# Intervals, Inequalities, and Absolute Values

Certain sets of real numbers, called intervals, occur frequently in calculus and correspond geometrically to line segments. For example, if a < b, the open interval from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

Notice that the endpoints of the interval, namely, a and b, are excluded. This is indicated by the round brackets () and by the open dots in Figure 1. The closed interval from a to b is the set

$$[a,b] = \{x \mid a \le x \le b\}$$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 2. It is also possible to include only one endpoint in an interval, as shown in Table 1.

We also need to consider infinite intervals such as

$$(a,\infty) = \{x \mid x > a\}$$

Table I lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that a < b.

# Table of Intervals

This does not mean that  $\infty$  ("infinity") is a number. The notation  $(a, \infty)$  stands for the set of all numbers that are greater than a, so the symbol  $\infty$  simply indicates that the interval extends indefinitely far in the positive direction.

Notation	Set description	Picture	Notation	Set description	Picture
( <i>a</i> , <i>b</i> )	$\{x \mid a < x < b\}$	o <u> </u>	$(a,\infty)$	$\{x \mid x > a\}$	
[a,b]	$\{x \mid a \le x \le b\}$	a h	$[a,\infty)$	$\{x \mid x \ge a\}$	a
[ <i>a</i> , <i>b</i> )	$\{x \mid a \le x < b\}$	a h	$(-\infty, b)$ $(-\infty, b]$	$ \begin{cases} x \mid x < b \\ x \mid x \le b \end{cases} $	b
(a,b]	$\{x \mid a < x \le b\}$	a b	$(-\infty,\infty)$	ℝ (set of all real numbers)	b



# Inequalities

When working with inequalities, note the following rules:

## **Rules for Inequalities**

I. If a < b, then a + c < b + c. **2.** If a < b and c < d, then a + c < b + d. **3.** If a < b and c > 0, then ac < bc. **4.** If a < b and c < 0, then ac > bc. **5.** If 0 < a < b, then 1/a > 1/b.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a positive number, but Rule 4 says that if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we take the inequality 3 < 5 and multiply by 2, we get 6 < 10, but if we multiply by -2, we get -6 > -10. Finally, Rule 5 says that if we take reciprocals, then we

**EXAMPLE I** Solve the inequality 1 + x < 7x + 5.

**SOLUTION** The given inequality is satisfied by some values of *x* but not by others. To *solve* an inequality means to determine the set of numbers *x* for which the inequality is true. This is called the *solution set*.

First we subtract 1 from each side of the inequality (using Rule 1 with c = -1):

reverse the direction of an inequality (provided the numbers are positive).

$$x < 7x + 4$$

Then we subtract 7x from both sides (Rule 1 with c = -7x):

$$-6x < 4$$

Now we divide both sides by -6 (Rule 4 with  $c = -\frac{1}{6}$ ):

$$x > -\frac{4}{6} = -\frac{2}{3}$$

All these steps can be reversed, so the solution set consists of all numbers greater than  $-\frac{2}{3}$ . In other words, the solution of the inequality is the interval  $(-\frac{2}{3}, \infty)$ .

**EXAMPLE 2** Solve the inequality  $x^2 - 5x + 6 \le 0$ .

**SOLUTION** First we factor the left side:

$$(x-2)(x-3) \le 0$$

We know that the corresponding equation (x - 2)(x - 3) = 0 has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

 $(-\infty, 2)$  (2,3) (3, $\infty$ )

On each of these intervals we determine the signs of the factors. For instance,

 $x \in (-\infty, 2) \Rightarrow x < 2 \Rightarrow x - 2 < 0$ 

Then we record these signs in a chart:

Interval	x = 2	<i>x</i> = 3	(x-2)(x-3)
x < 2	- 11	_	+
2 < x < 3	+	-	100
x > 3	+	÷	-

is to use a graphing device to graph the parabola  $y = x^2 - 5x + 6$  (as in Figure 3) and observe that the curve lies on or below the x-axis when  $2 \le x \le 3$ .  $y = x^2 - 5x + 6$ 

A visual method for solving Example 2

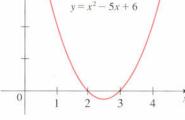


FIGURE 3

Another method for obtaining the information in the chart is to use *test* values. For instance, if we use the test value x = 1 for the interval  $(-\infty, 2)$ , then substitution in  $x^2 - 5x + 6$  gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial  $x^2 - 5x + 6$  does not change sign inside any of the three intervals, so we conclude that it is positive on  $(-\infty, 2)$ .

Then we read from the chart that (x - 2)(x - 3) is negative when 2 < x < 3. Thus, the solution of the inequality  $(x - 2)(x - 3) \le 0$  is

$${x | 2 \le x \le 3} = [2, 3]$$

Notice that we have included the endpoints 2 and 3 because we seek values of x such that the product is either negative or zero. The solution is illustrated in Figure 4.

**EXAMPLE 3** Solve  $x^3 + 3x^2 > 4x$ .

**SOLUTION** First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^{3} + 3x^{2} - 4x > 0$$
 or  $x(x - 1)(x + 4) > 0$ 

As in Example 2, we solve the corresponding equation x(x - 1)(x + 4) = 0and use the solutions x = -4, x = 0, and x = 1 to divide the real line into four intervals  $(-\infty, -4)$ , (-4, 0), (0, 1), and  $(1, \infty)$ . On each interval the product keeps a constant sign as shown in the chart:

Interval	X	x = 1	$\chi + 4$	x(x - 1)(x + 4)
x < -4	_		_	
-4 < x < 0			-6	t
0 < x < 1	+		192	
$x \ge 1$	+	đ	+	t

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

FIGURE 5

-4

The solution is illustrated in Figure 5.

# Absolute Value

0

1

The **absolute value** of a number a, denoted by |a|, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \ge 0$$
 for every number a

For example,

$$|3| = 3$$
  $|-3| = 3$   $|0| = 0$   
 $|\sqrt{2} - 1| = \sqrt{2} - 1$   $|3 - \pi| = \pi - 3$ 



#### APPENDIX A INTERVALS, INEQUALITIES, AND ABSOLUTE VALUES AS

In general, we have

Remember that if a is negative, then -a is positive.

a  = a	if $a \ge 0$
a  = -a	if $a < 0$

**EXAMPLE 4** Express |3x - 2| without using the absolute value symbol.

SOLUTION

2

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \ge 0\\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$
$$= \begin{cases} 3x - 2 & \text{if } x \ge \frac{2}{3}\\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

Recall that the symbol  $\sqrt{}$  means "the positive square root of." Thus,  $\sqrt{r} = s$ means  $s^2 = r$  and  $s \ge 0$ . Therefore, the equation  $\sqrt{a^2} = a$  is not always true. It is true only when  $a \ge 0$ . If a < 0, then -a > 0, so we have  $\sqrt{a^2} = -a$ . In view of (2), we then have the equation

$$\sqrt{a^2} = |a|$$

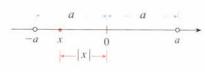
which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

Properties of Absolute Values Suppose *a* and *b* are any real numbers and *n* is an integer. Then 1. |ab| = |a||b|2.  $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$  ( $b \neq 0$ ) 3.  $|a^n| = |a|^n$ 

For solving equations or inequalities involving absolute values, it is often very helpful to use the following statements.

Suppose a > 0. Then **4.** |x| = a if and only if  $x = \pm a$  **5.** |x| < a if and only if -a < x < a**6.** |x| > a if and only if x > a or x < -a



For instance, the inequality |x| < a says that the distance from x to the origin is less than a, and you can see from Figure 6 that this is true if and only if x lies between -a and a.

FIGURE 6

#### APPENDIX A INTERVALS, INEQUALITIES, AND ABSOLUTE VALUES A6

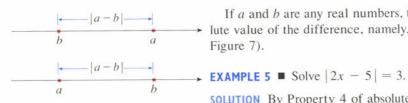


FIGURE 7 Length of a line segment = |a - b|

If a and b are any real numbers, then the distance between a and b is the abso-

**SOLUTION** By Property 4 of absolute values, |2x - 5| = 3 is equivalent to

2x - 5 = 3 or 2x - 5 = -3

So 2x = 8 or 2x = 2. Thus, x = 4 or x = 1.

**EXAMPLE 6** Solve |x - 5| < 2.

**SOLUTION 1** By Property 5 of absolute values, |x - 5| < 2 is equivalent to

-2 < x - 5 < 2

Therefore, adding 5 to each side, we have

3 < x < 7

and the solution set is the open interval (3,7).

. . . . . . .

**SOLUTION 2** Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 8 we see that this is the interval (3,7).

**EXAMPLE 7** Solve  $|3x + 2| \ge 4$ .

**SOLUTION** By Properties 4 and 6,  $|3x + 2| \ge 4$  is equivalent to

 $3x + 2 \ge 4$  or  $3x + 2 \le -4$ 

In the first case  $3x \ge 2$ , which gives  $x \ge \frac{2}{3}$ . In the second case  $3x \le -6$ , which gives  $x \leq -2$ . So the solution set is

$$\{x \mid x \le -2 \text{ or } x \ge \frac{2}{3}\} = (-\infty, -2] \cup \left[\frac{2}{3}, \infty\right)$$



Exercises •

**I−10** ■ Rewrite the expression without using the absolute value symbol.

<b>I.</b>  5 - 23	<b>2.</b> $ \pi - 2 $
<b>3.</b> $ \sqrt{5} - 5 $	<b>4.</b> $  -2  -  -3  $
<b>5.</b> $ x - 2 $ if $x < 2$	<b>6.</b> $ x - 2 $ if $x > 2$
<b>7.</b> $ x + 1 $	<b>8.</b> $ 2x - 1 $
<b>9.</b> $ x^2 + 1 $	<b>10.</b> $ 1 - 2x^2 $

11-26 Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

**12.**  $4 - 3x \ge 6$ 11. 2x + 7 > 314. 1 + 5x > 5 - 3x**13.**  $1 - x \le 2$ 

<b>15.</b> $0 \le 1 - x < 1$	<b>16.</b> $1 < 3x + 4 \le 16$
<b>17.</b> $(x - 1)(x - 2) > 0$	<b>18.</b> $x^2 < 2x + 8$
<b>19.</b> $x^2 < 3$	<b>20.</b> $x^2 \ge 5$
<b>21.</b> $x^3 - x^2 \le 0$	
<b>22.</b> $(x + 1)(x - 2)(x + 3)$	$\geq 0$
<b>23.</b> $x^3 > x$	<b>24.</b> $x^3 + 3x < 4x^2$
<b>25.</b> $\frac{1}{x} < 4$	<b>26.</b> $-3 < \frac{1}{x} \le 1$
a) A (A) (A) (A) (A)	(A) X X X X

27. The relationship between the Celsius and Fahrenheit temperature scales is given by  $C = \frac{5}{9}(F - 32)$ , where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on



the Celsius scale corresponds to the temperature range  $50 \le F \le 95$ ?

- **28.** Use the relationship between *C* and *F* given in Exercise 27 to find the interval on the Fahrenheit scale corresponding to the temperature range  $20 \le C \le 30$ .
- **29.** As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.
  - (a) If the ground temperature is 20 °C, write a formula for the temperature at height h.
  - (b) What range of temperature can be expected if an airplane takes off and reaches a maximum height of 5 km?
- **30.** If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height *h* above the ground *t* seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

**31–32** Solve the equation for x.

- **31.** |x + 3| = |2x + 1|**32.** |3x + 5| = 1
- **33–40** Solve the inequality.
- **33.** |x| < 3**34.**  $|x| \ge 3$ **35.** |x 4| < 1**36.** |x 6| < 0.1**37.**  $|x + 5| \ge 2$ **38.**  $|x + 1| \ge 3$ **39.**  $|2x 3| \le 0.4$ **40.** |5x 2| < 6
- **41.** Solve the inequality  $a(bx c) \ge bc$  for x assuming that a, b, and c are positive constants.
- 42. Solve the inequality ax + b < c for x assuming that a, b, and c are negative constants.</li>
- **43.** Prove that |ab| = |a||b|. [*Hint:* Use Equation 3.]
- 44. Show that if 0 < a < b, then  $a^2 < b^2$ .



Coordinate Geometry

The points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the *x*-axis; the other line is vertical with positive direction upward and is called the *y*-axis.

Any point *P* in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through *P* perpendicular to the *x*- and *y*-axes. These lines intersect the axes in points with coordinates *a* and *b* as shown in Figure 1. Then the point *P* is assigned the ordered pair (a, b). The first number *a* is called the *x*-coordinate of *P*; the second number *b* is called the *y*-coordinate of *P*. We say that *P* is the point with coordinates (a, b), and we denote the point by the symbol P(a, b). Several points are labeled with their coordinates in Figure 2.

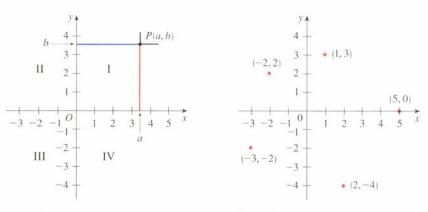


FIGURE I

FIGURE 2

By reversing the preceding process we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." [Although the notation used for an open interval (a, b) is the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system**, or the Cartesian coordinate system in honor of the French mathematician René Descartes (1596-1650), even though another Frenchman, Pierre Fermat (1601-1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the coordinate plane, or the Cartesian plane, and is denoted by  $\mathbb{R}^2$ .

The x- and y-axes are called the coordinate axes and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

**EXAMPLE I** Describe and sketch the regions given by the following sets.

(b)  $\{(x, y) | y = 1\}$  (c)  $\{(x, y) | | y | < 1\}$ (a)  $\{(x, y) | x \ge 0\}$ 

# SOLUTION

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it, as indicated by the shaded region in Figure 3(a).

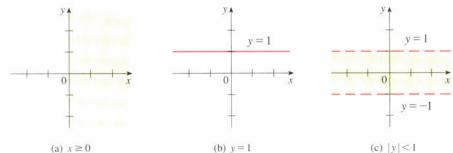


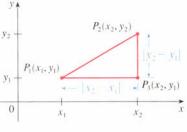
FIGURE 3

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 3(b)].

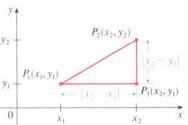
(c) Recall from Appendix A that

|y| < 1 if and only if -1 < y < 1

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus, the region consists of all points that lie between (but not on) the horizontal lines y = 1 and y = -1. [These lines are shown as broken lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.]



Recall from Appendix A that the distance between points a and b on a number line is |a - b| = |b - a|. Thus, the distance between points  $P_1(x_1, y_1)$  and  $P_3(x_2, y_1)$  on a horizontal line is  $|x_2 - x_1|$ , and the distance between  $P_2(x_2, y_2)$  and  $P_3(x_2, y_1)$  on a vertical line is  $|y_2 - y_1|$  (see Figure 4).





#### APPENDIX B COORDINATE GEOMETRY = A9

To find the distance  $|P_1P_2|$  between any two points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ , we note that triangle  $P_1P_2P_3$  in Figure 4 is a right triangle, and so by the Pythagorean Theorem we have

$$|P_1P_2| = \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Distance Formula** The distance between the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For instance, the distance between the points (1, -2) and (5, 3) is

$$\sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

# Circles

An equation of a curve is an equation satisfied by the coordinates of the points on the curve and by no other points. Let's use the distance formula to find the equation of a circle with radius r and center (h, k). By definition, the circle is the set of all points P(x, y) whose distance from the center C(h, k) is r (see Figure 5). Thus, P is on the circle if and only if |PC| = r. From the distance formula, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the desired equation.

**Equation of a Circle** An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin (0,0), the equation is

$$x^{2} + y^{2} = r^{2}$$

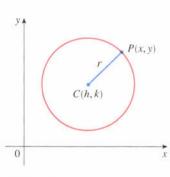
For instance, an equation of the circle with radius 3 and center (2, -5) is

$$(x-2)^2 + (y+5)^2 = 9$$

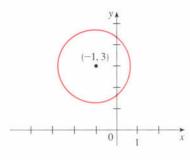
**EXAMPLE 2** Sketch the graph of the equation  $x^2 + y^2 + 2x - 6y + 7 = 0$  by first showing that it represents a circle and then finding its center and radius.

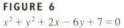
**SOLUTION** We first group the *x*-terms and *y*-terms as follows:

$$(x^{2} + 2x) + (y^{2} - 6y) = -7$$









Then we complete the square within each grouping, adding the appropriate constants (the squares of half the coefficients of x and y) to both sides of the equation:

$$x^{2} + 2x + 1) + (y^{2} - 6y + 9) = -7 + 1 + 9$$

or

$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle, we see that h = -1, k = 3, and  $r = \sqrt{3}$ , so the given equation represents a circle with center (-1, 3) and radius  $\sqrt{3}$ . It is sketched in Figure 6.

# Lines

To find the equation of a line L we use its *slope*, which is a measure of the steepness of the line.

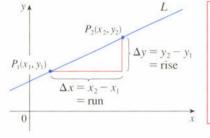
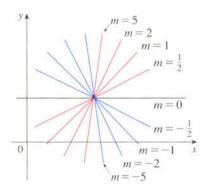


FIGURE 7



**Definition** The **slope** of a nonvertical line that passes through the points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Thus, the slope of a line is the ratio of the change in y,  $\Delta y$ , to the change in x,  $\Delta x$  (see Figure 7). The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

Figure 8 shows several lines labeled with their slopes. Lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

Now let's find an equation of the line that passes through a given point  $P_1(x_1, y_1)$  and has slope *m*. A point P(x, y) with  $x \neq x_1$  lies on this line if and only if the slope of the line through  $P_1$  and P is equal to *m*; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when  $x = x_1$  and  $y = y_1$ . Therefore, it is an equation of the required line.

**Point-Slope Form of the Equation of a Line** An equation of the line passing through the point  $P_1(x_1, y_1)$  and having slope *m* is

$$y - y_1 = m(x - x_1)$$



**EXAMPLE 3** Find an equation of the line through the points (-1, 2)and (3, -4).

**SOLUTION** The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form w 2, we obtain

$$y - 2 = -\frac{3}{2}(x + 1)$$

which can be written as

$$3x + 2y = 1$$

Suppose a nonvertical line has slope m and y-intercept b (see Figure 9). This means it intersects the y-axis at the point (0, b), so the point-slope form of the equation of the line, with  $x_1 = 0$  and  $y_1 = b$ , becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

Slope-Intercept Form of the Equation of a Line An equation of the line with slope m and y-intercept b is

v = mx + b

In particular, if a line is horizontal, its slope is m = 0, so its equation is y = b, where b is the y-intercept (see Figure 10). A vertical line does not have a slope, but we can write its equation as x = a, where a is the x-intercept, because the x-coordinate of every point on the line is a.

When we say that y is a linear function of x, we mean that the graph of the function is a line, so we can use the slope-intercept form to write a formula for the function as

y = f(x) = mx + b

# EXAMPLE 4

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20 °C and the temperature at a height of 1 km is 10 °C, express the temperature T (in  $^{\circ}$ C) as a function of the height h (in kilometers), assuming the function is linear.

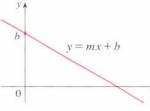
(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

# SOLUTION

(a) Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

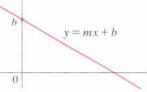




y A

h

0



y = b

X

x = a

a

$$3 - (-1)$$
 2  
with  $x_1 = -1$  and  $y_1 = -1$ 

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APPENDIX B COORDINATE GEOMETRY

We are given that T = 20 when h = 0, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is b = 20.

We are also given that T = 10 when h = 1, so

 $10 = m \cdot 1 + 20$ 

The slope of the line is therefore m = 10 - 20 = -10 and the required linear function is

$$T = -10h + 20$$

(b) The graph is sketched in Figure 11. The slope is m = -10 °C/km, and this represents the rate of change of temperature with respect to height.

(c) At a height of h = 2.5 km, the temperature is

$$T = -10(2.5) + 20 = -5$$
 °C

**EXAMPLE 5** Graph the inequality x + 2y > 5.

**SOLUTION** We are asked to sketch the graph of the set  $\{(x, y) | x + 2y > 5\}$ , and we begin by solving the inequality for y:

x

$$+ 2y > 5$$
$$2y > -x + 5$$
$$y > -\frac{1}{2}x + \frac{5}{4}$$

Compare this inequality with the equation  $y = -\frac{1}{2}x + \frac{5}{2}$ , which represents a line with slope  $-\frac{1}{2}$  and y-intercept  $\frac{5}{2}$ . We see that the given graph consists of points whose y-coordinates are *larger* than those on the line  $y = -\frac{1}{2}x + \frac{5}{2}$ . Thus, the graph is the region that lies *above* the line, as illustrated in Figure 12.

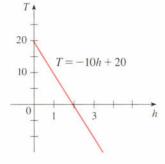
# Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus, Third Edition* by Stewart, Redlin, and Watson (Pacific Grove, CA: Brooks/Cole Publishing Co., 1998).

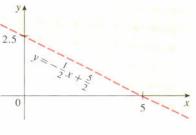
# **Parallel and Perpendicular Lines**

- 1. Two nonvertical lines are parallel if and only if they have the same slope.
- **2.** Two lines with slopes  $m_1$  and  $m_2$  are perpendicular if and only if  $m_1m_2 = -1$ , that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$









**EXAMPLE 6** Find an equation of the line through the point (5, 2) that is parallel to the line 4x + 6y + 5 = 0.

SOLUTION The given line can be written in the form

$$y = -\frac{2}{3}x - \frac{5}{6}$$

which is in slope-intercept form with  $m = -\frac{2}{3}$ . Parallel lines have the same slope, so the required line has slope  $-\frac{2}{3}$  and its equation in point-slope form is

$$y - 2 = -\frac{2}{3}(x - 5)$$

We can write this equation as 2x + 3y = 16.

**EXAMPLE 7** Show that the lines 2x + 3y = 1 and 6x - 4y - 1 = 0 are perpendicular.

**SOLUTION** The equations can be written as

$$y = -\frac{2}{3}x + \frac{1}{3}$$
 and  $y = \frac{3}{2}x - \frac{1}{4}$ 

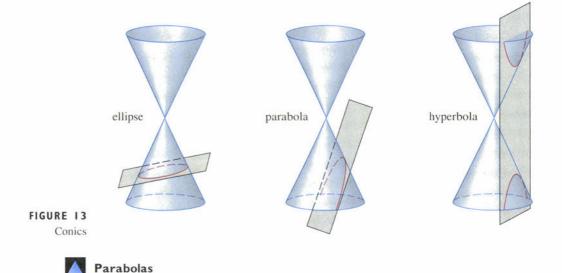
from which we see that the slopes are

$$m_1 = -\frac{2}{3}$$
 and  $m_2 = \frac{3}{2}$ 

Since  $m_1m_2 = -1$ , the lines are perpendicular.

# **Conic Sections**

Here we review the geometric definitions of parabolas, ellipses, and hyperbolas and their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 13.



A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illus-

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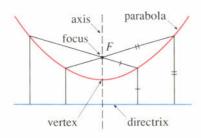




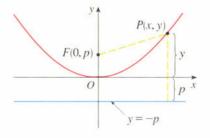
FIGURE 16

trated by Figure 14. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 14 on page 265 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 15. If the focus is the point (0, p), then the directrix has the equation y = -p and the parabola has the equation

$$x^2 = 4py$$



## FIGURE 15

 $= 2 x^{2}$ 

 $y = \frac{1}{2} x^2$ 

 $= -2 x^{2}$ 

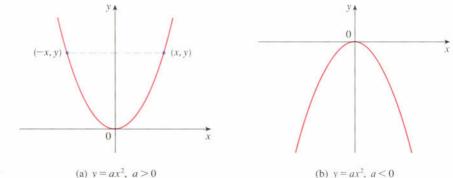
(See Exercise 53.)

If we write a = 1/(4p), then the equation of the parabola becomes



Figure 16 shows the graphs of several parabolas with equations of the form  $y = ax^2$  for various values of the number a. We see that the parabola  $y = ax^2$ opens upward if a > 0 and downward if a < 0 (as in Figure 17). The graph is symmetric with respect to the y-axis because its equation is unchanged when x is replaced by -x. This corresponds to the fact that the function  $f(x) = ax^2$  is an even function.

If we interchange x and y in the equation  $y = ax^2$ , the result is  $x = ay^2$ , which also represents a parabola. (Interchanging x and y amounts to reflecting about the



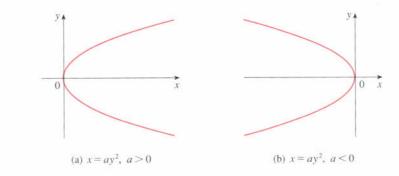




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diagonal line y = x.) The parabola  $x = ay^2$  opens to the right if a > 0 and to the left if a < 0 (see Figure 18). This time the parabola is symmetric with respect to the x-axis because the equation is unchanged when y is replaced by -y.



**EXAMPLE 8** Sketch the region bounded by the parabola  $x = 1 - y^2$  and the line x + y + 1 = 0.

**SOLUTION** First we find the points of intersection by solving the two equations. Substituting x = -y - 1 into the equation  $x = 1 - y^2$ , we get  $-y - 1 = 1 - y^2$ , which gives

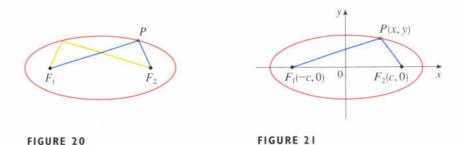
$$0 = y^{2} - y - 2 = (y - 2)(y + 1)$$

so y = 2 or -1. Thus, the points of intersection are (-3, 2) and (0, -1), and we draw the line x + y + 1 = 0 passing through these points.

To sketch the parabola  $x = 1 - y^2$  we start with the parabola  $x = -y^2$ in Figure 18(b) and shift one unit to the right. We also make sure it passes through the points (-3, 2) and (0, -1). The region bounded by  $x = 1 - y^2$  and x + y + 1 = 0 means the finite region whose boundaries are these curves. It is sketched in Figure 19.

# Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points  $F_1$  and  $F_2$  is a constant (see Figure 20). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the Sun at one focus.



In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points (-c, 0) and (c, 0), as in Figure 21, so that the origin is halfway

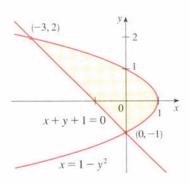
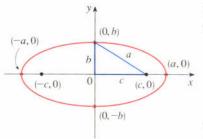
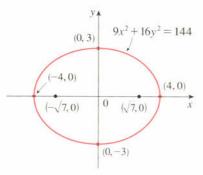


FIGURE 18

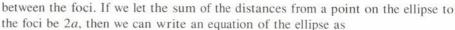
FIGURE 19











$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where  $c^2 = a^2 - b^2$ . (See Exercise 55 and Figure 22.) Notice that the *x*-intercepts are  $\pm a$ , the *y*-intercepts are  $\pm b$ , the foci are  $(\pm c, 0)$ , and the ellipse is symmetric with respect to both axes. If the foci of an ellipse are located on the *y*-axis at  $(0, \pm c)$ , then we can find its equation by interchanging *x* and *y* in (1).

**EXAMPLE 9** Sketch the graph of  $9x^2 + 16y^2 = 144$  and locate the foci.

SOLUTION Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have  $a^2 = 16$ ,  $b^2 = 9$ , a = 4, and b = 3. The x-intercepts are  $\pm 4$  and the y-intercepts are  $\pm 3$ . Also,  $c^2 = a^2 - b^2 = 7$ , so  $c = \sqrt{7}$  and the foci are  $(\pm\sqrt{7}, 0)$ . The graph is sketched in Figure 23.

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 61). This principle is used in *lithotripsy*, a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is positioned at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

# Hyperbolas

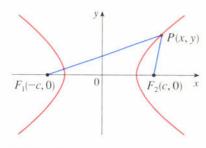
A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points  $F_1$  and  $F_2$  (the foci) is a constant. This definition is illustrated in Figure 24.

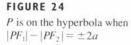
Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. It is left as Exercise 57 to show that when the foci are on the *x*-axis at  $(\pm c, 0)$  and the difference of distances is  $|PF_1| - |PF_2| = \pm 2a$ , then the equation of the hyperbola is

2

 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 

where  $c^2 = a^2 + b^2$ . Notice that the x-intercepts are again  $\pm a$ , but if we put x = 0 in Equation 2 we get  $y^2 = -b^2$ , which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.





To analyze the hyperbola further, we look at Equation 2 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \ge 1$$

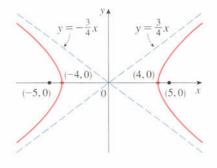
This shows that  $x^2 \ge a^2$ , so  $|x| = \sqrt{x^2} \ge a$ . Therefore, we have  $x \ge a$  or  $x \le -a$ . This means that the hyperbola consists of two parts, called its *branches*. When we draw a hyperbola it is useful to first draw its *asymptotes*, which are the lines y = (b/a)x and y = -(b/a)x shown in Figure 25. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. If the foci of a hyperbola are on the y-axis, we find its equation by reversing the roles of x and y.

**EXAMPLE 10** Find the foci and asymptotes of the hyperbola  $9x^2 - 16y^2 = 144$  and sketch its graph.

SOLUTION If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in (2) with a = 4 and b = 3. Since  $c^2 = 16 + 9 = 25$ , the foci are ( $\pm 5, 0$ ). The asymptotes are the lines  $y = \frac{3}{4}x$  and  $y = -\frac{3}{4}x$ . The graph is shown in Figure 26.







**1−2** ■ Find the distance between the given points.

Exercises ·

**1.** (1, 1), (4, 5) **2.** (1, -3), (5, 7)

**3-4** Find the slope of the line through P and Q.

**3.** P(-3,3), Q(-1,-6) **4.** P(-1,-4), Q(6,0)

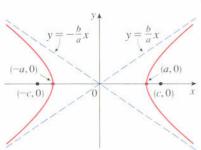
- 5. Show that the points (-2, 9), (4, 6), (1, 0), and (-5, 3) are the vertices of a square.
- 6. (a) Show that the points A(-1, 3), B(3, 11), and C(5, 15) are collinear (lie on the same line) by showing that |AB| + |BC| = |AC|.
  - (b) Use slopes to show that A, B, and C are collinear.

**7–10** ■ Sketch the graph of the equation.

**7.** x = 3 **8.** y = -2 **9.** xy = 0**10.** |y| = 1

**11–24** ■ Find an equation of the line that satisfies the given conditions.

- **II.** Through (2, -3), slope 6
- **12.** Through (-3, -5), slope  $-\frac{7}{2}$
- **13.** Through (2, 1) and (1, 6)
- **14.** Through (-1, -2) and (4, 3)





- **15.** Slope 3, y-intercept -2
- 16. Slope  $\frac{2}{5}$ , y-intercept 4
- **17.** *x*-intercept 1, *y*-intercept -3
- 18. x-intercept -8, y-intercept 6
- 19. Through (4, 5), parallel to the x-axis
- 20. Through (4, 5), parallel to the y-axis
- **21.** Through (1, -6), parallel to the line x + 2y = 6
- **22.** y-intercept 6, parallel to the line 2x + 3y + 4 = 0
- **23.** Through (-1, -2), perpendicular to the line 2x + 5y + 8 = 0
- **24.** Through  $(\frac{1}{2}, -\frac{2}{3})$ , perpendicular to the line 4x 8y = 1

**25–28** Find the slope and y-intercept of the line and draw its graph.

**25.** x + 3y = 0 **26.** 2x - 3y + 6 = 0

**27.** 
$$3x - 4y = 12$$
 **28.**  $4x + 5y = 10$ 

**29–36** Sketch the given region in the *xy*-plane.

- **29.**  $\{(x, y) | x < 0\}$  **30.**  $\{(x, y) | x \ge 1 \text{ and } y < 3\}$  **31.**  $\{(x, y) | |x| \le 2\}$  **32.**  $\{(x, y) | |x| < 3 \text{ and } |y| < 2\}$  **33.**  $\{(x, y) | 0 \le y \le 4 \text{ and } x \le 2\}$ **34.**  $\{(x, y) | y > 2x - 1\}$
- **35.**  $\{(x, y) | 1 + x \le y \le 1 2x\}$
- **36.**  $\{(x, y) \mid -x \le y < \frac{1}{2}(x + 3)\}$

**37–38** Find an equation of a circle that satisfies the given conditions.

**37.** Center (3, -1); radius 5

**38.** Center (-1, 5); passes through (-4, -6)

**39–40** ■ Show that the equation represents a circle and find the center and radius.

**39.**  $x^2 + y^2 - 4x + 10y + 13 = 0$ 

- **40.**  $x^2 + y^2 + 6y + 2 = 0$
- **41.** Show that the lines 2x y = 4 and 6x 2y = 10 are not parallel and find their point of intersection.
- **42.** Show that the lines 3x 5y + 19 = 0 and 10x + 6y 50 = 0 are perpendicular and find their point of intersection.

**43.** (a) Show that the midpoint of the line segment from  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$  is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

- (b) Find the midpoint of the line segment joining the points (1, 3) and (7, 15).
- **44.** Find an equation of the perpendicular bisector of the line segment joining the points A(1, 4) and B(7, -2).
- **45.** (a) Show that if the *x* and *y*-intercepts of a line are nonzero numbers *a* and *b*, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.

- (b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.
- **46.** The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the flea market, then the number y of spaces he can rent is given by the equation y = 200 4x.
  - (a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
  - (b) What do the slope, the *y*-intercept, and the *x*-intercept of the graph represent?
- **47.** The relationship between the Fahrenheit (*F*) and Celsius (*C*) temperature scales is given by the linear function  $F = \frac{9}{5}C + 32$ .
  - (a) Sketch a graph of this function.
  - (b) What is the slope of the graph and what does it represent? What is the *F*-intercept and what does it represent?
- **48.** Jason and Debbie leave Detroit at 2:00 P.M. and drive at a constant speed west along I-90. They pass Ann Arbor, 40 mi from Detroit, at 2:50 P.M.
  - (a) Express the distance traveled in terms of the time elapsed.
  - (b) Draw the graph of the equation in part (a).
  - (c) What is the slope of this line? What does it represent?
- **49.** Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 120 chirps per minute at 70 °F and 168 chirps per minute at 80 °F.
  - (a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.
  - (b) What is the slope of the graph? What does it represent?

- (c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.
- **50.** The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
  - (a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
  - (b) What is the slope of the graph and what does it represent?
  - (c) What is the *y*-intercept of the graph and what does it represent?
- 51. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in<sup>2</sup>. Below the surface, the water pressure increases by 4.34 lb/in<sup>2</sup> for every 10 ft of descent.
  - (a) Express the water pressure as a function of the depth below the ocean surface.
  - (b) At what depth is the pressure  $100 \text{ lb/in}^2$ ?
- 52. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
  - (a) Express the monthly cost *C* as a function of the distance driven *d*, assuming that a linear relationship gives a suitable model.
  - (b) Use part (a) to predict the cost of driving 1500 miles per month.
  - (c) Draw the graph of the linear function. What does the slope represent?
  - (d) What does the y-intercept represent?
  - (e) Why does a linear function give a suitable model in this situation?
- **53.** Suppose that P(x, y) is any point on the parabola with focus (0, p) and directrix y = -p. (See Figure 15.) Use the definition of a parabola to show that  $x^2 = 4py$ .

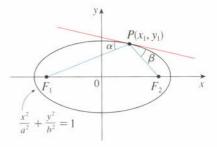
Trigonometry

- 54. Find the focus and directrix of the parabola  $y = x^2$ . Illustrate with a diagram.
- 55. Suppose an ellipse has foci (±c, 0) and the sum of the distances from any point P(x, y) on the ellipse to the foci is 2a. Show that the coordinates of P satisfy Equation 1.
- 56. Find the foci of the ellipse  $x^2 + 4y^2 = 4$  and sketch its graph.
- **57.** Use the definition of a hyperbola to derive Equation 2 for a hyperbola with foci  $(\pm c, 0)$ .
- **58.** (a) Find the foci and asymptotes of the hyperbola  $x^2 y^2 = 1$  and sketch its graph.
  - (b) Sketch the graph of  $y^2 x^2 = 1$ .
- **59–60** Sketch the region bounded by the curves.

**59.** x + 4y = 8 and  $x = 2y^2 - 8$ 

**60.**  $y = 4 - x^2$  and x - 2y = 2

**61.** Let  $P(x_1, y_1)$  be a point on the ellipse  $x^2/a^2 + y^2/b^2 = 1$  with foci  $F_1$  and  $F_2$  and let  $\alpha$  and  $\beta$  be the angles between the lines  $PF_1$ ,  $PF_2$  and the ellipse as in the figure. Prove that  $\alpha = \beta$ . This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula in Problem 13 on page 265 to show that tan  $\alpha = \tan \beta$ .]





Here we review the aspects of trigonometry that are used in calculus: radian measure, trigonometric functions, trigonometric identities, and inverse trigonometric functions.

. . . . . . . . .

# Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains  $360^\circ$ , which is the same as  $2\pi$  rad. Therefore

П

$$\pi$$
 rad = 180°

and

2 1 rad = 
$$\left(\frac{180}{\pi}\right)^{\circ} \approx 57.3^{\circ}$$
 1° =  $\frac{\pi}{180}$  rad  $\approx 0.017$  rad

## EXAMPLE I

(a) Find the radian measure of 60°.

(b) Express  $5\pi/4$  rad in degrees.

# SOLUTION

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by  $\pi/180$ . Therefore

$$60^\circ = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by  $180/\pi$ . Thus

$$\frac{5\pi}{4} \operatorname{rad} = \frac{5\pi}{4} \left( \frac{180}{\pi} \right) = 225^{\circ}$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	$0^{\circ}$	30°	45°	$60^{\circ}$	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\pi$	$\pi$	π	$\pi$	$2\pi$	3π	$5\pi$	π	3π	$2\pi$
100000000000000000000000000000000000000		6	4	3	2	3	4	6		2	

Figure 1 shows a sector of a circle with central angle  $\theta$  and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference  $2\pi r$  and central angle  $2\pi$ , we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi i}$$

Solving this equation for  $\theta$  and for a, we obtain



Remember that Equations 3 are valid only when  $\theta$  is measured in radians.

In particular, putting a = r in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

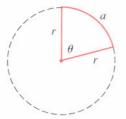
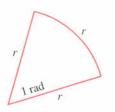


FIGURE I





# EXAMPLE 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm? (b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of  $3\pi/8$  rad?

### SOLUTION

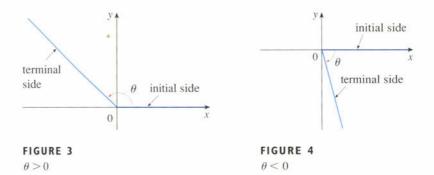
(a) Using Equation 3 with a = 6 and r = 5, we see that the angle is

$$\theta = \frac{6}{5} = 1.2$$
 rad

(b) With r = 3 cm and  $\theta = 3\pi/8$  rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8}$$
 cm

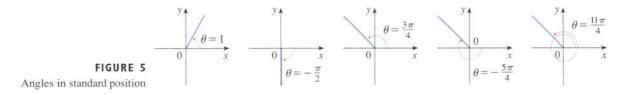
The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive *x*-axis as in Figure 3.



A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4. Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles  $3\pi/4$ ,  $-5\pi/4$ , and  $11\pi/4$  have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and  $2\pi$  rad represents a complete revolution.



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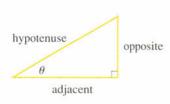


FIGURE 6

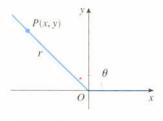


FIGURE 7

# The Trigonometric Functions

4

5

For an acute angle  $\theta$  the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

$\sin \theta = \frac{\text{opp}}{\text{hyp}}$	$\csc\theta = \frac{\mathrm{hyp}}{\mathrm{opp}}$
$\cos\theta = \frac{\mathrm{adj}}{\mathrm{hyp}}$	$\sec \theta = \frac{\text{hyp}}{\text{adj}}$
$\tan \theta = \frac{\mathrm{opp}}{\mathrm{adj}}$	$\cot \theta = \frac{\mathrm{adj}}{\mathrm{opp}}$

This definition does not apply to obtuse or negative angles, so for a general angle  $\theta$  in standard position we let P(x, y) be any point on the terminal side of  $\theta$  and we let *r* be the distance |OP| as in Figure 7. Then we define

$\sin\theta = \frac{y}{r}$	$\csc \theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$	$\sec \theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$	$\cot \theta = \frac{x}{y}$

Since division by 0 is not defined,  $\tan \theta$  and  $\sec \theta$  are undefined when x = 0 and  $\csc \theta$  and  $\cot \theta$  are undefined when y = 0. Notice that the definitions in (4) and (5) are consistent when  $\theta$  is an acute angle.

If  $\theta$  is a number, the convention is that  $\sin \theta$  means the sine of the angle whose *radian* measure is  $\theta$ . For example, the expression  $\sin 3$  implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle  $3^{\circ}$  we would write sin  $3^{\circ}$  and, with our calculator in degree mode, we find that

$$\sin 3^\circ \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 8. For instance,

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin \frac{\pi}{6} = \frac{1}{2} \qquad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos \frac{\pi}{3} = \frac{1}{2}$$
$$\tan \frac{\pi}{4} = 1 \qquad \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan \frac{\pi}{3} = \sqrt{3}$$

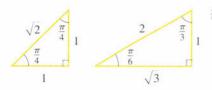
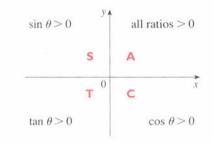


FIGURE 8

#### APPENDIX C TRIGONOMETRY A23

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" shown in Figure 9.



#### FIGURE 9



**SOLUTION** From Figure 10 we see that a point on the terminal side for  $\theta = 2\pi/3$  is  $P(-1,\sqrt{3})$ . Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos \frac{2\pi}{3} = -\frac{1}{2} \qquad \tan \frac{2\pi}{3} = -\sqrt{3}$$
$$\csc \frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec \frac{2\pi}{3} = -2 \qquad \cot \frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

The table gives some values of  $\sin \theta$  and  $\cos \theta$  found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	$2\pi$
sin $ heta$	Ö	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	ΞĪ.	0
cosθ	I	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	L

**EXAMPLE 4** If  $\cos \theta = \frac{2}{5}$  and  $0 < \theta < \pi/2$ , find the other five trigonometric functions of  $\theta$ .

**SOLUTION** Since  $\cos \theta = \frac{2}{5}$ , we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 11. If the opposite side has length x, then the Pythagorean Theorem gives  $x^2 + 4 = 25$  and so  $x^2 = 21$ , and  $x = \sqrt{21}$ . We can now use the diagram to write the other five trigonometric functions:

$$\sin \theta = \frac{\sqrt{21}}{5} \qquad \tan \theta = \frac{\sqrt{21}}{2}$$
$$\csc \theta = \frac{5}{\sqrt{21}} \qquad \sec \theta = \frac{5}{2} \qquad \cot \theta = \frac{2}{\sqrt{21}}$$

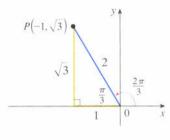
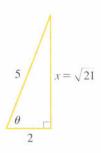


FIGURE 10





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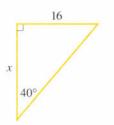


FIGURE 12

**EXAMPLE 5** Use a calculator to approximate the value of x in Figure 12.

**SOLUTION** From the diagram we see that

6

$$x = \frac{16}{\tan 40^\circ} \approx 19.07$$

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

 $\tan 40^\circ = \frac{16}{2}$ 

 $\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$  $\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$ 

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that  $x^2 + y^2 = r^2$ . Therefore

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1$$

If we now divide both sides of Equation 7 by  $\cos^2\theta$  and use Equations 6, we get

$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if we divide both sides of Equation 7 by  $\sin^2\theta$ , we get

$$1 + \cot^2 \theta = \csc^2 \theta$$

The identities

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

Odd functions and even functions are discussed in Section 1.1.

show that sin is an odd function and cos is an even function. They are easily proved by drawing a diagram showing  $\theta$  and  $-\theta$  in standard position (see Exercise 19). Since the angles  $\theta$  and  $\theta + 2\pi$  have the same terminal side, we have

$$\sin(\theta + 2\pi) = \sin \theta$$
  $\cos(\theta + 2\pi) = \cos \theta$ 

These identities show that the sine and cosine functions are periodic with period  $2\pi$ .

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

12a	$\sin(x + y) = \sin x \cos y + \cos x \sin y$
126	$\cos(x + y) = \cos x \cos y - \sin x \sin y$

The proofs of these addition formulas are outlined in Exercises 51, 52, and 53.

By substituting -y for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following subtraction formulas:

13a	$\sin(x - y) = \sin x \cos y - \cos x \sin y$
136	$\cos(x - y) = \cos x \cos y + \sin x \sin y$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for  $tan(x \pm y)$ :

Ida
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
Ida $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$ 

If we put y = x in the addition formulas (12), we get the **double-angle formulas**:

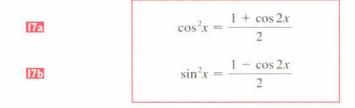
$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity  $\sin^2 x + \cos^2 x = 1$ , we obtain the following alternate forms of the double-angle formulas for  $\cos 2x$ :



If we now solve these equations for  $\cos^2 x$  and  $\sin^2 x$ , we get the following halfangle formulas, which are useful in integral calculus:



There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of them, remember that they can all be deduced from Equations 12a and 12b.

**EXAMPLE 6** Find all values of x in the interval  $[0, 2\pi]$  such that

$$\sin x = \sin 2x$$

SOLUTION Using the double-angle formula (15a), we rewrite the given equation as

 $\sin x = 2 \sin x \cos x$  or  $\sin x (1 - 2 \cos x) = 0$ 

Therefore, there are two possibilities:

$$\sin x = 0$$
 or  $1 - 2\cos x = 0$   
 $x = 0, \pi, 2\pi$   $\cos x = \frac{1}{2}$   
 $x = \frac{\pi}{3}, \frac{5\pi}{3}$ 

The given equation has five solutions: 0,  $\pi/3$ ,  $\pi$ ,  $5\pi/3$ , and  $2\pi$ .

# Graphs of the Trigonometric Functions

The graph of the function  $f(x) = \sin x$ , shown in Figure 13(a), is obtained by plotting points for  $0 \le x \le 2\pi$  and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the integer multiples of  $\pi$ , that is,



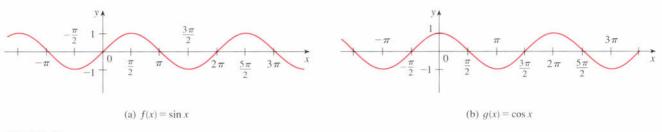


FIGURE 13

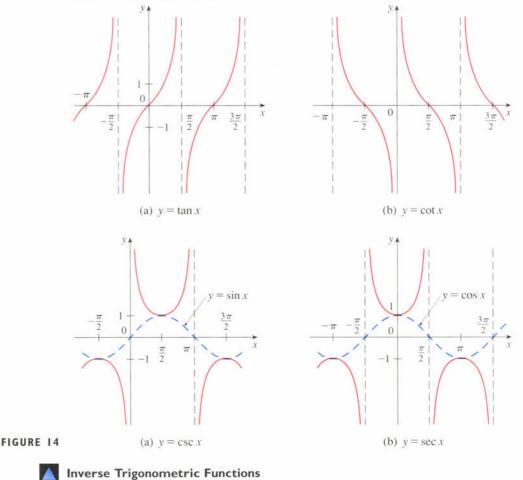
Because of the identity

$$\cos x = \sin \left( x + \frac{\pi}{2} \right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount  $\pi/2$  to the left [see Figure 13(b)]. Note that for both the sine and cosine functions the domain is  $(-\infty, \infty)$  and the range is the closed interval [-1, 1]. Thus, for all values of *x*, we have

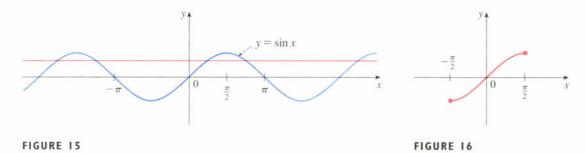
$$-1 \le \sin x \le 1$$
  $-1 \le \cos x \le 1$ 

The graphs of the remaining four trigonometric functions are shown in Figure 14 and their domains are indicated there. Notice that tangent and cotangent have range  $(-\infty, \infty)$ , whereas cosecant and secant have range  $(-\infty, -1] \cup [1, \infty)$ . All four functions are periodic: tangent and cotangent have period  $\pi$ , whereas cosecant and secant have period  $2\pi$ .



Inverse functions are reviewed in Section 1.6.

When we try to find the inverse functions of the trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one. You can see from Figure 15 that the sine function  $y = \sin x$  is not one-to-one (use the Horizontal Line Test). But the function  $f(x) = \sin x$ ,  $-\pi/2 \le x \le \pi/2$  (see Figure 16), *is* one-to-one. The inverse function of this restricted sine function f exists and is denoted by  $\sin^{-1}$  or arcsin. It is called the **inverse sine function** or the **arcsine function**.



Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\sin^{-1}x = y \iff \sin y = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

 $\bigotimes \sin^{-1} x \neq \frac{1}{\sin x}$ 

Thus, if  $-1 \le x \le 1$ ,  $\sin^{-1}x$  is the number between  $-\pi/2$  and  $\pi/2$  whose sine is x.

# **EXAMPLE 7** Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan(\arcsin \frac{1}{3})$ .

## SOLUTION

(a) We have

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

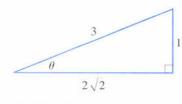
because  $\sin(\pi/6) = \frac{1}{2}$  and  $\pi/6$  lies between  $-\pi/2$  and  $\pi/2$ .

(b) Let  $\theta = \arcsin \frac{1}{3}$ . Then we can draw a right triangle with angle  $\theta$  as in Figure 17 and deduce from the Pythagorean Theorem that the third side has length  $\sqrt{9-1} = 2\sqrt{2}$ . This enables us to read from the triangle that

$$\tan(\arcsin\frac{1}{3}) = \tan\theta = \frac{1}{2\sqrt{2}}$$

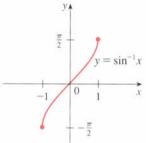
The cancellation equations for inverse functions [see (4) in Section 1.6] become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \le x \le 1$$





#### APPENDIX C TRIGONOMETRY = A29



its graph, shown in Figure 18, is obtained from that of the restricted sine function (Figure 16) by reflection about the line y = x. The tangent function can be made one-to-one by restricting it to the interval

 $(-\pi/2,\pi/2)$ . Thus, the **inverse tangent function** is defined as the inverse of the function  $f(x) = \tan x, -\pi/2 < x < \pi/2$  (see Figure 19). It is denoted by  $\tan^{-1}$  or arctan.

The inverse sine function,  $\sin^{-1}$ , has domain [-1, 1] and range  $[-\pi/2, \pi/2]$ , and

$$\tan^{-1}x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

0

 $\frac{\pi}{2}$ 

FIGURE 18

**FIGURE 19**  $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ 

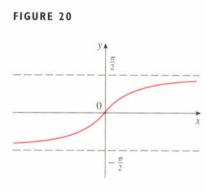
**EXAMPLE 8** Simplify the expression  $\cos(\tan^{-1}x)$ .

 $-\frac{\pi}{2}$ 

**SOLUTION 1** Let  $y = \tan^{-1}x$ . Then  $\tan y = x$  and  $-\pi/2 < y < \pi/2$ . We want to find  $\cos y$  but, since  $\tan y$  is known, it is easier to find  $\sec y$  first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$
  
$$\sec y = \sqrt{1 + x^2} \quad (\text{since sec } y > 0 \text{ for } -\pi/2 < y < \pi/2)$$
  
$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

Thus

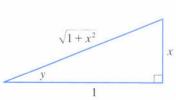


**FIGURE 2 I**  $y = \tan^{-1}x = \arctan x$ 

**SOLUTION 2** Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If  $y = \tan^{-1}x$ , then  $\tan y = x$ , and we can read from Figure 20 (which illustrates the case y > 0) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

The inverse tangent function,  $\tan^{-1} = \arctan$ , has domain  $\mathbb{R}$  and range  $(-\pi/2, \pi/2)$ . Its graph is shown in Figure 21. We know that the lines  $x = \pm \pi/2$  are vertical asymptotes of the graph of tan. Since the graph of  $\tan^{-1}$  is obtained by reflecting the graph of the restricted tangent function about the line y = x, it follows that the lines  $y = \pi/2$  and  $y = -\pi/2$  are horizontal asymptotes of the graph of  $\tan^{-1}$ .



Of the six inverse trigonometric functions, arcsin and arctan are the ones that are most useful for the purposes of calculus. The inverse cosine function is investigated in Exercise 46. The remaining inverse trigonometric functions don't arise as frequently.

# C Exercises · · · · ·

- 1-2 Convert from degrees to radians.
- **I.** (a) 210° (b) 9°
- **2.** (a)  $-315^{\circ}$  (b)  $36^{\circ}$
- 3-4 Convert from radians to degrees.
- **3.** (a)  $4\pi$
- (b)  $-\frac{3\pi}{8}$
- **4.** (a)  $-\frac{7\pi}{2}$  (b)  $\frac{8\pi}{3}$
- 5. Find the length of a circular arc subtended by an angle of  $\pi/12$  rad if the radius of the circle is 36 cm.
- **6.** If a circle has radius 10 cm, find the length of the arc subtended by a central angle of 72°.
- **7.** A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?
- 8. Find the radius of a circular sector with angle  $3\pi/4$  and arc length 6 cm.

• Draw, in standard position, the angle whose measure is given.

9. (a)  $315^{\circ}$  (b)  $-\frac{3\pi}{4}$  rad 10. (a)  $\frac{7\pi}{3}$  rad (b) -3 rad

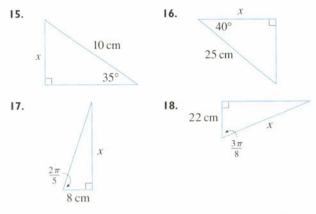
ti-12 ■ Find the exact trigonometric ratios for the angle whose radian measure is given.

11.  $\frac{3\pi}{4}$  12.  $\frac{4\pi}{3}$ 

13–14 ■ Find the remaining trigonometric ratios.

**13.** 
$$\sin \theta = \frac{3}{5}, \quad 0 < \theta < \frac{\pi}{2}$$
  
**14.**  $\tan \alpha = 2, \quad 0 < \alpha < \frac{\pi}{2}$ 

**15–18** Find, correct to five decimal places, the length of the side labeled x.



**19–20** ■ Prove the equation.

**19.** (a) Equation 10a (b) Equation 10b

**20.** (a) Equation 14a (b) Equation 14b

8 8 8 8 8 8 9 9 9 9 8 8 8

21–26 ■ Prove the identity.

**21.** 
$$\sin\left(\frac{\pi}{2} + x\right) = \cos x$$
 **22.**  $\sin(\pi - x) = \sin x$ 

**23.**  $\sin\theta\cot\theta = \cos\theta$ 

**24.**  $(\sin x + \cos x)^2 = 1 + \sin 2x$ 

**25.** 
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

**26.**  $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ 

**27–28** If  $\sin x = \frac{1}{3}$  and  $\sec y = \frac{5}{4}$ , where x and y lie between 0 and  $\pi/2$ , evaluate the expression.

**27.** sin(x + y) **28.** cos 2y

**29–32** • Find all values of x in the interval  $[0, 2\pi]$  that satisfy the equation.

<b>29.</b> $2\cos x - 1 = 0$	<b>30.</b> $2\sin^2 x = 1$
<b>31.</b> $\sin 2x = \cos x$	<b>32.</b> $ \tan x  = 1$

**33–36** Find all values of x in the interval  $[0, 2\pi]$  that satisfy the inequality.

**33.**  $\sin x \le \frac{1}{2}$  **34.**  $2\cos x + 1 > 0$ 
**35.**  $-1 < \tan x < 1$  **36.**  $\sin x > \cos x$ 

**37–40** ■ Graph the function by starting with the graphs in Figures 13 and 14 and applying the transformations of Section 1.2 where appropriate.

**37.** 
$$y = \cos\left(x - \frac{\pi}{3}\right)$$
  
**38.**  $y = \tan 2x$   
**39.**  $y = \frac{1}{3} \tan\left(x - \frac{\pi}{2}\right)$   
**40.**  $y = |\sin x|$ 

**41**−**44** ■ Find the exact value of each expression.

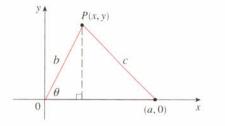
- **41.** (a)  $\sin^{-1}(0.5)$  (b)  $\arctan(-1)$
- **42.** (a)  $\tan^{-1}\sqrt{3}$  (b)  $\arcsin 1$
- **43.** (a)  $\sin(\sin^{-1}(0.7))$  (b)  $\arcsin\left(\sin\frac{5\pi}{4}\right)$
- **44.** (a)  $\sec(\arctan 2)$  (b)  $\sin(2\sin^{-1}(\frac{3}{5}))$
- **45.** Prove that  $\cos(\sin^{-1}x) = \sqrt{1 x^2}$ .
- 46. The inverse cosine function, cos<sup>-1</sup> = arccos, is defined as the inverse of the restricted cosine function
  - $f(x) = \cos x, \, 0 \le x \le \pi.$
  - (a) What are the domain and range of the inverse cosine function?
  - (b) Sketch the graph of arccos.
- 47. Find the domain and range of the function

 $g(x) = \sin^{-1}(3x + 1)$ 

- **48.** (a) Graph the function  $f(x) = \sin(\sin^{-1}x)$  and explain the appearance of the graph.
  - (b) Graph the function  $g(x) = \sin^{-1}(\sin x)$ . How do you explain the appearance of this graph?
  - 49. Prove the Law of Cosines: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

[*Hint*: Introduce a coordinate system so that  $\theta$  is in

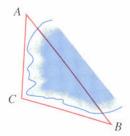


standard position as in the figure. Express *x* and *y* in terms of  $\theta$  and then use the distance formula to compute *c*.]

**50.** In order to find the distance |AB| across a small inlet, a point *C* is located as in the figure and the following measurements are recorded:

 $\angle C = 103^{\circ}$  |AC| = 820 m |BC| = 910 m

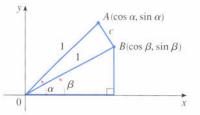
Use the Law of Cosines from Exercise 49 to find the required distance.



51. Use the figure to prove the subtraction formula

 $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ 

[*Hint:* Compute  $c^2$  in two ways (using the Law of Cosines from Exercise 49 and also using the distance formula) and compare the two expressions.]



- **52.** Use the formula in Exercise 51 to prove the addition formula for cosine (12b).
- 53. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \qquad \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$

- to prove the subtraction formula for the sine function.
- 54. (a) Show that the area of a triangle with sides of lengths *a* and *b* and with included angle  $\theta$  is

$$A = \frac{1}{2}ab\sin\theta$$

(b) Find the area of triangle *ABC*, correct to five decimal places, if

|AB| = 10 cm |BC| = 3 cm  $\angle ABC = 107^{\circ}$ 



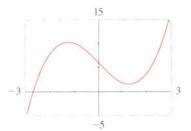
# Precise Definitions of Limits

as

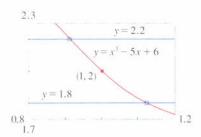
The definitions of limits that have been given in this book are appropriate for intuitive understanding of the basic concepts of calculus. For the purposes of deeper understanding and rigorous proofs, however, the precise definitions of this appendix are necessary. In particular, the definition of a limit given here is used in Appendix E to prove that the limit of a sum is the sum of the limits.

When we say that f(x) has the limit L as x approaches a, we mean, according to the intuitive definition in Section 2.2, that we can make f(x) arbitrarily close to L by taking x close enough to a (but not equal to a). A more precise definition is based on the idea of specifying just how small we need to make the distance |x - a| in order to make the distance |f(x) - L| less than some given number. The following example illustrates the idea.

It is traditional to use the Greek letter  $\delta$  (delta) in this situation.









**EXAMPLE I** Use a graph to find a number  $\delta$  such that

 $|(x^3 - 5x + 6) - 2| < 0.2$  whenever  $|x - 1| < \delta$ 

**SOLUTION** A graph of  $f(x) = x^3 - 5x + 6$  is shown in Figure 1; we are interested in the region near the point (1, 2). Notice that we can rewrite the inequality

$$|(x^{3} - 5x + 6) - 2| < 0.2$$
  
$$|x < x^{3} - 5x + 6 < 2.2$$

So we need to determine the values of x for which the curve  $y = x^3 - 5x + 6$ lies between the horizontal lines y = 1.8 and y = 2.2. Therefore, we graph the curves  $y = x^3 - 5x + 6$ , y = 1.8, and y = 2.2 near the point (1,2) in Figure 2. Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line y = 2.2 and the curve  $y = x^3 - 5x + 6$  is about 0.911. Similarly,  $y = x^3 - 5x + 6$  intersects the line y = 1.8 when  $x \approx 1.124$ . So, rounding to be safe, we can say that

 $1.8 < x^3 - 5x + 6 < 2.2$  whenever 0.92 < x < 1.12

The interval (0.92, 1.12) is not symmetric about x = 1. The distance from x = 1 to the left endpoint is 1 - 0.92 = 0.08 and the distance to the right endpoint is 0.12. We can choose  $\delta$  to be the smaller of these numbers, that is,  $\delta = 0.08$ . Then we can rewrite our inequalities in terms of distances as follows:

 $|(x^3 - 5x + 6) - 2| < 0.2$  whenever |x - 1| < 0.08

This just says that by keeping x within 0.08 of 1, we are able to keep f(x) within 0.2 of 2.

Although we chose  $\delta = 0.08$ , any smaller positive value of  $\delta$  would also have worked.

Using the same graphical procedure as in Example 1, but replacing the number 0.2 by smaller numbers, we find that

$$|(x^{3} - 5x + 6) - 2| < 0.1 \quad \text{whenever} \quad |x - 1| < 0.046$$
  
$$|(x^{3} - 5x + 6) - 2| < 0.05 \quad \text{whenever} \quad |x - 1| < 0.024$$
  
$$|(x^{3} - 5x + 6) - 2| < 0.01 \quad \text{whenever} \quad |x - 1| < 0.004$$

In each case we have found a number  $\delta$  such that the values of the function  $f(x) = x^3 - 5x + 6$  lie in successively smaller intervals centered at 2 if the distance from x to 1 is less than  $\delta$ . It turns out that it is always possible to find such a number  $\delta$ , no matter how small the interval is. In other words, for *any* positive number  $\varepsilon$ , no matter how small, there exists a positive number  $\delta$  such that

$$|(x^3 - 5x + 6) - 2| < \varepsilon$$
 whenever  $|x - 1| < \delta$ 

This indicates that

The condition 0 < |x - a| is just

another way of saying that  $x \neq a$ .

$$\lim_{x \to 1} (x^3 - 5x + 6) = 2$$

and suggests a more precise way of defining the limit of a general function.

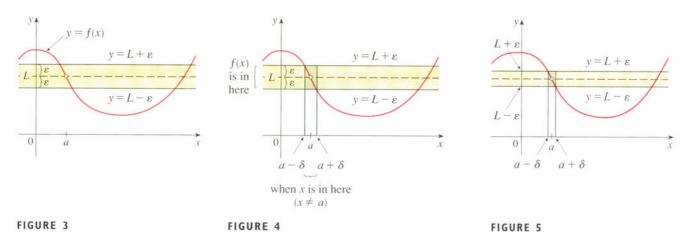
**Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** f(x) as x approaches a is L, and we write

$$\lim_{x \to \infty} f(x) = L$$

if for every number  $\varepsilon > 0$  there is a corresponding number  $\delta > 0$  such that

 $|f(x) - L| < \varepsilon$  whenever  $0 < |x - a| < \delta$ 

Definition 1 is illustrated in Figures 3–5. If  $\varepsilon > 0$  is given, then we draw the horizontal lines  $y = L + \varepsilon$  and  $y = L - \varepsilon$  and the graph of f (see Figure 3). If  $\lim_{x\to a} f(x) = L$ , then we can find a number  $\delta > 0$  such that if we restrict x to lie in the interval  $(a - \delta, a + \delta)$  and take  $x \neq a$ , then the curve y = f(x) lies between the lines  $y = L - \varepsilon$  and  $y = L + \varepsilon$  (see Figure 4). You can see that if such a  $\delta$  has been found, then any smaller  $\delta$  will also work.



It's important to realize that the process illustrated in Figures 3 and 4 must work for *every* positive number  $\varepsilon$  no matter how small it is chosen. Figure 5 shows that if a smaller  $\varepsilon$  is chosen, then a smaller  $\delta$  may be required.

**EXAMPLE 2** Use the  $\varepsilon$ ,  $\delta$  definition to prove that  $\lim x^2 = 0$ .

**SOLUTION** Let  $\varepsilon$  be a given positive number. According to Definition 1 with a = 0 and L = 0, we need to find a number  $\delta$  such that

$$|x^{2} - 0| < \varepsilon$$
 whenever  $0 < |x - 0| < \delta$   
 $x^{2} < \varepsilon$  whenever  $0 < |x| < \delta$ 

But, since the square root function is an increasing function, we know that

$$x^2 < \varepsilon \iff \sqrt{x^2} < \sqrt{\varepsilon} \iff |x| < \sqrt{\varepsilon}$$

So if we choose  $\delta = \sqrt{\varepsilon}$ , then  $x^2 < \varepsilon \iff |x| < \delta$  (see Figure 6). This shows that  $\lim_{x \to 0} x^2 = 0$ .

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number  $\varepsilon$ . Then you must be able to produce a suitable  $\delta$ . You have to be able to do this for *every*  $\varepsilon > 0$ , not just a particular  $\varepsilon$ .

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of f(x) to within a degree of accuracy  $\varepsilon$  (say, 0.01). B then responds by finding a number  $\delta$  such that  $|f(x) - L| < \varepsilon$  whenever  $0 < |x - a| < \delta$ . Then A may become more exacting and challenge B with a smaller value of  $\varepsilon$  (say, 0.0001). Again B has to respond by finding a corresponding  $\delta$ . Usually, the smaller the value of  $\varepsilon$ , the smaller the corresponding value of  $\delta$  must be. If B always wins, no matter how small A makes  $\varepsilon$ , then  $\lim_{x\to a} f(x) = L$ .

**EXAMPLE 3** Prove that  $\lim_{x \to -5} (4x - 5) = 7$ .

#### SOLUTION

I. Preliminary analysis of the problem (guessing a value for  $\delta$ ). Let  $\varepsilon$  be a given positive number. We want to find a number  $\delta$  such that

 $|(4x-5)-7| < \varepsilon$  whenever  $0 < |x-3| < \delta$ 

But |(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|. Therefore, we want

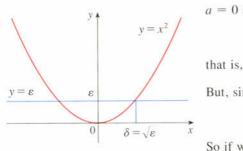
$$4|x-3| < \varepsilon$$
 whenever  $0 < |x-3| < \delta$ 

that is,  $|x-3| < \frac{\varepsilon}{4}$  whenever  $0 < |x-3| < \delta$ 

This suggests that we should choose  $\delta = \varepsilon/4$ .

**2.** Proof (showing that the  $\delta$  works). Given  $\varepsilon > 0$ , choose  $\delta = \varepsilon/4$ . If  $0 < |x - 3| < \delta$ , then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$





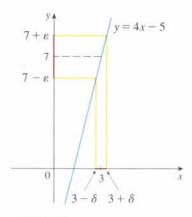


FIGURE 7

Thus

 $|(4x-5)-7| < \varepsilon$  whenever  $0 < |x-3| < \delta$ 

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

This example is illustrated by Figure 7.

Note that in the solution of Example 2 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for  $\delta$ . But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

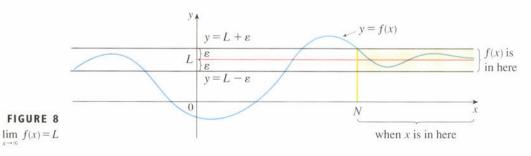
It's not always easy to prove that limit statements are true using the  $\varepsilon$ ,  $\delta$  definition. For a more complicated function such as  $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$ , a proof would require a great deal of ingenuity. Fortunately, this is not necessary because the Limit Laws stated in Section 2.3 can be proved using Definition 1, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

# Limits at Infinity

Infinite limits and limits at infinity can also be defined in a precise way. The following is a precise version of Definition 4 in Section 2.5.

**2 Definition** Let f be a function defined on some interval  $(a, \infty)$ . Then  $\lim_{x \to \infty} f(x) = L$ means that for every  $\varepsilon > 0$  there is a corresponding number N such that  $|f(x) - L| < \varepsilon$  whenever x > N

In words, this says that the values of f(x) can be made arbitrarily close to L (within a distance  $\varepsilon$ , where  $\varepsilon$  is any positive number) by taking x sufficiently large (larger than N, where N depends on  $\varepsilon$ ). Graphically, it says that by choosing x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines  $y = L - \varepsilon$  and  $y = L + \varepsilon$  as in Figure 8. This must be true



no matter how small we choose  $\varepsilon$ . If a smaller value of  $\varepsilon$  is chosen, then a larger value of N may be required.

In Example 5 in Section 2.5 we calculated that

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a graphing device to relate this statement to Definition 2 with  $L = \frac{3}{5}$  and  $\varepsilon = 0.1$ .

**EXAMPLE 4** Use a graph to find a number N such that

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \quad \text{whenever} \quad x > N$$

**SOLUTION** We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

We need to determine the values of x for which the given curve lies between the horizontal lines y = 0.5 and y = 0.7. So we graph the curve and these lines in Figure 9. Then we use the cursor to estimate that the curve crosses the line y = 0.5 when  $x \approx 6.7$ . To the right of this number the curve stays between the lines y = 0.5 and y = 0.7. Rounding to be safe, we can say that

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \quad \text{whenever} \quad x > 7$$

In other words, for  $\varepsilon = 0.1$  we can choose N = 7 (or any larger number) in Definition 2.

**EXAMPLE 5** Use Definition 2 to prove that  $\lim_{x \to \infty} \frac{1}{x} = 0$ .

**SOLUTION** Let  $\varepsilon$  be a given positive number. According to Definition 2, we want to find N such that

$$\left|\frac{1}{x} - 0\right| < \varepsilon$$
 whenever  $x > N$ 

In computing the limit we may assume x > 0, in which case

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = \frac{1}{x}$$

Therefore, we want

$$\frac{1}{x} < \varepsilon$$
 whenever  $x > N$ 

that is, 
$$x > \frac{1}{\varepsilon}$$
 whenever  $x > N$ 

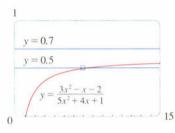
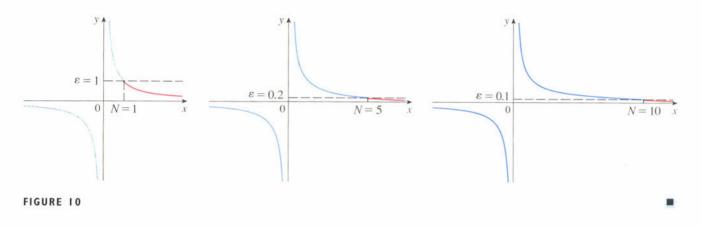


FIGURE 9

So if we choose  $N = 1/\varepsilon$ , then  $1/x < \varepsilon \iff x > N$ . This proves the desired limit. Figure 10 illustrates the proof by showing some values of  $\varepsilon$  and the corresponding values of N.



Infinite limits can also be formulated precisely. See Exercise 16.

Sequences

In Section 8.1 we used the notation

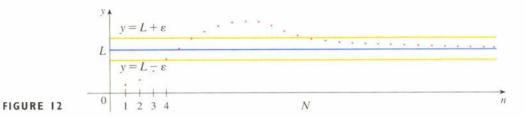
$$\lim_{n\to\infty}a_n=L$$

to mean that the terms of the sequence  $\{a_n\}$  approach *L* as *n* becomes large. Notice that the following precise definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity (Definition 2).

**E Definition** A sequence  $\{a_n\}$  has the **limit** L and we write  $\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$ if for every  $\varepsilon > 0$  there is a corresponding integer N such that  $|a_n - L| < \varepsilon$  whenever n > N

Definition 3 is illustrated by Figure 11, in which the terms  $a_1, a_2, a_3, \ldots$  are plotted on a number line. No matter how small an interval  $(L - \varepsilon, L + \varepsilon)$  is chosen, there exists an N such that all terms of the sequence from  $a_{N+1}$  onward must lie in that interval.

Another illustration of Definition 3 is given in Figure 12. The points on the graph of  $\{a_n\}$  must lie between the horizontal lines  $y = L + \varepsilon$  and  $y = L - \varepsilon$  if n > N. This picture must be valid no matter how small  $\varepsilon$  is chosen, but usually a smaller  $\varepsilon$  requires a larger N.



Comparison of Definitions 2 and 3 shows that the only difference between  $\lim_{n\to\infty} a_n = L$  and  $\lim_{x\to\infty} f(x) = L$  is that *n* is required to be an integer. The following definition shows how to make precise the idea that  $a_n$  becomes infinite as *n* becomes infinite.

**4 Definition**  $\lim_{n\to\infty} a_n = \infty$  means that for every positive number *M* there is an integer *N* such that

 $a_n > M$  whenever n > N

**EXAMPLE 6** Prove that  $\lim_{n \to \infty} \sqrt{n} = \infty$ .

SOLUTION Let M be any positive number. (Think of it as being very large.) Then

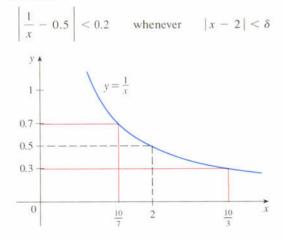
$$\sqrt{n} > M \iff n > M^2$$

So if we take  $N = M^2$ , then Definition 4 shows that  $\lim_{n\to\infty} \sqrt{n} = \infty$ .



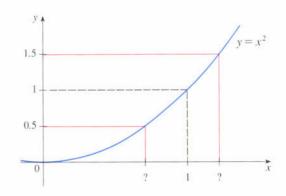
1. Use the given graph of f(x) = 1/x to find a number  $\delta$  such that

Exercises · · ·



2. Use the given graph of  $f(x) = x^2$  to find a number  $\delta$  such that

 $|x^2 - 1| < \frac{1}{2}$  whenever  $|x - 1| < \delta$ 



$$\sqrt{4x+1} - 3 | < 0.5$$
 whenever  $|x-2| < \delta$ 

4. Use a graph to find a number  $\delta$  such that

$$\sin x - \frac{1}{2} \Big| < 0.1$$
 whenever  $\left| x - \frac{\pi}{6} \right| < \delta$ 

.

**5.** For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 1 by finding values of  $\delta$  that correspond to  $\varepsilon = 1$  and  $\varepsilon = 0.1$ .

**6.** For the limit

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

illustrate Definition 1 by finding values of  $\delta$  that correspond to  $\varepsilon = 0.5$  and  $\varepsilon = 0.1$ .

- 7. Use Definition 1 to prove that  $\lim_{x\to 0} x^3 = 0$ .
- 8. (a) How would you formulate an ε, δ definition of the one-sided limit lim<sub>x→a<sup>+</sup></sub> f(x) = L?
  - (b) Use your definition in part (a) to prove that  $\lim_{x\to 0^+} \sqrt{x} = 0.$

**9–10** Prove each statement using the  $\varepsilon$ ,  $\delta$  definition of limit and illustrate with a diagram like Figure 7.

9. 
$$\lim_{x \to 2} (3x - 2) = 4$$
 10.  $\lim_{x \to 4} (5 - 2x) = -3$ 

- A machinist is required to manufacture a circular metal disk with area 1000 cm<sup>2</sup>.
  - (a) What radius produces such a disk?
  - (b) If the machinist is allowed an error tolerance of ±5 cm<sup>2</sup> in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
  - (c) In terms of the ε, δ definition of lim<sub>x→a</sub> f(x) = L, what is x? What is f(x)? What is a? What is L? What value of ε is given? What is the corresponding value of δ?
- 12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

- (a) How much power is needed to maintain the temperature at 200 °C?
- (b) If the temperature is allowed to vary from 200 °C by up to ±1 °C, what range of wattage is allowed for the input power?
- (c) In terms of the ε, δ definition of lim<sub>x→a</sub> f(x) = L, what is x? What is f(x)? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

# **13.** Use a graph to find a number N such that

$$\left| \frac{6x^2 + 5x - 3}{2x^2 - 1} - 3 \right| < 0.2 \quad \text{whenever} \quad x > N$$

14. For the limit

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 2 by finding values of N that correspond to  $\varepsilon = 0.5$  and  $\varepsilon = 0.1$ .

15. (a) Determine how large we have to take x so that

$$\frac{1}{x^2} < 0.0001$$

(b) Use Definition 2 to prove that

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

**16.** (a) For what values of x is it true that

$$\frac{1}{x^2} < 1,000,000$$

- (b) The precise definition of lim<sub>x→a</sub> f(x) = ∞ states that for every positive number *M* (no matter how large) there is a corresponding positive number δ such that f(x) > M whenever 0 < |x - a| < δ. Use this definition to prove that lim<sub>x→0</sub> (1/x<sup>2</sup>) = ∞.
- **17.** (a) Use a graph to guess the value of the limit

$$\lim_{n\to\infty}\frac{n^5}{n!}$$

- (b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to ε = 0.1 and ε = 0.001 in Definition 3.
- **18.** Use Definition 3 to prove that  $\lim_{n \to \infty} r^n = 0$  when |r| < 1.
- **19.** Use Definition 3 to prove that if  $\lim_{n \to \infty} |a_n| = 0$ , then  $\lim_{n \to \infty} a_n = 0$ .
- **20.** Use Definition 4 to prove that  $\lim_{n \to \infty} n^3 = \infty$ .

A Few Proofs



In this appendix we present proofs of some theorems that were stated in the main body of the text. We start by proving the Triangle Inequality, which is an important property of absolute value.

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**The Triangle Inequality** If *a* and *b* are any real numbers, then

 $|a+b| \le |a|+|b|$ 

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, then the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, and we can prove it as follows.

Notice that

$$-|a| \le a \le |a|$$

is always true because a equals either |a| or -|a|. The corresponding statement for b is

$$-|b| \le b \le |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \le a + b \le |a| + |b|$$

When combined, Properties 4 and 5 of absolute value (see Appendix A) say that

 $|x| \leq a \iff -a \leq x \leq a$ 

If we now apply Properties 4 and 5 of absolute value from Appendix A (with x replaced by a + b and a by |a| + |b|), we obtain

$$|a+b| \le |a|+|b|$$

which is what we wanted to show.

Next we use the Triangle Inequality to prove the Sum Law for limits.

The Sum Law was first stated in Section 2.3.

Sum Law If 
$$\lim_{x \to a} f(x) = L$$
 and  $\lim_{x \to a} g(x) = M$  both exist, then  
$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

**Proof** Let  $\varepsilon > 0$  be given. According to Definition 1 in Appendix D, we must find  $\delta > 0$  such that

$$|f(x) + g(x) - (L + M)| < \varepsilon$$
 whenever  $0 < |x - a| < \delta$ 

Using the Triangle Inequality, we can write

$$| f(x) + g(x) - (L + M) | = |(f(x) - L) + (g(x) - M)|$$
  
 
$$\leq | f(x) - L | + |g(x) - M |$$

We will make |f(x) + g(x) - (L + M)| less than  $\varepsilon$  by making each of the terms |f(x) - L| and |g(x) - M| less than  $\varepsilon/2$ .

Since  $\varepsilon/2 > 0$  and  $\lim_{x\to a} f(x) = L$ , there exists a number  $\delta_1 > 0$  such that

$$|f(x) - L| < \frac{\varepsilon}{2}$$
 whenever  $0 < |x - a| < \delta_1$ 

Similarly, since  $\lim_{x\to a} g(x) = M$ , there exists a number  $\delta_2 > 0$  such that

$$|g(x) - M| < \frac{\varepsilon}{2}$$
 whenever  $0 < |x - a| < \delta_2$ 

Let  $\delta = \min{\{\delta_1, \delta_2\}}$ . Notice that

if 
$$0 < |x - a| < \delta$$
 then  $0 < |x - a| < \delta_1$  and  $0 < |x - a| < \delta_2$ 

and so 
$$|f(x) - L| < \frac{\varepsilon}{2}$$
 and  $|g(x) - M| < \frac{\varepsilon}{2}$ 

Therefore, by (1),

$$|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M|$$
  
 $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ 

To summarize,

$$|f(x) + g(x) - (L + M)| < \varepsilon$$
 whenever  $0 < |x - a| < \delta$ 

Thus, by the definition of a limit,

$$\lim_{x \to a} \left[ f(x) + g(x) \right] = L + M$$

Fermat's Theorem was discussed in Section 4.2.

**Fermat's Theorem** If f has a local maximum or minimum at c, and if f'(c) exists, then f'(c) = 0.

**Proof** Suppose, for the sake of definiteness, that f has a local maximum at c. Then  $f(c) \ge f(x)$  if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \ge f(c + h)$$

and therefore

$$f(c+h) - f(c) \le 0$$

2

We can divide both sides of an inequality by a positive number. Thus, if h > 0 and h is sufficiently small, we have

$$\frac{f(c+h) - f(c)}{h} \le 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2 in Section 2.3), we get

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \le \lim_{h \to 0^+} 0 = 0$$

But since f'(c) exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

and so we have shown that  $f'(c) \leq 0$ .

If h < 0, then the direction of the inequality (2) is reversed when we divide by h:

$$\frac{f(c+h) - f(c)}{h} \ge 0 \qquad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \ge 0$$

We have shown that  $f'(c) \ge 0$  and also that  $f'(c) \le 0$ . Since both of these inequalities must be true, the only possibility is that f'(c) = 0.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner.



# Integration of Rational Functions by Partial Fractions

In this appendix we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. To illustrate the method, observe that by taking the fractions 2/(x - 1) and 1/(x + 2) to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2} \, dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$
$$= 2\ln|x-1| - \ln|x+2| + C$$

#### APPENDIX F INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS A43

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It is possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper*. Recall that if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where  $a_n \neq 0$ , then the degree of P is n and we write deg(P) = n.

If f is improper, that is,  $\deg(P) \ge \deg(Q)$ , then we must take the preliminary step of dividing Q into P (by long division) until a remainder R(x) is obtained such that  $\deg(R) < \deg(Q)$ . The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

As the following example illustrates, sometimes this preliminary step is all that is required.

**EXAMPLE I** Find 
$$\int \frac{x^3 + x}{x - 1} dx$$
.

**SOLUTION** Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3 + x}{x - 1} dx = \int \left( x^2 + x + 2 + \frac{2}{x - 1} \right) dx$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x - 1| + C$$

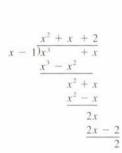
The next step is to factor the denominator Q(x) as far as possible. It can be shown that any polynomial Q can be factored as a product of linear factors (of the form ax + b) and irreducible quadratic factors (of the form  $ax^2 + bx + c$ , where  $b^2 - 4ac < 0$ ). For instance, if  $Q(x) = x^4 - 16$ , we could factor it as

$$Q(x) = (x^{2} - 4)(x^{2} + 4) = (x - 2)(x + 2)(x^{2} + 4)$$

The third step is to express the proper rational function R(x)/Q(x) (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^{i}} \quad \text{or} \quad \frac{Ax+B}{(ax^{2}+bx+c)^{j}}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.



**CASE I** • The denominator Q(x) is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated. In this case the partial fraction theorem states that there exist constants  $A_1, A_2, \ldots, A_k$  such that

2 
$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}$$

These constants can be determined as in the following example.

**EXAMPLE 2** Evaluate 
$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$

**SOLUTION** Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^{3} + 3x^{2} - 2x = x(2x^{2} + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, x(2x - 1)(x + 2), obtaining

$$4 \quad x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

$$x^{2} + 2x - 1 = (2A + B + 2C)x^{2} + (3A + 2B - C)x - 2A$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of  $x^2$  on the right side, 2A + B + 2C, must equal the coefficient of  $x^2$  on the left side—namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for A, B, and C:

$$2A + B + 2C = 1$$
$$3A + 2B - C = 2$$
$$-2A = -1$$

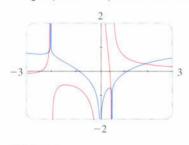
Solving, we get  $A = \frac{1}{2}$ ,  $B = \frac{1}{5}$ , and  $C = -\frac{1}{10}$ , and so

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx = \int \left[ \frac{1}{2} \frac{1}{x} + \frac{1}{5} \frac{1}{2x - 1} - \frac{1}{10} \frac{1}{x + 2} \right] dx$$
$$= \frac{1}{2} \ln|x| + \frac{1}{10} \ln|2x - 1| - \frac{1}{10} \ln|x + 2| + K$$

Another method for finding A, B, and C is given in the note after this example.

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Figure 1 shows the graphs of the integrand in Example 2 and its indefinite integral (with K = 0). Which is which?





In integrating the middle term we have made the mental substitution u = 2x - 1, which gives du = 2 dx and dx = du/2.

**NOTE** • We can use an alternative method to find the coefficients *A*, *B*, and *C* in Example 2. Equation 4 is an identity; it is true for every value of *x*. Let's choose values of *x* that simplify the equation. If we put x = 0 in Equation 4, then the second and third terms on the right side vanish and the equation then becomes -2A = -1, or  $A = \frac{1}{2}$ . Likewise,  $x = \frac{1}{2}$  gives  $5B/4 = \frac{1}{4}$  and x = -2 gives 10C = -1, so  $B = \frac{1}{5}$  and  $C = -\frac{1}{10}$ . (You may object that Equation 3 is not valid for  $x = 0, \frac{1}{2}$ , or -2, so why should Equation 4 be valid for those values? In fact, Equation 4 is true for all values of *x*, even  $x = 0, \frac{1}{2}$ , and -2. See Exercise 35 for the reason.)

**EXAMPLE 3** Find 
$$\int \frac{dx}{x^2 - a^2}$$
, where  $a \neq 0$ .

**SOLUTION** The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x+a) + B(x-a) = 1$$

Using the method of the preceding note, we put x = a in this equation and get A(2a) = 1, so A = 1/(2a). If we put x = -a, we get B(-2a) = 1, so B = -1/(2a). Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left[ \frac{1}{x - a} - \frac{1}{x + a} \right] dx$$
$$= \frac{1}{2a} [\ln|x - a| - \ln|x + a|] + C$$

Since  $\ln x - \ln y = \ln(x/y)$ , we can write the integral as

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

**CASE II** • Q(x) is a product of linear factors, some of which are repeated.

Suppose the first linear factor  $(a_1x + b_1)$  is repeated r times; that is,  $(a_1x + b_1)^r$  occurs in the factorization of Q(x). Then, instead of the single term  $A_1/(a_1x + b_1)$  in Equation 2, we would use

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$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \dots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

but we prefer to work out in detail a simpler example.

**EXAMPLE 4** Find 
$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$$
.

SOLUTION The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator  $Q(x) = x^3 - x^2 - x + 1$ . Since Q(1) = 0, we know that x - 1 is a factor and we obtain

$$x^{3} - x^{2} - x + 1 = (x - 1)(x^{2} - 1) = (x - 1)(x - 1)(x + 1)$$
$$= (x - 1)^{2}(x + 1)$$

Since the linear factor x - 1 occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator,  $(x - 1)^2(x + 1)$ , we get

7 
$$4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^{2}$$
$$= (A + C)x^{2} + (B - 2C)x + (-A + B + C)$$

Another method for finding the coefficients:

Put x = 1 in (7): B = 2. Put x = -1: C = -1. Put x = 0: A = B + C = 1. Now we equate coefficients:

$$A + C = 0$$
$$B - 2C = 4$$
$$-A + B + C = 0$$

Solving, we obtain A = 1, B = 2, and C = -1, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \, dx = \int \left[ x + 1 + \frac{1}{x - 1} + \frac{2}{(x - 1)^2} - \frac{1}{x + 1} \right] \, dx$$
$$= \frac{x^2}{2} + x + \ln|x - 1| - \frac{2}{x - 1} - \ln|x + 1| + K$$
$$= \frac{x^2}{2} + x - \frac{2}{x - 1} + \ln\left|\frac{x - 1}{x + 1}\right| + K$$

# **CASE III** • Q(x) contains irreducible quadratic factors, none of which is repeated.

If Q(x) has the factor  $ax^2 + bx + c$ , where  $b^2 - 4ac < 0$ , then, in addition to the partial fractions in Equations 2 and 6, the expression for R(x)/Q(x) will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

where A and B are constants to be determined. For instance, the function given by  $f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$  has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

The term given in (8) can be integrated by completing the square and using the formula

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + C$$

**EXAMPLE 5** Evaluate 
$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx.$$

**SOLUTION** Since  $x^3 + 4x = x(x^2 + 4)$  can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by  $x(x^2 + 4)$ , we have

$$2x^{2} - x + 4 = A(x^{2} + 4) + (Bx + C)x$$
$$= (A + B)x^{2} + Cx + 4A$$

Equating coefficients, we obtain

 $A + B = 2 \qquad C = -1 \qquad 4A = 4$ 

Thus A = 1, B = 1, and C = -1 and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} \, dx = \int \left[ \frac{1}{x} + \frac{x - 1}{x^2 + 4} \right] dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x-1}{x^2+4} \, dx = \int \frac{x}{x^2+4} \, dx - \int \frac{1}{x^2+4} \, dx$$

We make the substitution  $u = x^2 + 4$  in the first of these integrals so that du = 2x dx. We evaluate the second integral by means of Formula 9 with a = 2:

$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)} \, dx = \int \frac{1}{x} \, dx + \int \frac{x}{x^2 + 4} \, dx - \int \frac{1}{x^2 + 4} \, dx$$
$$= \ln|x| + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2}\tan^{-1}(x/2) + K$$

**EXAMPLE 6** Evaluate  $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx.$ 

**SOLUTION** Since the degree of the numerator is not less than the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x - 1}{4x^2 - 4x + 3}$$

Notice that the quadratic  $4x^2 - 4x + 3$  is irreducible because its discriminant is  $b^2 - 4ac = -32 < 0$ . This means it can't be factored, so we do not need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x - 1)^2 + 2$$

This suggests that we make the substitution u = 2x - 1. Then, du = 2 dx and x = (u + 1)/2, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} \, dx = \int \left( 1 + \frac{x - 1}{4x^2 - 4x + 3} \right) dx$$
  

$$= x + \frac{1}{2} \int \frac{\frac{1}{2}(u + 1) - 1}{u^2 + 2} \, du$$
  

$$= x + \frac{1}{4} \int \frac{u - 1}{u^2 + 2} \, du$$
  

$$= x + \frac{1}{4} \int \frac{u}{u^2 + 2} \, du - \frac{1}{4} \int \frac{1}{u^2 + 2} \, du$$
  

$$= x + \frac{1}{8} \ln(u^2 + 2) - \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$
  

$$= x + \frac{1}{8} \ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}} \tan^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right) + C$$

**NOTE** • Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c} \qquad \text{where } b^2 - 4ac < 0$$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu+D}{u^2+a^2} \, du = C \int \frac{u}{u^2+a^2} \, du + D \int \frac{1}{u^2+a^2} \, du$$

Then the first integral is a logarithm and the second is expressed in terms of tan<sup>-1</sup>.

# CASE IV • Q(x) contains a repeated irreducible quadratic factor.

If Q(x) has the factor  $(ax^2 + bx + c)^r$ , where  $b^2 - 4ac < 0$ , then instead of the single partial fraction (8), the sum

APPENDIX F INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS 
A49

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of R(x)/Q(x). Each of the terms in (10) can be integrated by completing the square.

**EXAMPLE 7** Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$
  
=  $\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2 + x + 1} + \frac{Ex+F}{x^2 + 1} + \frac{Gx+H}{(x^2 + 1)^2} + \frac{Ix+J}{(x^2 + 1)^3}$ 

**EXAMPLE 8** Evaluate 
$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx$$

 $I = -\frac{1}{2}, J = \frac{1}{2}$  SOLUTION The form of the partial fraction decomposition is

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by  $x(x^2 + 1)^2$ , we have

$$-x^{3} + 2x^{2} - x + 1 = A(x^{2} + 1)^{2} + (Bx + C)x(x^{2} + 1) + (Dx + E)x$$
$$= A(x^{4} + 2x^{2} + 1) + B(x^{4} + x^{2}) + C(x^{3} + x) + Dx^{2} + Ex$$
$$= (A + B)x^{4} + Cx^{3} + (2A + B + D)x^{2} + (C + E)x + A$$

If we equate coefficients, we get the system

$$A + B = 0$$
  $C = -1$   $2A + B + D = 2$   $C + E = -1$   $A = 1$ 

which has the solution A = 1, B = -1, C = -1, D = 1, and E = 0. Thus

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx$$
  
=  $\int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx$   
=  $\int \frac{dx}{x} - \int \frac{x}{x^2+1} dx - \int \frac{dx}{x^2+1} + \int \frac{x dx}{(x^2+1)^2}$   
=  $\ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$ 

It would be extremely tedious to work out by hand the numerical values of the coefficients in Example 7. Most computer algebra systems, however, can find the numerical values very quickly. For instance, the Maple command

convert(f, parfrac, x)

or the Mathematica command

Apart[f]

gives the following values:  $A = -1, \quad B = \frac{1}{8}, \quad C = D = -1,$   $E = -\frac{1}{8}, \quad F = \frac{15}{8}, \quad G = H = \frac{3}{4},$  $I = -\frac{1}{2}, \quad J = \frac{1}{2}$ 



Exercises · · · · · · · · · · · · · ·

1-10 • Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. 
$$\frac{5}{2x^2 - 3x - 2}$$
  
3.  $\frac{1}{x^4 - x^3}$   
5.  $\frac{x^2 + 1}{x^2 - 1}$   
7.  $\frac{x^2 - 2}{x(x^2 + 2)}$   
9.  $\frac{x^3 + x^2 + 1}{x^4 + x^3 + 2x^2}$   
2.  $\frac{x^2 + 9x - 12}{(3x - 1)(x + 6)^2}$   
4.  $\frac{x^4 + x^3 - x^2 - x + 1}{x^3 - x}$   
6.  $\frac{x^3 - 4x^2 + 2}{(x^2 + 1)(x^2 + 2)}$   
8.  $\frac{x^4 + x^2 + 1}{(x^2 + 1)(x^2 + 4)^2}$   
10.  $\frac{19x}{(x - 1)^3(4x^2 + 5x + 3)^2}$ 

**11–28** ■ Evaluate the integral.

11. 
$$\int \frac{x^2}{x+1} dx$$
  
12. 
$$\int \frac{x}{x-5} dx$$
  
13. 
$$\int_2^4 \frac{4x-1}{(x-1)(x+2)} dx$$
  
14. 
$$\int_3^7 \frac{1}{(x+1)(x-2)} dx$$
  
15. 
$$\int_0^1 \frac{2x+3}{(x+1)^2} dx$$
  
16. 
$$\int_0^2 \frac{x^3+x^2-12x+1}{x^2+x-12} dx$$
  
17. 
$$\int_2^3 \frac{6x^2+5x-3}{x^3+2x^2-3x} dx$$
  
18. 
$$\int_0^1 \frac{x}{x^2+4x+4} dx$$
  
19. 
$$\int \frac{5x^2+3x-2}{x^3+2x^2} dx$$
  
20. 
$$\int \frac{x^2}{(x-3)(x+2)^2} dx$$
  
21. 
$$\int_0^1 \frac{x}{x^2+x+1} dx$$
  
22. 
$$\int_0^1 \frac{x-1}{x^2+2x+2} dx$$
  
23. 
$$\int \frac{3x^2-4x+5}{(x-1)(x^2+1)} dx$$
  
24. 
$$\int \frac{2x+3}{x^3+3x} dx$$
  
25. 
$$\int \frac{1}{x^3-1} dx$$
  
26. 
$$\int \frac{x^4}{x^4-1} dx$$
  
28. 
$$\int \frac{x^4+1}{x(x^2+1)^2} dx$$

🚰 29. Use a graph of

$$f(x) = \frac{1}{x^2 - 2x - 3}$$

to decide whether  $\int_0^2 f(x) dx$  is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

- **30.** Graph both  $y = 1/(x^3 2x^2)$  and an antiderivative on the same screen.
  - 31. Find the area of the region under the curve

$$y = \frac{1}{x^2 - 6x + 8} \qquad 5 \le x \le 10$$

by completing the square and using the result of Example 3.

32. The region under the curve

$$y = \frac{1}{x^2 + 3x + 2}$$

from x = 0 to x = 1 is rotated about the x-axis. Find the volume of the resulting solid.

**33.** (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

- (b) Use part (a) to find  $\int f(x) dx$  (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.
- **34.** (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

- (b) Use part (a) to find  $\int f(x) dx$  and graph f and its indefinite integral on the same screen.
- (c) Use the graph of f to discover the main features of the graph of  $\int f(x) dx$ .
- **35.** Suppose that F, G, and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when Q(x) = 0. Prove that F(x) = G(x) for all x. [*Hint*: Use continuity.]

**36.** If f is a quadratic function such that f(0) = 1 and

$$\int \frac{f(x)}{x^2(x+1)^3} \, dx$$

is a rational function, find the value of f'(0).



Polar coordinates offer an alternative way of locating points in a plane. They are useful because, for certain types of regions and curves, polar coordinates provide very simple descriptions and equations. The principal applications of this idea occur in multivariable calculus: the evaluation of double integrals and the derivation of Kepler's laws of planetary motion.



Curves in Polar Coordinates

FIGURE I

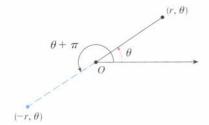


FIGURE 2

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

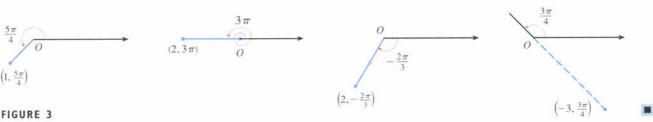
If *P* is any other point in the plane, let *r* be the distance from *O* to *P* and let  $\theta$  be the angle (usually measured in radians) between the polar axis and the line *OP* as in Figure 1. Then the point *P* is represented by the ordered pair  $(r, \theta)$  and *r*,  $\theta$  are called **polar coordinates** of *P*. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If P = O, then r = 0 and we agree that  $(0, \theta)$  represents the pole for any value of  $\theta$ .

We extend the meaning of polar coordinates  $(r, \theta)$  to the case in which r is negative by agreeing that, as in Figure 2, the points  $(-r, \theta)$  and  $(r, \theta)$  lie on the same line through O and at the same distance |r| from O, but on opposite sides of O. If r > 0, the point  $(r, \theta)$  lies in the same quadrant as  $\theta$ ; if r < 0, it lies in the quadrant on the opposite side of the pole. Notice that  $(-r, \theta)$  represents the same point as  $(r, \theta + \pi)$ .

**EXAMPLE 1** Plot the points whose polar coordinates are given:

(a)  $(1, 5\pi/4)$  (b)  $(2, 3\pi)$  (c)  $(2, -2\pi/3)$  (d)  $(-3, 3\pi/4)$ 

**SOLUTION** The points are plotted in Figure 3. In part (d) the point  $(-3, 3\pi/4)$  is located three units from the pole in the fourth quadrant because the angle  $3\pi/4$  is in the second quadrant and r = -3 is negative.



In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point  $(1, 5\pi/4)$  in Example 1(a) could be written as  $(1, -3\pi/4)$  or  $(1, 13\pi/4)$  or  $(-1, \pi/4)$  (see Figure 4).

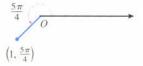
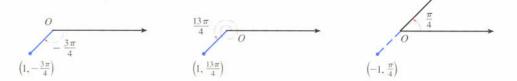
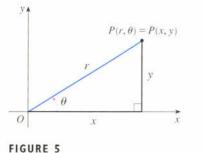


FIGURE 4



In fact, since a complete counterclockwise rotation is given by an angle  $2\pi$ , the point represented by polar coordinates  $(r, \theta)$  is also represented by

 $(r, \theta + 2n\pi)$  and  $(-r, \theta + (2n + 1)\pi)$ 



where *n* is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates (x, y) and polar coordinates  $(r, \theta)$ , then, from the figure, we have

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

and so

П

$$x = r\cos\theta \qquad \qquad y = r\sin\theta$$

Although Equations 1 were deduced from Figure 5, which illustrates the case where r > 0 and  $0 < \theta < \pi/2$ , these equations are valid for all values of r and  $\theta$ . (See the general definitions of  $\sin \theta$  and  $\cos \theta$  in Appendix C.)

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and  $\theta$  when x and y are known, we use the equations

2

$$r^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x}$$

which can be deduced from Equations 1 or simply read from Figure 5.

**EXAMPLE 2** Convert the point  $(2, \pi/3)$  from polar to Cartesian coordinates. SOLUTION Since r = 2 and  $\theta = \pi/3$ , Equations 1 give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$
$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is  $(1, \sqrt{3})$  in Cartesian coordinates.

**EXAMPLE 3** Represent the point with Cartesian coordinates (1, -1) in terms of polar coordinates.

**SOLUTION** If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
  $\tan \theta = \frac{y}{x} = \frac{-1}{1} = -1$ 

Since the point (1, -1) lies in the fourth quadrant, we can choose  $\theta = -\pi/4$  or  $\theta = 7\pi/4$ . Thus, one possible answer is  $(\sqrt{2}, -\pi/4)$ ; another is  $(\sqrt{2}, 7\pi/4)$ .

**NOTE** • Equations 2 don't uniquely determine  $\theta$  when x and y are given because, as  $\theta$  increases through the interval  $0 \le \theta < 2\pi$ , each value of  $\tan \theta$  occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and  $\theta$  that satisfy Equations 2. As in Example 3, we must choose  $\theta$  so that the point  $(r, \theta)$  lies in the correct quadrant.

The graph of a polar equation  $r = f(\theta)$ , or more generally  $F(r, \theta) = 0$ , consists of all points *P* that have at least one polar representation  $(r, \theta)$  whose coordinates satisfy the equation.

**EXAMPLE 4** What curve is represented by the polar equation r = 2?

**SOLUTION** The curve consists of all points  $(r, \theta)$  with r = 2. Since r represents the distance from the point to the pole, the curve r = 2 represents the circle with center O and radius 2. In general, the equation r = a represents a circle with center O and radius |a| (see Figure 6).

**EXAMPLE 5** Sketch the polar curve  $\theta = 1$ .

**SOLUTION** This curve consists of all points  $(r, \theta)$  such that the polar angle  $\theta$  is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points (r, 1) on the line with r > 0 are in the first quadrant, whereas those with r < 0 are in the third quadrant.

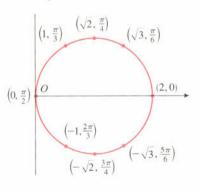
### EXAMPLE 6

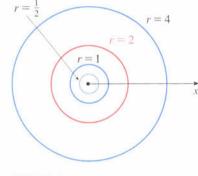
- (a) Sketch the curve with polar equation  $r = 2\cos\theta$ .
- (b) Find a Cartesian equation for this curve.

#### SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of  $\theta$  and plot the corresponding points  $(r, \theta)$ . Then we join these points to sketch the curve,

 $\theta$  $r = 2\cos\theta$ 2 0 V3  $\pi/6$  $\sqrt{2}$  $\pi/4$ 1  $\pi/3$ 0  $\pi/2$  $2\pi/3$  $3\pi/4$ V3  $5\pi/6$  $\overline{\pi}$ -2







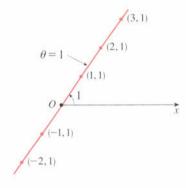




FIGURE 8 Table of values and graph of  $r = 2 \cos \theta$ 

which appears to be a circle. We have used only values of  $\theta$  between 0 and  $\pi$ , since if we let  $\theta$  increase beyond  $\pi$ , we obtain the same points again.

(b) To convert the given equation into a Cartesian equation we use Equations 1 and 2. From  $x = r \cos \theta$  we have  $\cos \theta = x/r$ , so the equation  $r = 2 \cos \theta$  becomes r = 2x/r, which gives

$$2x = r^2 = x^2 + y^2$$
 or  $x^2 + y^2 - 2x = 0$ 

Completing the square, we obtain

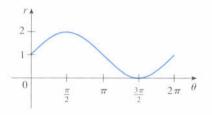
$$(x - 1)^2 + y^2 = 1$$

which is the equation of a circle with center (1, 0) and radius 1.

Figure 9 shows a geometrical illustration that the circle in Example 6 has the equation  $r = 2 \cos \theta$ . The angle *OPQ* is a right angle (Why?) and so  $r/2 = \cos \theta$ .

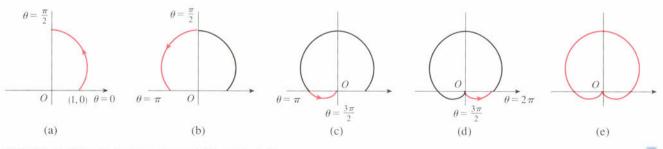
**EXAMPLE 7** Sketch the curve  $r = 1 + \sin \theta$ .

**SOLUTION** Instead of plotting points as in Example 6, we first sketch the graph of  $r = 1 + \sin \theta$  in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of *r* that correspond to increasing values of  $\theta$ .





For instance, we see that as  $\theta$  increases from 0 to  $\pi/2$ , r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As  $\theta$  increases from  $\pi/2$  to  $\pi$ , Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As  $\theta$  increases from  $\pi$  to  $3\pi/2$ , r decreases from 1 to 0 as shown in part (c). Finally, as  $\theta$  increases from  $3\pi/2$  to  $2\pi$ , r increases from 0 to 1 as shown in part (d). If we let  $\theta$  increase beyond  $2\pi$  or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in Figure 11(e). It is called a **cardioid** because it's shaped like a heart.





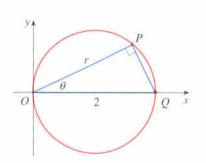
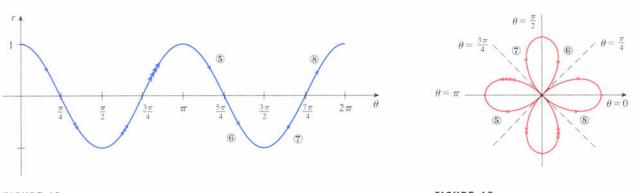


FIGURE 9

# **EXAMPLE 8** Sketch the curve $r = \cos 2\theta$ .

**SOLUTION** As in Example 7, we first sketch  $r = \cos 2\theta$ ,  $0 \le \theta \le 2\pi$ , in Cartesian coordinates in Figure 12. As  $\theta$  increases from 0 to  $\pi/4$ , Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by a single arrow). As  $\theta$  increases from  $\pi/4$  to  $\pi/2$ , r goes from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by a double arrow) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

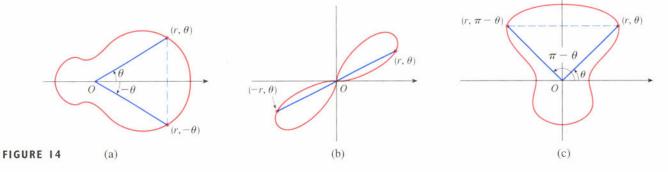


**FIGURE 12**  $r = \cos 2\theta$  in Cartesian coordinates

**FIGURE 13** Four-leaved rose  $r = \cos 2\theta$ 

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

- (a) If a polar equation is unchanged when  $\theta$  is replaced by  $-\theta$ , the curve is symmetric about the polar axis.
- (b) If the equation is unchanged when r is replaced by -r, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)
- (c) If the equation is unchanged when  $\theta$  is replaced by  $\pi \theta$ , the curve is symmetric about the vertical line  $\theta = \pi/2$ .



The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since  $\cos(-\theta) = \cos \theta$ . The curves in Examples 7 and 8 are symmetric about  $\theta = \pi/2$  because  $\sin(\pi - \theta) = \sin \theta$  and  $\cos 2(\pi - \theta) = \cos 2\theta$ . The four-leaved

rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves. For instance, in Example 6 we need only have plotted points for  $0 \le \theta \le \pi/2$  and then reflected about the polar axis to obtain the complete circle.

# **Tangents to Polar Curves**

To find a tangent line to a polar curve  $r = f(\theta)$  we regard  $\theta$  as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta$$
  $y = r\sin\theta = f(\theta)\sin\theta$ 

Then, using the method for finding slopes of parametric curves (Equation 7 in Section 3.5) and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We locate horizontal tangents by finding the points where  $dy/d\theta = 0$  (provided that  $dx/d\theta \neq 0$ ). Likewise, we locate vertical tangents at the points where  $dx/d\theta = 0$  (provided that  $dy/d\theta \neq 0$ ).

Notice that if we are looking for tangent lines at the pole, then r = 0 and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan \theta$$
 if  $\frac{dr}{d\theta} \neq 0$ 

For instance, in Example 8 we found that  $r = \cos 2\theta = 0$  when  $\theta = \pi/4$  or  $3\pi/4$ . This means that the lines  $\theta = \pi/4$  and  $\theta = 3\pi/4$  (or y = x and y = -x) are tangent lines to  $r = \cos 2\theta$  at the origin.

## EXAMPLE 9

3

(a) For the cardioid  $r = 1 + \sin \theta$  of Example 7, find the slope of the tangent line when  $\theta = \pi/3$ .

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

**SOLUTION** Using Equation 3 with  $r = 1 + \sin \theta$ , we have

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$
$$= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

(a) The slope of the tangent at the point where  $\theta = \pi/3$  is

$$\frac{dy}{dx}\Big|_{\theta=\pi/3} = \frac{\cos(\pi/3)\left(1+2\sin(\pi/3)\right)}{\left(1+\sin(\pi/3)\right)\left(1-2\sin(\pi/3)\right)}$$
$$= \frac{\frac{1}{2}\left(1+\sqrt{3}\right)}{\left(1+\sqrt{3}/2\right)\left(1-\sqrt{3}\right)} = \frac{1+\sqrt{3}}{\left(2+\sqrt{3}\right)\left(1-\sqrt{3}\right)}$$
$$= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta \left(1 + 2\sin\theta\right) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$
$$\frac{dx}{d\theta} = \left(1 + \sin\theta\right) \left(1 - 2\sin\theta\right) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore, there are horizontal tangents at the points  $(2, \pi/2)$ ,  $(\frac{1}{2}, 7\pi/6)$ ,  $(\frac{1}{2}, 11\pi/6)$  and vertical tangents at  $(\frac{3}{2}, \pi/6)$  and  $(\frac{3}{2}, 5\pi/6)$ . When  $\theta = 3\pi/2$ , both  $dy/d\theta$  and  $dx/d\theta$  are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta \to (3\pi/2)^{-}} \frac{dy}{dx} = \lim_{\theta \to (3\pi/2)^{-}} \frac{1+2\sin\theta}{1-2\sin\theta} \lim_{\theta \to (3\pi/2)^{-}} \frac{\cos\theta}{1+\sin\theta}$$
$$= -\frac{1}{3} \lim_{\theta \to (3\pi/2)^{-}} \frac{-\sin\theta}{\cos\theta} = \infty$$
By symmetry,
$$\lim_{\theta \to (3\pi/2)^{+}} \frac{dy}{dx} = -\infty$$

Thus, there is a vertical tangent line at the pole (see Figure 15).

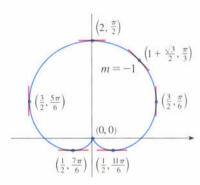
**NOTE** • Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin2\theta$$
$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^{2}\theta$$
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos2\theta} = \frac{\cos\theta + \sin2\theta}{-\sin\theta + \cos2\theta}$$

# Graphing Polar Curves with Graphing Devices

Although it is useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the one shown in Figure 16.

Some graphing devices have commands that enable us to graph polar curves directly. With other machines it is necessary to convert to parametric equations



**FIGURE 15** Tangent lines for  $r = 1 + \sin \theta$ 

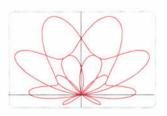


FIGURE 16  $r = \sin \theta + \sin^3(5\theta/2)$ 

first. In the latter case we take the polar equation  $r = f(\theta)$  and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta$$
  $y = r\sin\theta = f(\theta)\sin\theta$ 

Some machines require that the parameter be called t rather than  $\theta$ .

**EXAMPLE 10** Graph the curve  $r = \sin(8\theta/5)$ .

**SOLUTION** Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r \cos \theta = \sin(8\theta/5) \cos \theta$$
  $y = r \sin \theta = \sin(8\theta/5) \sin \theta$ 

In any case we need to determine the domain for  $\theta$ . So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is *n*, then

$$\sin\frac{8(\theta+2n\pi)}{5} = \sin\frac{8\theta}{5}$$

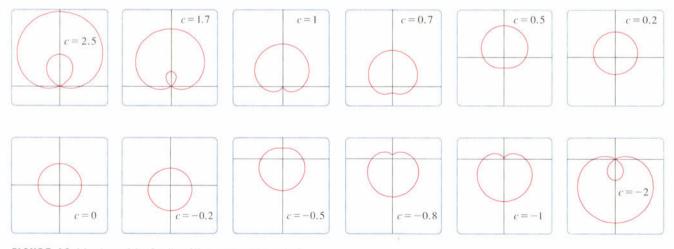
and so we require that  $16n\pi/5$  be an even multiple of  $\pi$ . This will first occur when n = 5. Therefore, we will graph the entire curve if we specify that  $0 \le \theta \le 10\pi$ . Switching from  $\theta$  to t, we have the equations

$$x = \sin(8t/5)\cos t$$
  $y = \sin(8t/5)\sin t$   $0 \le t \le 10\pi$ 

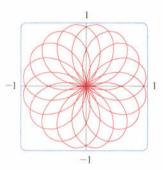
and Figure 17 shows the resulting curve. Notice that this rose has 16 loops.

**EXAMPLE II** Investigate the family of polar curves given by  $r = 1 + c \sin \theta$ . How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

**SOLUTION** Figure 18 shows computer-drawn graphs for various values of c. For c > 1 there is a loop that decreases in size as c decreases. When c = 1 the loop



**FIGURE 18** Members of the family of limaçons  $r = 1 + c \sin \theta$ 



In Exercise 45 you are asked to prove

analytically what we have discovered

FIGURE 17  $r = \sin(8\theta/5)$ 

from the graphs.

disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and  $\frac{1}{2}$  the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from  $\frac{1}{2}$  to 0, the limaçon is shaped like an oval. This oval becomes more circular as  $c \rightarrow 0$ , and when c = 0 the curve is just the circle r = 1.

The remaining parts of Figure 18 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c.

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Exercises

**1–2** Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with r > 0 and one with r < 0.

1. (a)  $(1, \pi/2)$ (b)  $(-1, \pi/5)$ (c) (3, 2)2. (a) (3, 0)(b)  $(2, -\pi/7)$ (c)  $(-1, \pi)$ 

**3-4** ■ Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3.	(a) (	$\sqrt{2}, \pi$	7/4)	(b)	(1.5	$3\pi/2$	2)	(c)	(-1,	$\pi/3)$	
4.	(a) (	$2, 2\pi/2$	/3)	(b)	(4,	3 <i>π</i> )		(c)	(-2,	$-5\pi$	(6)
×.	1943	*	S.	25	10	24	69 (	25	(4) (4)	25	1.47

**5–6** The Cartesian coordinates of a point are given. Find the polar coordinates  $(r, \theta)$  of the point, where r > 0 and  $0 \le \theta < 2\pi$ .

<b>5.</b> (a) (-1, 1)	(b) $(2\sqrt{3}, -2)$
<b>6.</b> (a) $(-1, -\sqrt{3})$	(b) (3, 4)
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**7–12** ■ Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

**7.** r > 1 **8.**  $0 \le \theta \le \pi/3$  **9.**  $0 \le r \le 2$ ,  $\pi/2 \le \theta \le \pi$  **10.**  $1 \le r < 3$ ,  $-\pi/4 \le \theta \le \pi/4$  **11.** 3 < r < 4,  $-\pi/2 \le \theta \le \pi$  **12.**  $-1 \le r \le 1$ ,  $\pi/4 \le \theta \le 3\pi/4$ **13.** 16 Find a Cartesian equation for the curve

**13–16** ■ Find a Cartesian equation for the curve described by the given polar equation.

**13.**  $r \sin \theta = 2$  **14.**  $r = 2 \sin \theta$  **15.**  $r = \frac{1}{1 - \cos \theta}$ **16.**  $r^2 = \sin 2\theta$ 

**17–20** ■ Find a polar equation for the curve represented by the given Cartesian equation.

**17.** y = 5 **18.** y = x + 1

**19.**  $x^2 + y^2 = 25$  **20.**  $x^2 - y^2 = 1$ **21-32** Sketch the curve with the given polar equation.

- **21.** r = 5**22.**  $\theta = 3\pi/4$ **23.**  $r = 2\sin\theta$ **24.**  $r = 1 + \cos\theta$ **25.**  $r = \theta, \ \theta \ge 0$ **26.**  $r = 1/\theta$ **27.**  $r = 1 2\cos\theta$ **28.**  $r = 2 + \cos\theta$ **29.**  $r = 2\cos 4\theta$ **30.**  $r = 2\cos 3\theta$ **31.**  $r^2 = 4\cos 2\theta$ **32.**  $r = 2\cos(3\theta/2)$
- **33.** Show that the polar curve  $r = 4 + 2 \sec \theta$  (called a **conchoid**) has the line x = 2 as a vertical asymptote by showing that  $\lim_{r \to \pm \infty} x = 2$ . Use this fact to help sketch the conchoid.
- **34.** Show that the curve  $r = \sin \theta \tan \theta$  (called a **cissoid** of **Diocles**) has the line x = 1 as a vertical asymptote. Show also that the curve lies entirely within the vertical strip  $0 \le x < 1$ . Use these facts to help sketch the cissoid.

**35–38** Find the slope of the tangent line to the given polar curve at the point specified by the value of  $\theta$ .

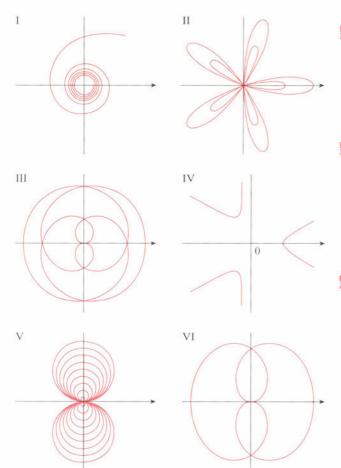
**35.**  $r = 3\cos\theta$ ,  $\theta = \pi/3$  **36.**  $r = \cos\theta + \sin\theta$ ,  $\theta = \pi/4$  **37.**  $r = \theta$ ,  $\theta = \pi/2$  **38.**  $r = \sin 3\theta$ ,  $\theta = \pi/6$ **39-42** In Find the points on the given curve where the tangent line is horizontal or vertical.

<b>39.</b> $r = \cos 2\theta$						<b>40.</b> $r = \cos \theta + \sin \theta$				
41.	r =	1 + 0	$\cos \theta$			42.	r = e	θ		
			1			22	32	<u>1</u> 27	2	2

- **43.** Show that the polar equation  $r = a \sin \theta + b \cos \theta$ , where  $ab \neq 0$ , represents a circle and find its center and radius.
- **44.** Show that the curves  $r = a \sin \theta$  and  $r = a \cos \theta$  intersect at right angles.

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- 45. (a) In Example 11 the graphs suggest that the limaçon r = 1 + c sin θ has an inner loop when |c| > 1. Prove that this is true, and find the values of θ that correspond to the inner loop.
  - (b) From Figure 18 it appears that the limaçon loses its dimple when  $c = \frac{1}{2}$ . Prove this.
- **46.** Match the polar equations with the graphs labeled I–VI. Give reasons for your choices.
  - (a)  $r = \sin(\theta/2)$  (b)  $r = \sin(\theta/4)$
  - (c)  $r = \sec(3\theta)$  (d)  $r = \theta \sin \theta$
  - (e)  $r = 1 + 4\cos 5\theta$  (f)  $r = 1/\sqrt{\theta}$



**47–50** Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

- **47.**  $r = 1 + 2\sin(\theta/2)$  (nephroid of Freeth)
- **48.**  $r = \sqrt{1 0.8 \sin^2 \theta}$  (hippopede)
- **49.**  $r = \sin(9\theta/4)$
- **50.**  $r = 1 + 4\cos(\theta/3)$

- **51.** How are the graphs of  $r = 1 + \sin(\theta \pi/6)$ and  $r = 1 + \sin(\theta - \pi/3)$  related to the graph of  $r = 1 + \sin\theta$ ? In general, how is the graph of  $r = f(\theta - \alpha)$  related to the graph of  $r = f(\theta)$ ?
- **52.** Use a graph to estimate the *y*-coordinate of the highest points on the curve  $r = \sin 2\theta$ . Then use calculus to find the exact value.
- **53.** (a) Investigate the family of curves defined by the polar equations  $r = \sin n\theta$ , where *n* is a positive integer. How is the number of loops related to *n*?
  - (b) What happens if the equation in part (a) is replaced by  $r = |\sin n\theta|$ ?

**54.** A family of curves is given by the equations

$$r = 1 + c \sin n\theta$$

where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

🚰 55. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

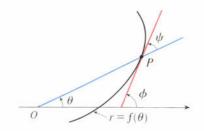
Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

**56.** The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

**57.** Let *P* be any point (except the origin) on the curve  $r = f(\theta)$ . If  $\psi$  is the angle between the tangent line



(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where  $\theta = 0$  and  $\pi/2$ . (c) Prove that any polar curve  $r = f(\theta)$  with the property that the angle  $\psi$  between the radial line and

 $r = Ce^{k\theta}$ , where C and k are constants.

the tangent line is a constant must be of the form

at P and the radial line OP, show that

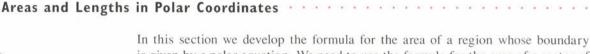
$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint*: Observe that  $\psi = \phi - \theta$  in the figure.]

58. (a) Use Exercise 57 to show that the angle between the tangent line and the radial line is  $\psi = \pi/4$  at every point on the curve  $r = e^{\theta}$ .



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is given by a polar equation. We need to use the formula for the area of a sector of a circle

$$A = \frac{1}{2}r^2\theta$$

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where, as in Figure 1, r is the radius and  $\theta$  is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle:  $A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta$ .

Let  $\mathcal{R}$  be the region, illustrated in Figure 2, bounded by the polar curve  $r = f(\theta)$  and by the rays  $\theta = a$  and  $\theta = b$ , where f is a positive continuous function and  $0 < b - a \le 2\pi$ . We divide the interval [a, b] into subintervals with endpoints  $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$  and equal width  $\Delta \theta$ . The rays  $\theta = \theta_i$  then divide  $\Re$  into *n* smaller regions with central angle  $\Delta \theta = \theta_i - \theta_{i-1}$ . If we choose  $\theta_i^*$  in the *i*th subinterval  $[\theta_{i-1}, \theta_i]$ , then the area  $\Delta A_i$  of the *i*th region is approximated by the area of the sector of a circle with central angle  $\Delta \theta$  and radius  $f(\theta_i^*)$  (see Figure 3).

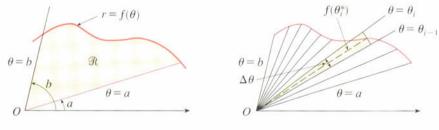


FIGURE 2



Thus, from Formula 1 we have

$$\Delta A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta$$

and so an approximation to the total area A of  $\mathcal{R}$  is

2 
$$A \approx \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta$$

It appears from Figure 3 that the approximation in (2) improves as  $n \to \infty$ . But the sums in (2) are Riemann sums for the function  $q(\theta) = \frac{1}{2} [f(\theta)]^2$ , so

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

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It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region  $\Re$  is

$$A = \int_{a}^{b} \frac{1}{2} [f(\theta)]^{2} d\theta$$

Formula 3 is often written as

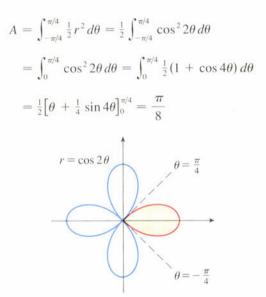
$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

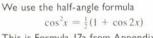
with the understanding that  $r = f(\theta)$ . Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

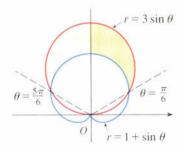
**EXAMPLE 1** Find the area enclosed by one loop of the four-leaved rose  $r = \cos 2\theta$ .

**SOLUTION** The curve  $r = \cos 2\theta$  was sketched in Example 8 in Section G.1. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from  $\theta = -\pi/4$  to  $\theta = \pi/4$ . Therefore, Formula 4 gives





This is Formula 17a from Appendix C. Alternatively, we could use Formula 64 in the Table of Integrals.





# FIGURE 4

**EXAMPLE 2** Find the area of the region that lies inside the circle  $r = 3 \sin \theta$  and outside the cardioid  $r = 1 + \sin \theta$ .

**SOLUTION** The cardioid (see Example 7 in Section G.1) and the circle are sketched in Figure 5 and the desired region is shaded. The values of *a* and *b* in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when  $3\sin\theta = 1 + \sin\theta$ , which gives  $\sin\theta = \frac{1}{2}$ , so  $\theta = \pi/6$ ,  $5\pi/6$ . The desired area can be found by subtracting the area inside

the cardioid between  $\theta = \pi/6$  and  $\theta = 5\pi/6$  from the area inside the circle from  $\pi/6$  to  $5\pi/6$ . Thus

$$A = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 d\theta - \frac{1}{2} \int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2 d\theta$$

Since the region is symmetric about the vertical axis  $\theta = \pi/2$ , we can write

$$A = 2 \left[ \frac{1}{2} \int_{\pi/6}^{\pi/2} 9 \sin^2 \theta \, d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/2} (1 + 2 \sin \theta + \sin^2 \theta) \, d\theta \right]$$
  
=  $\int_{\pi/6}^{\pi/2} (8 \sin^2 \theta - 1 - 2 \sin \theta) \, d\theta$   
=  $\int_{\pi/6}^{\pi/2} (3 - 4 \cos 2\theta - 2 \sin \theta) \, d\theta$   
=  $3\theta - 2 \sin 2\theta + 2 \cos \theta \right]_{\pi/2}^{\pi/2} = \pi$ 

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let  $\Re$  be a region, as illustrated in Figure 6, that is bounded by curves with polar equations  $r = f(\theta)$ ,  $r = g(\theta)$ ,  $\theta = a$ , and  $\theta = b$ , where  $f(\theta) \ge g(\theta) \ge 0$  and  $0 < b - a \le 2\pi$ . The area A of  $\Re$  is found by subtracting the area inside  $r = g(\theta)$  from the area inside  $r = f(\theta)$ , so using Formula 3 we have

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta - \int_a^b \frac{1}{2} [g(\theta)]^2 d\theta$$
$$= \frac{1}{2} \int_a^b ([f(\theta)]^2 - [g(\theta)]^2) d\theta$$

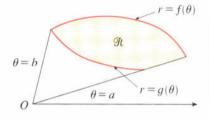
**CAUTION** • The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations  $r = 3 \sin \theta$  and  $r = 1 + \sin \theta$  and found only two such points,  $(\frac{3}{2}, \pi/6)$  and  $(\frac{3}{2}, 5\pi/6)$ . The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as (0, 0) or  $(0, \pi)$ , the origin satisfies  $r = 3 \sin \theta$  and so it lies on the circle; when represented as  $(0, 3\pi/2)$ , it satisfies  $r = 1 + \sin \theta$  and so it lies on the cardioid. Think of two points moving along the curves as the parameter value  $\theta$  increases from 0 to  $2\pi$ . On one curve the origin is reached at  $\theta = 0$  and  $\theta = \pi$ ; on the other curve it is reached at  $\theta = 3\pi/2$ . The points do not collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

**EXAMPLE 3** Find all points of intersection of the curves  $r = \cos 2\theta$  and  $r = \frac{1}{2}$ .

**SOLUTION** If we solve the equations  $r = \cos 2\theta$  and  $r = \frac{1}{2}$ , we get  $\cos 2\theta = \frac{1}{2}$  and, therefore,  $2\theta = \frac{\pi}{3}$ ,  $\frac{5\pi}{3}$ ,  $\frac{7\pi}{3}$ ,  $\frac{11\pi}{3}$ . Thus, the values of  $\theta$  between 0 and  $2\pi$ 

Here we use the identity  $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ (See Formula 17b in Appendix C.) Alternatively, we could use Formula 63 in the Table of Integrals.



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that satisfy both equations are  $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$ . We have found four points of intersection:  $(\frac{1}{2}, \pi/6), (\frac{1}{2}, 5\pi/6), (\frac{1}{2}, 7\pi/6)$ , and  $(\frac{1}{2}, 11\pi/6)$ .

However, you can see from Figure 7 that the curves have four other points of intersection—namely,  $(\frac{1}{2}, \pi/3)$ ,  $(\frac{1}{2}, 2\pi/3)$ ,  $(\frac{1}{2}, 4\pi/3)$ , and  $(\frac{1}{2}, 5\pi/3)$ . These can be found using symmetry or by noticing that another equation of the circle is  $r = -\frac{1}{2}$  and then solving the equations  $r = \cos 2\theta$  and  $r = -\frac{1}{2}$ .

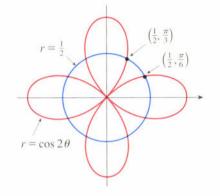


FIGURE 7

# Arc Length

To find the length of a polar curve  $r = f(\theta)$ ,  $a \le \theta \le b$ , we regard  $\theta$  as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta$$
  $y = r\sin\theta = f(\theta)\sin\theta$ 

Using the Product Rule and differentiating with respect to  $\theta$ , we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that f' is continuous, we can use Formula 1 in Section 6.3 to write the arc length as

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^{2} + \left(\frac{dy}{d\theta}\right)^{2}} \ d\theta$$

Therefore, the length of a curve with polar equation  $r = f(\theta)$ ,  $a \le \theta \le b$ , is

$$L = \int_{a}^{b} \sqrt{r^{2} + \left(\frac{dr}{d\theta}\right)^{2}} \ d\theta$$

5

**EXAMPLE 4** Find the length of the cardioid  $r = 1 + \sin \theta$ .

**SOLUTION** This cardioid was sketched in Figure 11 in Section G.1. Its full length is given by the parameter interval  $0 \le \theta \le 2\pi$ , so Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta} \, d\theta$$
$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \, d\theta$$

Using a computer algebra system to evaluate this integral, we find that the length of the cardioid is L = 8.



.2 Exercises · · · ·

**I**−4 ■ Find the area of the region that is bounded by the given curve and lies in the specified sector.

 $I. r = \theta, \quad 0 \le \theta \le \pi$ 

**2.** 
$$r = e^{\theta}, \quad -\pi/2 \le \theta \le \pi/2$$

**3.**  $r = 2\cos\theta$ ,  $0 \le \theta \le \pi/6$ 

4. 
$$r = 1/\theta$$
,  $\pi/6 \le \theta \le 5\pi/6$ 

. . . . . . . . . . .

**5–8** Sketch the curve and find the area that it encloses.

**5.**  $r^2 = 4\cos 2\theta$  **6.**  $r = 4(1 - \cos \theta)$ 

**7.**  $r = 4 - \sin \theta$  **8.**  $r = \sin 3\theta$ 

- **9.** Graph the curve  $r = 2 + \cos 6\theta$  and find the area that it encloses.
- **10.** The curve with polar equation  $r = 2 \sin \theta \cos^2 \theta$  is called a **bifolium**. Graph it and find the area that it encloses.

**11–14** Find the area of the region enclosed by one loop of the curve.

**11.**  $r = \sin 5\theta$  **12.**  $r = 2\cos 4\theta$ 

**13.**  $r = 1 + 2\sin\theta$  (inner loop)

14.  $r = 2 + 3\cos\theta$  (inner loop)

**15–18** Find the area of the region that lies inside the first curve and outside the second curve.

**15.**  $r = 1 - \cos \theta$ ,  $r = \frac{3}{2}$ 

16.  $r = 3\cos\theta$ ,  $r = 2 - \cos\theta$ 

**17.**  $r = 4 \sin \theta$ , r = 2

**18.**  $r = 1 + \cos \theta$ ,  $r = 3 \cos \theta$ 

**19–22** Find the area of the region that lies inside both curves.

**19.**  $r = \sin \theta$ ,  $r = \cos \theta$  **20.**  $r = \sin 2\theta$ ,  $r = \sin \theta$ 

- **21.**  $r = \sin 2\theta$ ,  $r = \cos 2\theta$  **22.**  $r^2 = 2 \sin 2\theta$ , r = 1
- **23.** Find the area inside the larger loop and outside the smaller loop of the limaçon  $r = \frac{1}{2} + \cos \theta$ .
- **24.** Graph the hippopede  $r = \sqrt{1 0.8 \sin^2 \theta}$  and the circle  $r = \sin \theta$ , and find the exact area of the region that lies inside both curves.

**25-28** Find all points of intersection of the given curves.

- **25.**  $r = \cos \theta$ ,  $r = 1 \cos \theta$
- **26.**  $r = \cos 3\theta$ ,  $r = \sin 3\theta$
- **27.**  $r = \sin \theta$ ,  $r = \sin 2\theta$
- **28.**  $r^2 = \sin 2\theta$ ,  $r^2 = \cos 2\theta$
- **29.** The points of intersection of the cardioid  $r = 1 + \sin \theta$ and the spiral loop  $r = 2\theta$ ,  $-\pi/2 \le \theta \le \pi/2$ , can't be found exactly. Use a graphing device to find the approximate values of  $\theta$  at which they intersect. Then use these values to estimate the area that lies inside both curves.
- **30.** Use a graph to estimate the values of  $\theta$  for which the curves  $r = 3 + \sin 5\theta$  and  $r = 6 \sin \theta$  intersect. Then estimate the area that lies inside both curves.

**31–34** ■ Find the exact length of the polar curve.

**31.**  $r = 2^{\theta}$ ,  $0 \le \theta \le 2\pi$ 

**32.**  $r = e^{-\theta}$ ,  $0 \le \theta \le 3\pi$ 

**CAS** 33.  $r = \theta^2$ ,  $0 \le \theta \le 2\pi$ 

 $\textbf{CAS} \textbf{ 34. } r = \theta, \quad 0 \le \theta \le 2\pi$ 

**35–36** ■ Use a calculator or computer to find the length of the loop correct to four decimal places.

**35.** One loop of the four-leaved rose  $r = \cos 2\theta$ 

**36.** The loop of the conchoid  $r = 4 + 2 \sec \theta$ 

. . . . . . . . . . . . .



#### **Conic Sections in Polar Coordinates**

In this project we give a unified treatment of all three types of conic sections in terms of a focus and directrix. We will see that if we place the focus at the origin, then a conic section has a simple polar equation. In Chapter 10 we will use the polar equation of an ellipse to derive Kepler's laws of planetary motion.

Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). Let C be the set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e). Notice that if the eccentricity is e = 1, then |PF| = |Pl| and so the given condition simply becomes the definition of a parabola as given in Appendix B.

1. If we place the focus F at the origin and the directrix parallel to the y-axis and d units to the right, then the directrix has equation x = d and is perpendicular to the polar axis. If the point P has polar coordinates  $(r, \theta)$ , use Figure 1 to show that

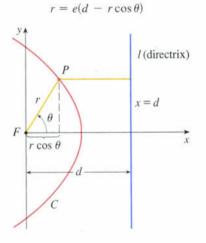


FIGURE I

- **2.** By converting the polar equation in Problem 1 to rectangular coordinates, show that the curve C is an ellipse if e < 1. (See Appendix B for a discussion of ellipses.)
- **3.** Show that *C* is a hyperbola if e > 1.
- 4. Show that the polar equation

$$r = \frac{ed}{1 + e\cos\theta}$$

represents an ellipse if e < 1, a parabola if e = 1, or a hyperbola if e > 1.

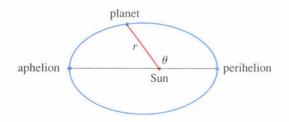
**5.** For each of the following conics, find the eccentricity and directrix. Then identify and sketch the conic.

(a) 
$$r = \frac{4}{1 + 3\cos\theta}$$
 (b)  $r = \frac{8}{3 + 3\cos\theta}$  (c)  $r = \frac{2}{2 + \cos\theta}$ 

- 6. Graph the conics r = e/(1 e cos θ) with e = 0.4, 0.6, 0.8, and 1.0 on a common screen. How does the value of e affect the shape of the curve?
- **7.** (a) Show that the polar equation of an ellipse with directrix x = d can be written in the form

$$r = \frac{a(1-e^2)}{1-e\cos\theta}$$

- (b) Find an approximate polar equation for the elliptical orbit of the planet Earth around the Sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about  $2.99 \times 10^8$  km.
- 8. (a) The planets move around the Sun in elliptical orbits with the Sun at one focus. The positions of a planet that are closest to and farthest from the Sun are called its *perihelion* and *aphelion*, respectively. (See Figure 2.) Use Problem 7(a) to show that the perihelion distance from a planet to the Sun is a(1 e) and the aphelion distance is a(1 + e).



#### FIGURE 2

**Complex Numbers** 

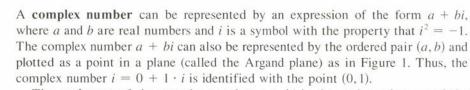
3 - 2i

2 + 3i

- (b) Use the data of Problem 7(b) to find the distances from Earth to the Sun at perihelion and at aphelion.
- (a) The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the Sun is 4.6 × 10<sup>7</sup> km. Use the results of Problem 8(a) to find its maximum distance from the Sun.
  - (b) Find the distance traveled by the planet Mercury during one complete orbit around the Sun. (Use your calculator or computer algebra system to evaluate the definite integral.)



- 2i •



The **real part** of the complex number a + bi is the real number a and the **imaginary part** is the real number b. Thus, the real part of 4 - 3i is 4 and the imaginary part is -3. Two complex numbers a + bi and c + di are **equal** if a = c and b = d, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the x-axis is called the real axis and the y-axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or sub-

FIGURE 1 Complex numbers as points in the Argand plane

tracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$(a + bi)(c + di) = a(c + di) + (bi)(c + di)$$
$$= ac + adi + bci + bdi2$$

Since  $i^2 = -1$ , this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

#### EXAMPLE I

$$(-1+3i)(2-5i) = (-1)(2-5i) + 3i(2-5i)$$
$$= -2 + 5i + 6i - 15(-1) = 13 + 11i$$

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number z = a + bi, we define its **complex** conjugate to be  $\overline{z} = a - bi$ . To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

**EXAMPLE 2** Express the number 
$$\frac{-1+3i}{2+5i}$$
 in the form  $a+bi$ .

**SOLUTION** We multiply numerator and denominator by the complex conjugate of 2 + 5i, namely 2 - 5i, and we take advantage of the result of Example 1:

$$\frac{-1+3i}{2+5i} = \frac{-1+3i}{2+5i} \cdot \frac{2-5i}{2-5i} = \frac{13+11i}{2^2+5^2} = \frac{13}{29} + \frac{11}{29}i$$

The geometric interpretation of the complex conjugate is shown in Figure 2:  $\overline{z}$  is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

Properties of Conjugates  $\overline{z + w} = \overline{z} + \overline{w}$   $\overline{zw} = \overline{z}\overline{w}$   $\overline{z^n} = \overline{z}^n$ 

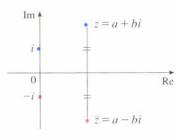
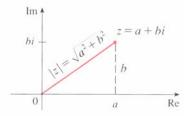


FIGURE 2



The **modulus**, or **absolute value**, |z| of a complex number z = a + bi is its distance from the origin. From Figure 3 we see that if z = a + bi, then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

FIGURE 3

$$z\overline{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$
  
 $z\overline{z} = |z|^2$ 

and so

$$\frac{z}{w} = \frac{zw}{w\overline{w}} = \frac{zw}{|w|^2}$$

Since  $i^2 = -1$ , we can think of *i* as a square root of -1. But we also have  $(-i)^2 = i^2 = -1$  and so -i is also a square root of -1. We say that *i* is the **principal square root** of -1 and write  $\sqrt{-1} = i$ . In general, if *c* is any positive number, we write

$$\sqrt{-c} = \sqrt{c} i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation  $ax^2 + bx + c = 0$  are valid even when  $b^2 - 4ac < 0$ :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**EXAMPLE 3** Find the roots of the equation  $x^2 + x + 1 = 0$ .

SOLUTION Using the quadratic formula, we have

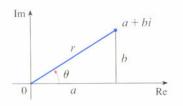
$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3} i}{2}$$

Notice that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation  $ax^2 + bx + c = 0$  with real coefficients a, b, and c are always complex conjugates. (If z is real,  $\overline{z} = z$ , so z is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.





# Polar Form

We know that any complex number z = a + bi can be considered as a point (a, b) and that any such point can be represented by polar coordinates  $(r, \theta)$  with  $r \ge 0$ . In fact,

 $a = r\cos\theta$   $b = r\sin\theta$ 

as in Figure 4. Therefore, we have

 $z = a + bi = (r\cos\theta) + (r\sin\theta)i$ 

Thus, we can write any complex number z in the form

$$z = r(\cos\theta + i\sin\theta)$$

where

$$r = |z| = \sqrt{a^2 + b^2}$$
 and  $\tan \theta = \frac{b}{a}$ 

The angle  $\theta$  is called the **argument** of z and we write  $\theta = \arg(z)$ . Note that  $\arg(z)$  is not unique; any two arguments of z differ by an integer multiple of  $2\pi$ .

**EXAMPLE 4** Write each of the following numbers in polar form. (a) z = 1 + i (b)  $w = \sqrt{3} - i$ 

#### SOLUTION

(a) We have  $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$  and  $\tan \theta = 1$ , so we can take  $\theta = \pi/4$ . Therefore, the polar form is

$$z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

(b) Here we have  $r = |w| = \sqrt{3 + 1} = 2$  and  $\tan \theta = -1/\sqrt{3}$ . Since w lies in the fourth quadrant, we take  $\theta = -\pi/6$  and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers z and w are shown in Figure 5.

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2)$$
  
=  $r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$ 

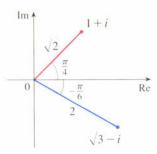


FIGURE 5

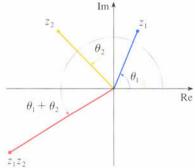
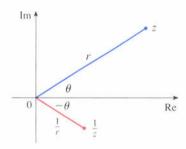


FIGURE 6





Therefore, using the addition formulas for cosine and sine, we have

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

This formula says that to multiply two complex numbers we multiply the moduli and add the arguments. (See Figure 6.)

A similar argument using the subtraction formulas for sine and cosine shows that to divide two complex numbers we divide the moduli and subtract the arguments.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0$$

In particular, taking  $z_1 = 1$  and  $z_2 = z$  (and therefore  $\theta_1 = 0$  and  $\theta_2 = \theta$ ), we have the following, which is illustrated in Figure 7.

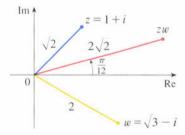
If 
$$z = r(\cos \theta + i \sin \theta)$$
, then  $\frac{1}{z} = \frac{1}{r}(\cos \theta - i \sin \theta)$ .

**EXAMPLE 5** Find the product of the complex numbers 1 + i and  $\sqrt{3} - i$  in polar form.

**SOLUTION** From Example 4 we have

$$1 + i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$
$$\sqrt{3} - i = 2 \left[ \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right]$$

and



So, by Equation 1,

$$(1+i)(\sqrt{3}-i) = 2\sqrt{2} \left[ \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \right] \\ = 2\sqrt{2} \left( \cos\frac{\pi}{12} + i\sin\frac{\pi}{12} \right)$$

FIGURE 8

This is illustrated in Figure 8.

Repeated use of Formula 1 shows how to compute powers of a complex number. If

	$z = r(\cos\theta + i\sin\theta)$
then	$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$
and	$z^3 = zz^2 = r^3(\cos 3\theta + i\sin 3\theta)$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

**2** De Moivre's Theorem If  $z = r(\cos \theta + i \sin \theta)$  and *n* is a positive integer, then

 $z^{n} = [r(\cos\theta + i\sin\theta)]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$ 

This says that to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n.

**EXAMPLE 6** Find  $(\frac{1}{2} + \frac{1}{2}i)^{10}$ .

**SOLUTION** Since  $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$ , it follows from Example 4(a) that  $\frac{1}{2} + \frac{1}{2}i$  has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2} \left( \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \right)$$

So by De Moivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10} \left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$
$$= \frac{2^5}{2^{10}} \left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An nth root of the complex number z is a complex number w such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

 $w = s(\cos \phi + i \sin \phi)$  and  $z = r(\cos \theta + i \sin \theta)$ 

and using De Moivre's Theorem, we get

$$s^{n}(\cos n\phi + i\sin n\phi) = r(\cos \theta + i\sin \theta)$$

The equality of these two complex numbers shows that

 $s^n = r$  or  $s = r^{1/n}$ 

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and  $\cos n\phi = \cos \theta$  and  $\sin n\phi = \sin \theta$ 

From the fact that sine and cosine have period  $2\pi$  it follows that

$$n\phi = \theta + 2k\pi \quad \text{or} \quad \phi = \frac{\theta + 2k\pi}{n}$$
  
Thus  $w = r^{1/n} \left[ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right) \right]$ 

Since this expression gives differents values of w for k = 0, 1, 2, ..., n - 1, we have the following.

**E** Roots of a Complex Number Let  $z = r(\cos \theta + i \sin \theta)$  and let *n* be a positive integer. Then *z* has the *n* distinct *n*th roots

$$w_k = r^{1/n} \left[ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right) \right]$$

where k = 0, 1, 2, ..., n - 1.

Notice that each of the *n*th roots of *z* has modulus  $|w_k| = r^{1/n}$ . Thus, all the *n*th roots of *z* lie on the circle of radius  $r^{1/n}$  in the complex plane. Also, since the argument of each successive *n*th root exceeds the argument of the previous root by  $2\pi/n$ , we see that the *n*th roots of *z* are equally spaced on this circle.

**EXAMPLE 7** Find the six sixth roots of z = -8 and graph these roots in the complex plane.

**SOLUTION** In trigonometric form,  $z = 8(\cos \pi + i \sin \pi)$ . Applying Equation 3 with n = 6, we get

$$w_k = 8^{1/6} \left( \cos \frac{\pi + 2k\pi}{6} + i \sin \frac{\pi + 2k\pi}{6} \right)$$

We get the six sixth roots of -8 by taking k = 0, 1, 2, 3, 4, 5 in this formula:

$$w_{0} = 8^{1/6} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \sqrt{2} \left( \frac{\sqrt{3}}{2} + \frac{1}{2} i \right)$$

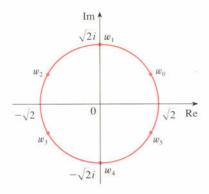
$$w_{1} = 8^{1/6} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = \sqrt{2} i$$

$$w_{2} = 8^{1/6} \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \sqrt{2} \left( -\frac{\sqrt{3}}{2} + \frac{1}{2} i \right)$$

$$w_{3} = 8^{1/6} \left( \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) = \sqrt{2} \left( -\frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$$

$$w_{4} = 8^{1/6} \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = -\sqrt{2} i$$

$$w_{5} = 8^{1/6} \left( \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right) = \sqrt{2} \left( -\frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$$



**FIGURE 9** The six sixth roots of z = -8

All these points lie on the circle of radius  $\sqrt{2}$  as shown in Figure 9.

4

# **Complex Exponentials**

We also need to give a meaning to the expression  $e^z$  when z = x + iy is a complex number. The theory of infinite series as developed in Chapter 8 can be extended to the case where the terms are complex numbers. Using the Taylor series for  $e^x$ (Equation 11 in Section 8.7) as our guide, we define

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^{n}}{n!} = 1 + z + \frac{z^{2}}{2!} + \frac{z^{3}}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

If we put z = iy, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1$$
,  $i^3 = i^2 i = -i$ ,  $i^4 = 1$ ,  $i^5 = i$ , ...

we get

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \dots$$
  
=  $1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \dots$   
=  $\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)$   
=  $\cos y + i \sin y$ 

Here we have used the Taylor series for cos y and sin y (Equations 16 and 15 in Section 8.7). The result is a famous formula called Euler's formula:

 $e^{iy} = \cos y + i \sin y$ 

Combining Euler's formula with Equation 5, we get

$$e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

(b)  $e^{-1+i\pi/2}$ **EXAMPLE 8** ■ Evaluate: (a)  $e^{i\pi}$ 

#### SOLUTION

7

(a) From Euler's formula (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos\theta + i\sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

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Exercises · · · · ·

**1–14** Evaluate the expression and write your answer in the form a + bi.

 1. (3 + 2i) + (7 - 3i) 2. (1 + i) - (2 - 3i) 

 3. (3 - i)(4 + i) 4. (4 - 7i)(1 + 3i) 

 5.  $\overline{12 + 7i}$  6.  $\overline{2i(\frac{1}{2} - i)}$  

 7.  $\frac{2 + 3i}{1 - 5i}$  8.  $\frac{5 - i}{3 + 4i}$  

 9.  $\frac{1}{1 + i}$  10.  $\frac{3}{4 - 3i}$  

 11.  $i^3$  12.  $i^{100}$  

 13.  $\sqrt{-25}$  14.  $\sqrt{-3}\sqrt{-12}$ 

**15–17** Find the complex conjugate and the modulus of the given number.

- **15.** 3 + 4i **16.**  $\sqrt{3} i$  **17.** -4i
- 18. Prove the following properties of complex numbers.
  - (a)  $\overline{z + w} = \overline{z} + \overline{w}$ (b)  $\overline{zw} = \overline{z} \overline{w}$
  - (c)  $\overline{z^n} = \overline{z}^n$ , where *n* is a positive integer [*Hint*: Write z = a + bi, w = c + di.]

**19–24** Find all solutions of the equation.

<b>19.</b> $4x^2 + 9 = 0$	<b>20.</b> $x^4 = 1$
<b>21.</b> $x^2 - 8x + 17 = 0$	<b>22.</b> $x^2 - 4x + 5 = 0$
<b>23.</b> $z^2 + z + 2 = 0$	<b>24.</b> $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$
x x x x x x	1 A A A A A A A A A A A A A A A A A A A

**25–28** Write the number in polar form with argument between 0 and  $2\pi$ .

<b>25.</b> $-3 + 3i$	<b>26.</b> $1 - \sqrt{3} i$
<b>27.</b> 3 + 4 <i>i</i>	<b>28.</b> 8 <i>i</i>

**29–32** Find polar forms for zw, z/w, and 1/z by first putting z and w into polar form.

**29.** 
$$z = \sqrt{3} + i$$
,  $w = 1 + \sqrt{3} i$   
**30.**  $z = 4\sqrt{3} - 4i$ ,  $w = 8i$   
**31.**  $z = 2\sqrt{3} - 2i$ ,  $w = -1 + i$   
**32.**  $z = 4(\sqrt{3} + i)$ ,  $w = -3 - 3i$ 

**33–36** Find the indicated power using De Moivre's Theorem.

**33.**  $(1 + i)^{20}$  **34.**  $(1 - \sqrt{3}i)^5$  **35.**  $(2\sqrt{3} + 2i)^5$ **36.**  $(1 - i)^8$ 

**37–40** Find the indicated roots. Sketch the roots in the complex plane.

37. The eigh	th roots of 1	<b>38.</b> The fifth roots of 32
<b>39.</b> The cube	e roots of <i>i</i>	<b>40.</b> The cube roots of $1 + i$
253 85 98	53 B.	8 (8) 8 8 9 8 8
<b>41–46</b> ■ Wri	te the number	r in the form $a + bi$ .
<b>41.</b> $e^{i\pi/2}$	<b>42.</b> $e^{2\pi i}$	<b>43.</b> $e^{i3\pi/4}$
<b>44.</b> $e^{-i\pi}$	<b>45.</b> $e^{2+i\pi}$	<b>46.</b> $e^{1+2i}$

- **47.** Use De Moivre's Theorem with n = 3 to express  $\cos 3\theta$  and  $\sin 3\theta$  in terms of  $\cos \theta$  and  $\sin \theta$ .
- **48.** Use Euler's formula to prove the following formulas for cos *x* and sin *x*:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- **49.** If u(x) = f(x) + ig(x) is a complex-valued function of a real variable x and the real and imaginary parts f(x) and g(x) are differentiable functions of x, then the derivative of u is defined to be u'(x) = f'(x) + ig'(x). Use this together with Equation 7 to prove that if  $F(x) = e^{rx}$ , then  $F'(x) = re^{rx}$  when r = a + bi is a complex number.
- **50.** (a) If *u* is a complex-valued function of a real variable, its indefinite integral  $\int u(x) dx$  is an antiderivative of *u*. Evaluate

$$\int e^{(1+i)x} dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

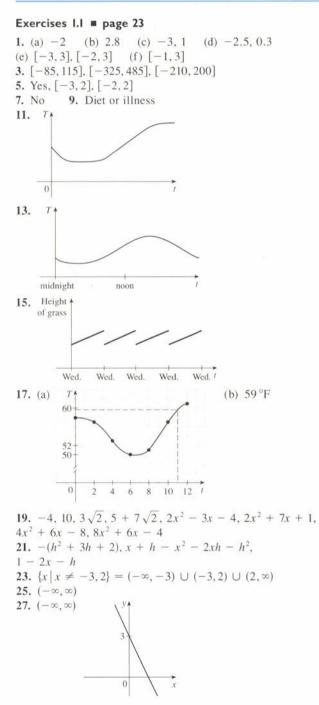
 $\int e^x \cos x \, dx \qquad \text{and} \qquad \int e^x \sin x \, dx$ 

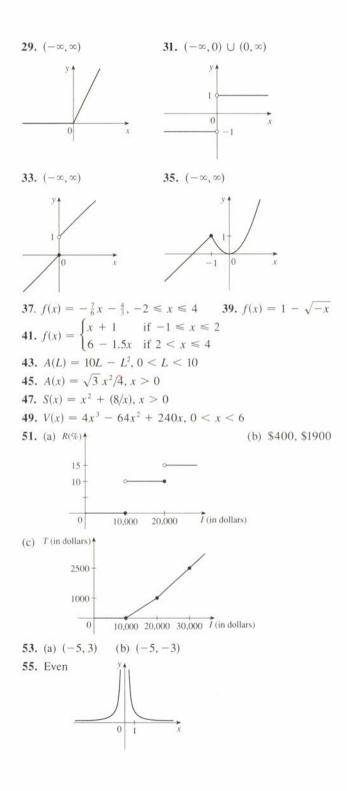
Compare with the method used in Example 4 in Section 5.6.

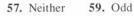


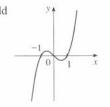
Answers to Odd-Numbered Exercises · · · · ·

#### CHAPTER I







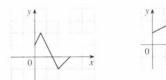


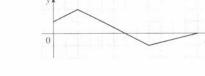
## Exercises I.2 = page 38

(a) Root
 (b) Algebraic
 (c) Polynomial (degree 9)
 (d) Rational
 (e) Trigonometric
 (f) Logarithmic

**3.** (a) g (b) h (c) f

5. (a) y = f(x) + 3 (b) y = f(x) - 3(c) y = f(x - 3) (d) y = f(x + 3) (e) y = -f(x)(f) y = f(-x) (g) y = 3f(x) (h)  $y = \frac{1}{3}f(x)$ 7. (a) (b)

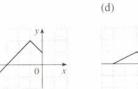


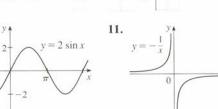


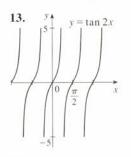
0

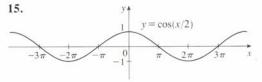
(c)

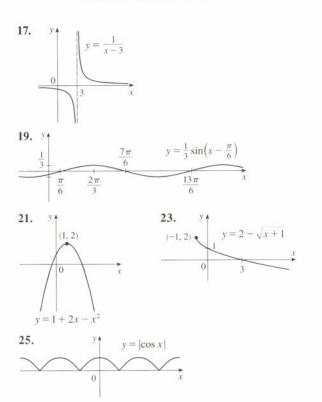
9.



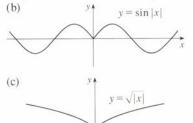




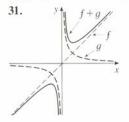




27. (a) The portion of the graph of y = f(x) to the right of the y-axis is reflected in the y-axis.



**29.**  $(f + g)(x) = x^3 + 5x^2 - 1, (-\infty, \infty)$  $(f - g)(x) = x^3 - x^2 + 1, (-\infty, \infty)$  $(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2, (-\infty, \infty)$  $(f/g)(x) = (x^3 + 2x^2)/(3x^2 - 1), \{x \mid x \neq \pm 1/\sqrt{3}\}$ 



**33.**  $(f \circ g)(x) = 3(6x^2 + 7x + 2), (-\infty, \infty)$   $(g \circ f)(x) = 6x^2 - 3x + 2, (-\infty, \infty)$   $(f \circ f)(x) = 8x^4 - 8x^3 + x, (-\infty, \infty)$  $(g \circ g)(x) = 9x + 8, (-\infty, \infty)$ 

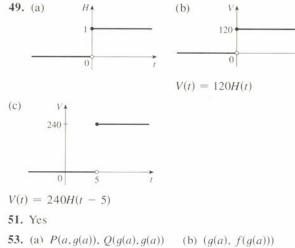
**35.** 
$$(f \circ g)(x) = \sqrt{-x}, (-\infty, 0]$$
  
 $(g \circ f)(x) = \sqrt{1 - \sqrt{x^2 - 1}}, [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$   
 $(f \circ f)(x) = \sqrt{x^2 - 2}, (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$   
 $(g \circ g)(x) = \sqrt{1 - \sqrt{1 - x}}, [0, 1]$   
**37.**  $(f \circ g \circ h)(x) = 1/(x^2 + 2)^3$   
**39.**  $g(x) = x - 9, f(x) = x^5$   
**41.**  $g(x) = x^2, f(x) = x/(x + 4)$   
**43.**  $h(x) = x^2, g(x) = 3^x, f(x) = 1 - x$   
**45.** (a) 4 (b) 3 (c) 0 (d) Does not exist;  $f(6) =$   
is not in the domain of g. (e) 4 (f) -2

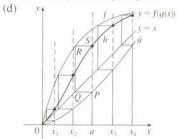
47. (a) r(t) = 60t

(b)  $(A \circ r)(t) = 3600\pi t^2$ ; the area of the circle as a function of time

6

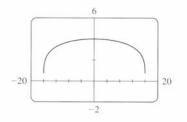
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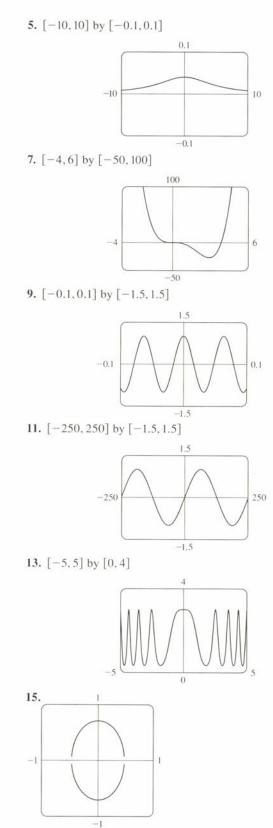


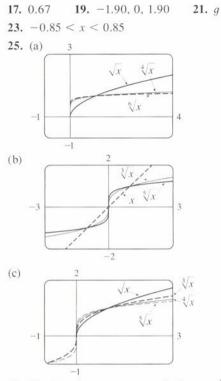


# Exercises I.3 ■ page 47

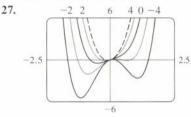
- 1. (c)
- **3.** [−20, 20] by [−2, 6]







(d) Graphs of even roots are similar to  $\sqrt{x}$ , graphs of odd roots are similar to  $\sqrt[3]{x}$ . As *n* increases, the graph of  $y = \sqrt[n]{x}$  becomes steeper near 0 and flatter for x > 1.

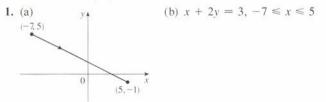


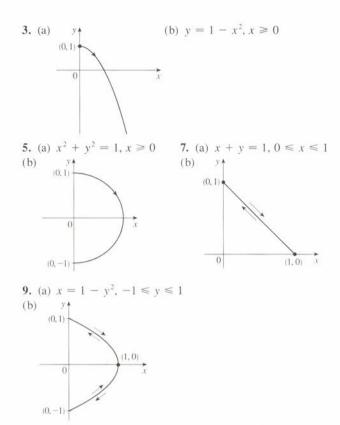
If c < 0, the graph has three humps: two minimum points and a maximum point. These humps get flatter as cincreases until at c = 0 two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

29. The hump gets larger and moves to the right.

**31.** If c < 0, the loop is to the right of the origin; if c > 0, the loop is to the left. The closer c is to 0, the larger the loop.

#### Exercises I.4 = page 53



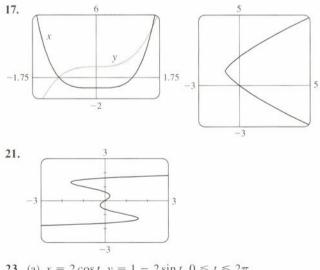


11. Moves counterclockwise along the circle  $x^2 + y^2 = 1$ from (-1,0) to (1,0)

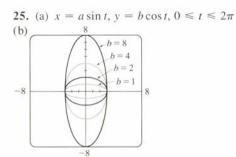
**13.** Moves once clockwise around the ellipse

 $(x^{2}/4) + (y^{2}/9) = 1$ , starting and ending at (0, 3)

**15.** Moves down the first quadrant branch of the hyperbola xy = 1 from  $(\frac{1}{2}, 1)$  to  $(\sin 1, \csc 1)$ 

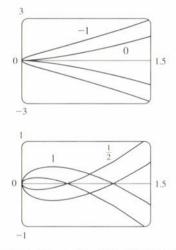


**23.** (a)  $x = 2\cos t$ ,  $y = 1 - 2\sin t$ ,  $0 \le t \le 2\pi$ (b)  $x = 2\cos t$ ,  $y = 1 + 2\sin t$ ,  $0 \le t \le 6\pi$ (c)  $x = 2\cos t$ ,  $y = 1 + 2\sin t$ ,  $\pi/2 \le t \le 3\pi/2$ 



(c) As b increases, the ellipse stretches vertically. **29.**  $x = a \cos \theta$ ,  $y = b \sin \theta$ ;  $(x^2/a^2) + (y^2/b^2) = 1$ , ellipse **31.**  y + 2a y + 2a y + 3a y + 3ay

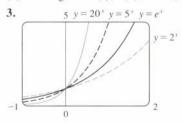
**33.** For c = 0, there is a cusp; for c > 0, there is a loop whose size increases as c increases.



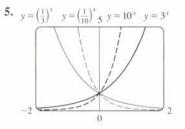
**35.** As *n* increases, the number of oscillations increases; *a* and *b* determine the width and height.

#### Exercises I.5 = page 62

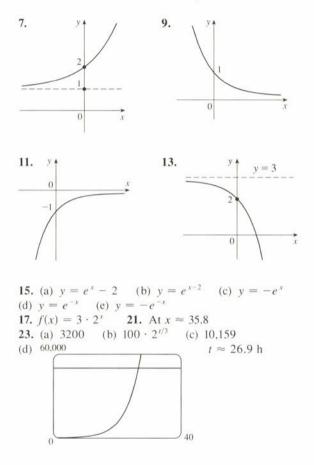
**1.** (a)  $f(x) = a^x$ , a > 0 (b)  $\mathbb{R}$  (c)  $(0, \infty)$ (d) See Figures 4(c), 4(b), and 4(a), respectively.



All approach 0 as  $x \to -\infty$ , all pass through (0, 1), and all increase. The larger the base, the faster the rate of increase for x > 0.

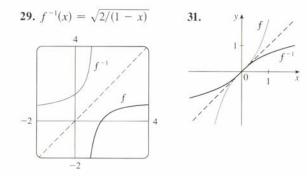


The functions with base greater than 1 are increasing and those with base less than 1 are decreasing. The latter are reflections of the former about the y-axis.



# Exercises 1.6 = page 73

1. (a) See Definition 1. (b) It must pass the Horizontal Line Test. 3. No. 5. No 7. Yes 9. Yes 11. No 15. No 19. 0 13. No 17. 2 **21.**  $F = \frac{9}{5}C + 32$ ; the Fahrenheit temperature as a function of the Celsius temperature;  $(-273.15, \infty)$ **23.**  $f^{-1}(x) = (5x - 1)/(2x + 3)$ **25.**  $f^{-1}(x) = (x^2 - 2)/5, x \ge 0$ **27.**  $y = e^x - 3$ 

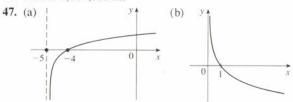


**33.** (a) It's defined as the inverse of the exponential function with base *a*, that is,  $\log_a x = y \iff a^y = x$ .

(b)  $(0, \infty)$  (c)  $\mathbb{R}$  (d) See Figure 13. **35.** (a) 6 (b) -2 **37.** (a) 2 (b) 2 **39.**  $3 \ln 2$  **41.** (a) 2.321928 (b) 2.025563 **43.**   $y = \log_{1.5} x$   $y = \log_{1.5} x$   $y = \log_{1.5} x$   $y = \log_{1.5} x$  $y = \log_{1.5} x$ 

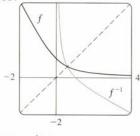
All graphs approach  $-\infty$  as  $x \to 0^+$ , all pass through (1,0), and all increase. The larger the base, the slower the rate of increase for x > 0.

45. About 1,084,588 mi



**49.** (a) 4 ln 2 (b) 1/e

**51.** (a)  $5 + \log_2 3$  or  $5 + (\ln 3)/\ln 2$  (b)  $\frac{1}{2}(1 + \sqrt{1 + 4e})$ **53.** 4



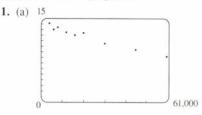
55.  $f^{-1}(x) =$ 

 $-(\sqrt[3]{4}/6)(\sqrt[3]{A}-27x^{2}+20)-\sqrt[3]{A}+27x^{2}-20)+\sqrt[3]{2}),$ where  $A = 3\sqrt{3}\sqrt{27x^{4}-40x^{2}+16}$ ; two of the expressions are complex.

57. (a)  $f^{-1}(n) = (3/\ln 2) \ln(n/100)$ ; the time elapsed when there are *n* bacteria (b) After about 26.9 h

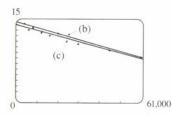
**59.** (a)  $y = \ln x + 3$  (b)  $y = \ln(x + 3)$  (c)  $y = -\ln x$ (d)  $y = \ln(-x)$  (e)  $y = e^x$  (f)  $y = e^{-x}$ (g)  $y = -e^x$  (h)  $y = e^x - 3$ 

#### Exercises 1.7 ■ page 82



Yes, appropriate

(b) y = -0.000105357x + 14.521429



(c) y = -0.0000997855x + 13.950764 [See graph in (b).] (d) About 11.5 per 100 population (e) About 6% **3.** (a)  $y = 301.813054e^{-0.198762x}$ ;  $y = -0.00243042x^4 + 0.135159x^3 - 2.014322x^2 - 4.055294x + 199.092227$ (b) Exponential model: about 202.8 million tons in 1972, 27.8 million tons in 1982. Polynomial model: about 184.0 million tons in 1972, 43.5 million tons in 1982

**5.**  $y = 0.0272238976x^3 - 162.1725931x^2 + 322.017.833x - 213.136,407.3;$ **1982 1992 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995 1995**

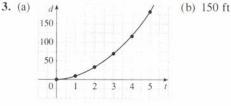
# Chapter | Review = page 84

## True-False Quiz

1. False 3. False 5. True 7. True

#### Exercises

(a) 2.7 (b) 2.3, 5.6 (c) [-6, 6] (d) [-4, 4]
 (e) [-4, 4] (f) No; *f* fails the Horizontal Line Test.
 (g) Odd; its graph is symmetric about the origin.



**5.**  $\left[-2\sqrt{3}/3, 2\sqrt{3}/3\right], \left[0, 2\right]$  **7.**  $(-\infty, \infty), (0, 1]$ 

9. (a) Shift the graph 8 units upward.

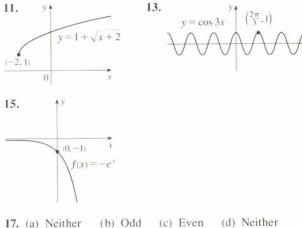
(b) Shift the graph 8 units to the left.

(c) Stretch the graph vertically by a factor of 2, then shift it 1 unit upward.

(d) Shift the graph 2 units to the right and 2 units downward.

(e) Reflect the graph about the *x*-axis.

(f) Reflect the graph about the line y = x (assuming f is one-to-one).



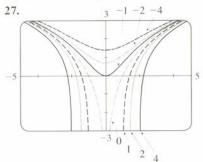
17. (a) Neither (b) Odd (c) Even (d) Neither 19.  $(f \circ g)(x) = \ln(x^2 - 9), (-\infty, -3) \cup (3, \infty)$   $(g \circ f)(x) = (\ln x)^2 - 9, (0, \infty)$   $(f \circ f)(x) = \ln \ln x, (1, \infty)$  $(g \circ g)(x) = (x^2 - 9)^2 - 9, (-\infty, \infty)$ 

**21.** 1 **23.** (a) 9 (b) 2

25. (a)  $\frac{1}{16}$  g (b)  $m(t) = 2^{-t/4}$ 

(c)  $t(m) = -4 \log_2 m$ ; the time elapsed when there are m grams of <sup>100</sup> Pd

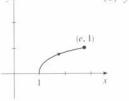
(d) About 26.6 days

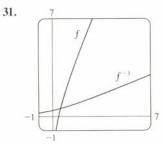


For c < 0, *f* is defined everywhere. As *c* increases, the dip at x = 0 becomes deeper. For  $c \ge 0$ , the graph has asymptotes at  $x = \pm \sqrt{c}$ .

29. (a) y

(b)  $y = \sqrt{\ln x}$ 

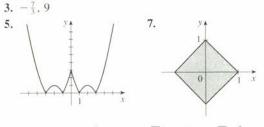




**33.** y = 0.263x - 450.034; about 76.0 years

#### Principles of Problem Solving = page 92

1.  $a = 4\sqrt{h^2 - 16}/h$ , where *a* is the length of the altitude and *h* is the length of the hypotenuse



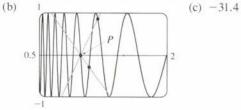
**9.** 5 **11.**  $x \in [-1, 1 - \sqrt{3}] \cup (1 + \sqrt{3}, 3]$ **17.**  $f_n(x) = x^{2^{n+1}}$ 

#### CHAPTER 2

#### Exercises 2.1 ■ page 100

**1.** (a) -0.43, -0.35, 0.2, 0.8, 1.1 (b) 0.5 (c) 0.573. (a) (i) 0.236068 (ii) 0.242641 (iii) 0.248457 (iv) 0.249844 (v) 0.249984 (vi) 0.267949 (vii) 0.258343 (viii) 0.251582 (ix) 0.250156 (x) 0.250016 (b)  $\frac{1}{4}$  (c)  $y = \frac{1}{4}x + 1$ 5. (a) (i) -32 ft/s (ii) -25.6 ft/s (iii) -24.8 ft/s (iv) -24.16 ft/s (b) -24 ft/s 7. (a) (i)  $\frac{13}{6}$  ft/s (ii)  $\frac{7}{6}$  ft/s (iii)  $\frac{19}{24}$  ft/s (iv)  $\frac{331}{600}$  ft/s (b)  $\frac{1}{2}$  ft/s (c) 54 4-(i) (ii) 3 (iii) 2 (iv) 1.5 2 3 t (d) 2 tangent 1.5 2

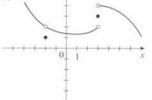
**9.** (a) 0, 1.7321, -1.0847, -2.7433, 4.3301, -2.8173, 0, -2.1651, -2.6061, -5, 3.4202; no



# Exercises 2.2 = page 109

#### 1. Yes

**3.** (a) 3 (b) 2 (c) -2 (d) Does not exist (e) 1 (f) -1 (g) -1 (h) -1 (i) -3**5.** (a) 1 (b) 0 (c) Does not exist **7.**  $y_{1}$ 

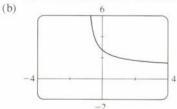


**9.** 0.806452, 0.641026, 0.510204, 0.409836, 0.369004, 0.336689, 0.165563, 0.193798, 0.229358, 0.274725, 0.302115, 0.330022;  $\frac{1}{3}$ 

**11.** 0.459698, 0.489670, 0.493369, 0.496261, 0.498336, 0.499583, 0.499896, 0.499996;  $\frac{1}{2}$ 

**13.** (a) 4

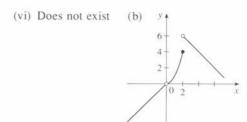
**15.** (a) 2.71828; it is the value of e



17. (a) 0.998000, 0.638259, 0.358484, 0.158680, 0.038851, 0.008928, 0.001465; 0
(b) 0.000572, -0.000614, -0.000907, -0.000978, -0.000993, -0.001000; -0.001
19. Within 0.182; within 0.095

# Exercises 2.3 = page II8

**1.** (a) 5 (b) 9 (c) 2 (d)  $-\frac{1}{3}$  (e)  $-\frac{3}{8}$  (f) 0 (g) Does not exist (h)  $-\frac{6}{11}$ **3.** 75 **5.** -3 **7.** 0 **9.** Does not exist **11.** -10**13.** 6 **15.**  $-\sqrt{2}/4$  **17.**  $\frac{1}{2}$  **21.** 1 **25.** 0 **27.** Does not exist **29.** (a) (i) 0 (ii) 0 (iii) 1 (iv) 4 (v) 6

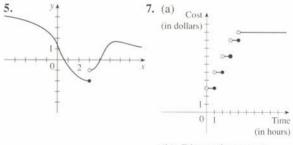


**31.** (a) (i) -2 (ii) Does not exist (iii) -3 (b) (i) n - 1 (ii) n (c) a is not an integer. **39.** 15; -1

#### Exercises 2.4 ■ page 128

1.  $\lim_{x \to 4} f(x) = f(4)$ 

3. (a) -5 (jump), -3 (infinite), -1 (undefined),
3 (removable), 5 (infinite), 8 (jump), 10 (undefined)
(b) -5, left; -3, left; -1, neither; 3, neither; 5, neither;
8, right; 10, neither

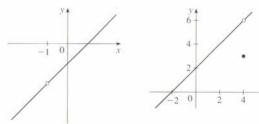


(b) Discontinuous at 
$$t = 1, 2, 3, 4$$

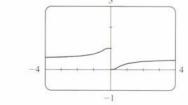
6 .

**11.** f(-1) is not defined

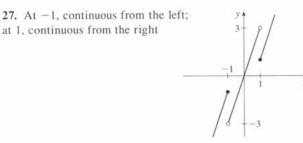
**13.**  $\lim_{x \to 4} f(x) \neq f(4)$ 



**15.**  $\{x \mid x \neq -\frac{1}{2}, \frac{1}{3}\}$  **17.**  $\mathbb{R}$  **19.**  $(-\infty, -1) \cup (1, \infty)$ **21.** x = 0 3



**23.**  $\frac{7}{3}$  **25.** 1



#### 29. $\frac{1}{2}$ **37.** (b) (0.44, 0.45)

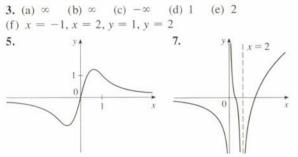
at 1, continuous from the right

43. Yes 39. (b) 5.016

#### Exercises 2.5 ■ page 140

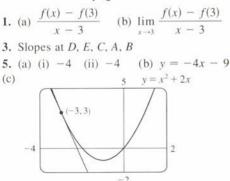
1. (a) As x approaches 2, f(x) becomes large. (b) As x approaches 1 from the right, f(x) becomes large negative.

- (c) As x becomes large, f(x) approaches 5.
- (d) As x becomes large negative, f(x) approaches 3.

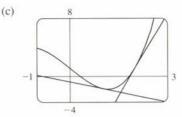


**13.**  $x \approx -1.62, x \approx 0.62, x = 1, y = 1$ 9. 0 11. 1.5 21. -3 23.  $\frac{3}{2}$ 15. œ 17.  $-\infty$ 19. 0 25. Does not exist 27. ∞ 29. 0 **31.** x = -1, x = 1, y = 133. (a) IV (b) III (c) II (d) VI (e) I (f) V **35.**  $(2 - x)/[x^2(x - 3)]$ **37.** (a) 0 (b)  $\infty$  or  $-\infty$ **39.** 4 **41.** (b) Concentration approaches that of the brine. **43.** (b) 23.03 (c) Yes,  $x > 10 \ln 10$ 

#### Exercises 2.6 ■ page 149

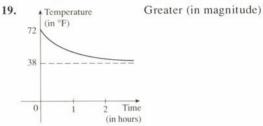


7.  $y = \frac{1}{2}x + \frac{1}{2}$  9.  $y = \frac{1}{4}x + \frac{3}{4}$ **11.** (a)  $3a^2 - 4$  (b) y = -x - 1, y = 8x - 15



**13.** (a) 0 (b) C (c) Speeding up, slowing down, (d) The car stopped. neither

**17.**  $12a^2 + 6$ , 18 m/s, 54 m/s, 114 m/s 15. -24 ft/s

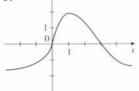


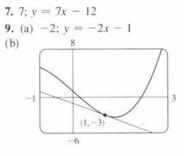
**21.** (a) (i) -1.2 °C/h (ii) -1.25 °C/h (iii) -1.3 °C/h (b) -1.9 °C/h

23. (a) (i) \$20.25/unit (ii) \$20.05/unit (b) \$20/unit

# Exercises 2.7 = page 156

**1.** The line from (2, f(2)) to (2 + h, f(2 + h))**3.** g'(0), 0, g'(4), g'(2), g'(-2)5. y A





**11.** 3.296 **13.** 1 - 4a **15.**  $-(a^2 + 1)/(a^2 - 1)^2$ **17.**  $f(x) = \sqrt{x}, a = 1$  **19.**  $f(x) = x^9, a = 1$ **21.**  $f(x) = \sin x$ ,  $a = \pi/2$  **23.** -2 m/s

25. (a) The rate at which the cost is changing per ounce of gold produced; dollars per ounce

(b) When the 800th ounce of gold is produced, the cost of production is \$17/oz.

(c) Decrease in the short term; increase in the long term

**27.** (a) The rate at which the fuel consumption is changing with respect to speed; gal/mi

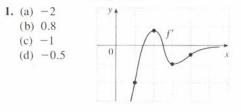
(b) The fuel consumption is decreasing by 0.05 gal/mi as the car's speed reaches 20 mi/h.

29. The price of coffee beans was rising by about

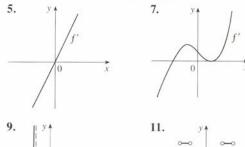
\$0.54/kg/year in 1983 and falling by about \$0.065/kg/year in 1990.

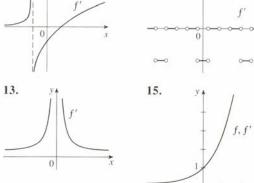
31. Does not exist

# Exercises 2.8 ■ page 168









 $f'(x) = e^{x}$  **17.** (a) 0, 1, 2, 4 (b) -1, -2, -4 (c) f'(x) = 2x **19.** f'(x) = 5,  $\mathbb{R}$ ,  $\mathbb{R}$  **21.**  $g'(x) = 1/\sqrt{1+2x}$ ,  $\left[-\frac{1}{2},\infty\right)$ ,  $\left(-\frac{1}{2},\infty\right)$  **23.**  $f'(x) = -2/(x-1)^{2}$ ,  $\{x \mid x \neq 1\}$ ,  $\{x \mid x \neq 1\}$  **25.** (a)  $f'(x) = 1 + 2/x^{2}$  **27.** (a) The rate at which the unemployment rate is

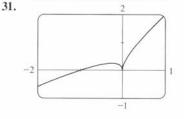
0

changing, in percent unemployed per year

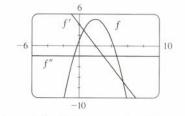
t	1983	1984	1985	1986	1987
U'(t)	-2.1	-1.2	-0.25	-0.5	-0.75

t	1988	1989	1990	1991	1992
U'(t)	-0.45	0	0.7	0.95	0.7

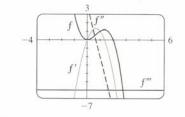
**29.** 4 (discontinuity); 8 (corner); -1, 11 (vertical tangents)

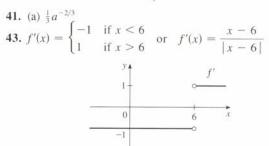


Differentiable at -1; not differentiable at 0 33. a = f, b = f', c = f''35. a =acceleration, b = velocity, c = position 37. f'(x) = 4 - 2x, f''(x) = -2



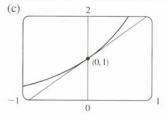
**39.** 
$$f'(x) = 4x - 3x^2$$
,  $f''(x) = 4 - 6x$ ,  $f'''(x) = -6$ ,  $f^{(4)}(x) = 0$ 





Exercises 2.9 = page 174

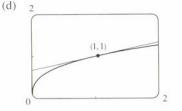
**1.** (a) 1.0986 (b) 1.0549, 1.1099



Less; the tangent line lies below the curve.

**3.** (a)  $\frac{1}{3}$  (b)  $\frac{1}{3}x + \frac{2}{3}$ 

(c) 0.83333, 0.96667, 0.99667, 1.00333, 1.03333, 1.16667, 1.33333; overestimates; those for 0.99 and 1.01



The tangent lines lie above the curve.

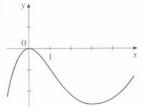
**5.** (a) 2 (b) 0.8, 0.9, 0.98, 1.02, 1.1, 1.2; underestimates (c) 2



7. 148°F; underestimate
9. 32.5%, 35%
11. (a) 4.8, 5.2 (b) Too large

#### Exercises 2.10 = page 180

(a) Increasing on (-∞, 0) and (3,∞); decreasing on (0, 3)
 (b) Local maximum at 0, local minimum at 3
 (c) y↑



Increasing on (2, 5); decreasing on (-∞, 2) and (5,∞)
 If D(t) is the size of the deficit as a function of time, then at the time of the speech D'(t) > 0, but D"(t) < 0.</li>
 (a) The rate starts small, grows rapidly, levels off, then decreases and becomes negative.

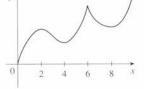
(b) (1932, 2.5) and (1937, 4.3); the rate of change of population density starts to decrease in 1932 and starts to increase in 1937.

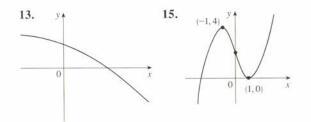
9. K(3) - K(2); concave downward

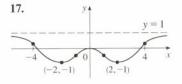
**11.** (a) Increasing on (0, 2), (4, 6), and  $(8, \infty)$ ; decreasing on (2, 4) and (6, 8)

(b) Local maxima at x = 2, 6; minima at x = 4, 8

(c) Concave upward on (3, 6) and  $(6, \infty)$ , concave downward on (0, 3) (d) 3 (e)  $y \uparrow$ 

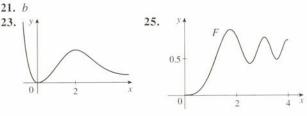






**19.** (a) Increasing on  $(0, \infty)$ ; decreasing on  $(-\infty, 0)$ 

(b) Minimum at x = 0



#### Chapter 2 Review page 182

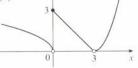
#### **True-False Quiz**

 1. False
 3. True
 5. False
 7. True
 9. False

 11. True
 13. False
 15. False
 15. False
 15. False

#### Exercises

1. (a) (i) 3 (ii) 0 (iii) Does not exist;  $\lim_{x\to -3^-} f(x) \neq \lim_{x\to -3^+} f(x)$ (iv) 2 (v)  $\infty$  (vi)  $-\infty$  (vii) 4 (viii) -17.0 9. ∞ 11. -1 13. 0 3. 0 5. 2 17. x = 0, y = 019. 1 15. 0 21. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 0 (v) 0 (vi) 0 (b) At 0 and 3 (c)



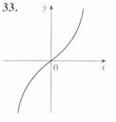
**25.** (a) (i) 3 m/s (ii) 2.75 m/s (iii) 2.625 m/s (iv) 2.525 m/s (b) 2.5 m/s **27.** f''(5), 0, f'(5), f'(2), 1, f'(3)**29.** (a) -0.736 (b)  $y \approx -0.736x + 1.104$ (c) 1.5

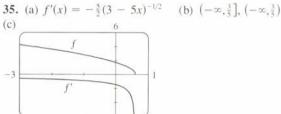


**31.** (a) The rate at which the cost changes with respect to the interest rate; dollars/(percent per year)

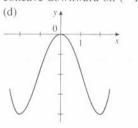
(b) As the interest rate increases past 10%, the cost is increasing at a rate of 1200/(percent per year).

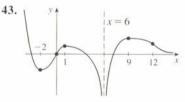
(c) Always positive





- **37.** -4 (discontinuity), 1 (corner), 2 (discontinuity), 5 (vertical tangent)
- **39.** (a) 1 (b) x + 1 (c) 0.8, 0.9, 0.99, 1.01, 1.1, 1.2 (d) Underestimates; those for  $e^{-0.01}$  and  $e^{0.01}$
- **41.** (a) Increasing on (-2, 0) and  $(2, \infty)$ ; decreasing on  $(-\infty, -2)$  and (0, 2)
- (b) Maria (0, 2) and (0, 2)
- (b) Maximum at 0; minima at -2 and 2
  (c) Concave upward on (-∞, -1) and (1,∞); concave downward on (-1, 1)





45. (a) About 35 ft/s (b) About (8, 180)(c) The point at which the car's velocity is maximized

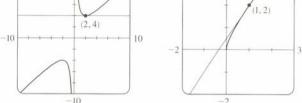
#### Focus on Problem Solving = page 187

**1.**  $\frac{2}{3}$  **3.** -4 **5.** 1 **7.**  $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ **9.** (b) Yes (c) Yes; no **11.**  $(\pm\sqrt{3}/2, \frac{1}{4})$ **13.** (a) 0 (b) 1 (c)  $f'(x) = x^2 + 1$  **15.**  $\frac{3}{4}$ 

#### CHAPTER 3

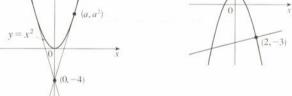
#### Exercises 3.1 page 199

1. (a) See Definition of the Number *e* (page 198). (b) 0.99, 1.03; 2.7 < *e* < 2.8 3.  $y' = 8x^7$  5.  $y' = -\frac{2}{5}x^{-7/5}$  7. f'(x) = 2x - 109.  $V'(r) = 4\pi r^2$  11.  $Y'(t) = -54t^{-10}$ 13.  $F'(x) = 12,288x^2$  15.  $g'(x) = 2x - (2/x^3)$ 17.  $y' = \frac{3}{2}\sqrt{x} + (2/\sqrt{x}) - 3/(2x\sqrt{x})$  19.  $y' = 3 + 2e^x$ 21.  $4x - 4x^3$  23.  $45x^{14} - 15x^2$  25.  $1 - x^{-2/3}$ 27. (a) 0.264 (b)  $2^{2/5}/5 \approx 0.263902$ 29. y = 4 31.  $y = \frac{3}{2}x + \frac{1}{2}$ 



- **33.**  $f'(x) = 4x^3 9x^2 + 16$ ,  $f''(x) = 12x^2 18x$ **35.**  $f'(x) = 2 - \frac{15}{4}x^{-1/4}$ ,  $f''(x) = \frac{15}{16}x^{-5/4}$
- **35.**  $f(x) = 2 \frac{1}{4}x^{-1}$ ,  $f'(x) = \frac{1}{16}x^{-1}$
- **37.** (a)  $v(t) = 3t^2 3$ , a(t) = 6t (b)  $12 \text{ m/s}^2$
- (c)  $a(1) = 6 \text{ m/s}^2$
- 39. (a) 16 million/year; 80 million/year
- (b)  $P'(t) = 3at^2 + 2bt + c$
- (c) 14.0 million/year (smaller); 78.8 million/year (smaller)
- (d) 86.5 million/year
- **41.**  $(\ln \frac{3}{2}, \infty)$  **43.**  $(1, 0), (-\frac{1}{3}, \frac{32}{27})$

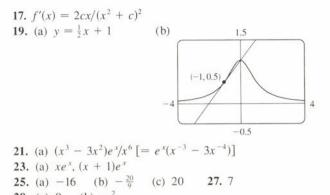
**47.** 
$$(\pm 2, 4)$$
  
**49.**  $y = \frac{1}{4}x - \frac{7}{2}$   
**49.**  $y = \frac{1}{4}x - \frac{7}{2}$ 



**53.**  $P(x) = x^2 - x + 3$  **55.** (a)  $F(x) = \frac{1}{3}x^3 + C$ , *C* any real number; infinitely many (b)  $F(x) = \frac{1}{4}x^4 + C$ ,  $\frac{1}{5}x^5 + C$ , *C* any real number (c)  $F(x) = x^{n+1}/(n+1) + C$ , *C* any real number **57.**  $y = 2x^2 - x$  **59.** 1000

#### Exercises 3.2 page 206

1.  $y' = 5x^4 + 3x^2 + 2x$ 3.  $f'(x) = x(x + 2)e^x$ 5.  $y' = (x - 2)e^x/x^3$ 7.  $h'(x) = -3/(x - 1)^2$ 9.  $G'(s) = (2s + 1)(s^2 + 2) + (s^2 + s + 1)(2s)$   $[= 4s^3 + 3s^2 + 6s + 2]$ 11.  $y' = \frac{3}{2}\sqrt{x} + (2/\sqrt{x}) - 3/(2x\sqrt{x})$ 13.  $y' = (r^2 - 2)e^r$ 15.  $y' = -(4x^3 + 2x)/(x^4 + x^2 + 1)^2$ 



**29.** (a) 0 (b)  $-\frac{2}{3}$  **31.** \$7.322 billion per year **33.**  $(-3,\infty)$  **35.** Two,  $(-2 \pm \sqrt{3}, (1 \mp \sqrt{3})/2)$  **39.**  $(x^2 + 2x)e^x$ ,  $(x^2 + 4x + 2)e^x$ ,  $(x^2 + 6x + 6)e^x$ ,  $(x^2 + 8x + 12)e^x$ ,  $(x^2 + 10x + 20)e^x$ ;  $f^{(m)}(x) = [x^2 + 2nx + n(n - 1)]e^x$ 

# Exercises 3.3 page 217

1. (a)  $3t^2 - 24t + 36$  (b) -9 m/s (c) t = 2, 6(d)  $0 \le t < 2, t > 6$  (e) 96 m(f)  $t = 6, \qquad t = 2, \\ s = 0, \qquad s = 0, \\ s = 0, \\ 0 \qquad t = 2, \\ s = 32, \\ s = 3$ 

- (i) Speeding up when 2 < t < 4 or t > 6; slowing down when  $0 \le t < 2$  or 4 < t < 6
- **3.** (a) t = 4 s

-25

(b) t = 1.5 s; the velocity has an absolute minimum.

5. (a) 30 mm<sup>2</sup>/mm; the rate at which the area is increasing with respect to side length as x reaches 15 mm

- (b)  $\Delta A \approx 2x \Delta x$
- 7. (a) (i)  $5\pi$  (ii)  $4.5\pi$  (iii)  $4.1\pi$  (b)  $4\pi$
- (c)  $\Delta A \approx 2\pi r \,\Delta r$
- 9. (a)  $8\pi \text{ ft}^2/\text{ft}$  (b)  $16\pi \text{ ft}^2/\text{ft}$

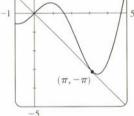
(c)  $24\pi$  ft<sup>2</sup>/ft The rate increases as the radius increases. **11.** (a) 6 kg/m (b) 12 kg/m (c) 18 kg/m;

- At the right end; at the left end
- **13.** (a) 4.75 A (b) 5 A;  $t = \frac{2}{3}$  s
- 15. (a)  $dV/dP = -C/P^2$  (b) At the beginning
- 17. (a)  $a^2k/(akt + 1)^2$
- (c) It approaches a moles/L
- (d) It approaches 0
- (e) The reaction virtually stops.

- **19.** (a) 0.926 cm/s; 0.694 cm/s; 0 (b) 0; -92.6 (cm/s)/cm; -185.2 (cm/s)/cm (c) At the center; at the edge **21.** (a)  $C'(x) = 3 + 0.02x + 0.0006x^2$ (b) \$11/yd, the rate at which the cost is changing as the 100th yard is being produced (c) \$11.07/yd **23.** (a)  $[xp'(x) - p(x)]/x^2$ ; the average productivity increases as new workers are added. **25.** -0.2436 K/min **27.** (a) 0 and 0 (b) C = 0
- (c) (0,0), (500, 50); it is possible for the species to coexist.

#### Exercises 3.4 page 225

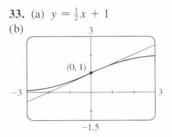
1.  $\cos x - \sin x$ 3.  $2x \cos x - x^2 \sin x$ 5.  $-2 \csc^2 x - \sqrt{x} \sec x \tan x - \frac{1}{2} x^{-1/2} \sec x$ 7.  $(x \sec^2 x - \tan x)/x^2$ 9.  $(\sin x + \cos x + x \sin x - x \cos x)/(1 + \sin 2x)$ 11.  $e^x (\tan^2 x + \tan x - x)$ 17.  $y = 2x + 1 - \pi/2$ 19. (a) y = -x (b) -1 -1 -1 -1 -1 -1 -1 -5



**21.** (a)  $2 - \csc^2 x$  **23.**  $g'(s) = 2s\cos s - s^2\sin s$ ,  $g''(s) = (2 - s^2)\cos s - 4s\sin s$  **25.**  $(2n + 1)\pi \pm \pi/3$ , *n* an integer **27.**  $(\pi/3, 5\pi/3)$  **29.** (a)  $v(t) = 8\cos t$ ,  $a(t) = -8\sin t$ (b)  $4\sqrt{3}$ , -4,  $-4\sqrt{3}$ ; to the left; speeding up **31.** 5 ft/rad **33.**  $-\cos x$  **35.**  $A = -\frac{3}{10}$ ,  $B = -\frac{1}{10}$  **37.** 4 **39.**  $\frac{1}{2}$ **41.** 1

## Exercises 3.5 ■ page 234

1.  $10(x^2 + 4x + 6)^4(x + 2)$ 3.  $-\sin(\tan x) \sec^2 x$ 5.  $e^{\sqrt{x}}/(2\sqrt{x})$ 7.  $g'(x) = (2x - 7)/(2\sqrt{x^2 - 7x})$ 9.  $y' = -3x^2\sin(x^3)$ 11.  $y' = 5^{-1/x}(\ln 5)/x^2$ 13.  $y' = e^{-x^2}(1 - 2x^2)$ 15.  $G'(x) = 6(3x - 2)^9(5x^2 - x + 1)^{11}(85x^2 - 51x + 9)$ 17.  $y' = (\cos x - x \sin x)e^{x\cos x}$ 19.  $F'(y) = 39(y - 6)^2/(y + 7)^4$ 21.  $f'(z) = -\frac{2}{5}(2z - 1)^{-6/5}$ 23.  $y' = \sin(1/x) - \cos(1/x)/x$ 25.  $y' = 6x^2\tan(x^3)\sec^2(x^3)$ 27.  $y' = [1 + 1/(2\sqrt{x})]/(2\sqrt{x + \sqrt{x}})$ 29.  $y' = \cos(\tan\sqrt{\sin x})(\sec^2\sqrt{\sin x})[1/(2\sqrt{\sin x})](\cos x)$ 31.  $y = -\frac{3}{16}x + \frac{11}{4}$ 



**35.** (a)  $-1/(x^2\sqrt{1-x^2})$  **37.** 28 **39.** (a)  $\frac{3}{4}$  (b) Does not exist (c) -2 **41.** -17.4 **43.** (a)  $(0,\infty)$  (b)  $G'(x) = h'(\sqrt{x})/(2\sqrt{x})$  **45.** (a)  $F'(x) = e^x f'(e^x)$  (b)  $G'(x) = e^{f(x)} f'(x)$  **47.**  $x = 2n\pi$  or  $(2n + 1)\pi \pm \pi/3$ , *n* any integer **51.**  $-2^{50} \cos 2x$ **53.**  $v(t) = (5\pi/2) \cos(10\pi t)$  cm/s,

 $a(t) = -25\pi^2 \sin(10\pi t) \text{ cm/s}^2$ 

**55.** (a)  $dB/dt = (7\pi/54)\cos(2\pi t/5.4)$  (b) 0.16

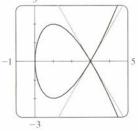
**57.** dv/dt is the rate of change of velocity with respect to time; dv/ds is the rate of change of velocity with respect to displacement.

**59.** (a)  $y \approx 100.012437 e^{-10.005531t}$  (b)  $-670.625828 \ \mu \text{A}$ 

**61.**  $y = (1/\pi)x - \pi$ 

**63.** (a) 
$$y = \sqrt{3}x - 3\sqrt{3}, y = -\sqrt{3}x + 3\sqrt{3}$$

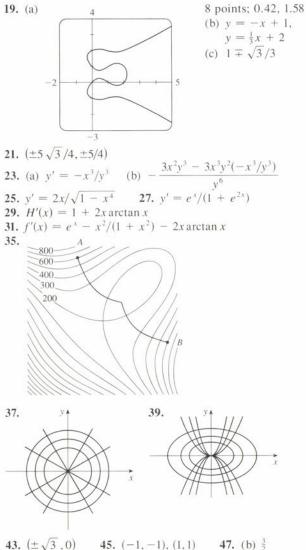
- (b) Horizontal at  $(1, \pm 2)$ ; vertical at (0, 0)
- (c)



65. (b) The factored form 67. (b)  $-n\cos^{n-1}x\sin[(n + 1)x]$ 

#### Exercises 3.6 ■ page 245

1. (a) 
$$y' = -(2x + y + 3)/x$$
  
(b)  $y = (5/x) - x - 3$ ,  $y' = -(5/x^2) - 1$   
3.  $(y - 2x)/(3y^2 - x)$  5.  $-x^3/y^3$   
7.  $1 + [e^x(1 + x)]/\sin(x - y)$  9.  $-y/x$   
11.  $y = -\frac{5}{4}x - 4$  13.  $y = x$  15.  $y = -\frac{9}{13}x + \frac{40}{13}$   
17. (a)  $y = \frac{9}{2}x - \frac{5}{2}$   
(b) 5  
 $-2 \sqrt{(1,2)} -2$ 

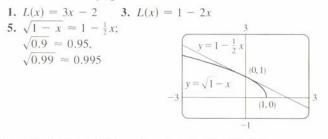


**49.** (a) 0 (b)  $-\frac{1}{2}$ 

# Exercises 3.7 ■ page 252

1. The differentiation formula is simplest. 3.  $f'(\theta) = -\tan \theta$  5.  $f'(x) = 2x/[(x^2 - 4) \ln 3]$ 7.  $g'(x) = -2a/(a^2 - x^2)$  9. F'(x) = 1/(2x)11.  $f'(x) = (2 + \ln x)/(2\sqrt{x})$ 13. y' = (3x - 2)/[x(x - 1)] 15. y' = -x/(1 + x)17.  $y' = 1/(x \ln 10), y'' = -1/(x^2 \ln 10)$ 19.  $f'(x) = 2x \ln(1 - x^2) - 2x^3/(1 - x^2), (-1, 1)$  21. 0 23. (a) (0, 1/e) (b)  $(0, \infty)$ 25.  $y' = (3x - 7)^4(8x^2 - 1)^3[12/(3x - 7) + 48x/(8x^2 - 1)]$ 27.  $y' = \frac{(x + 1)^4(x - 5)^3}{(x - 3)^8} \left[\frac{4}{x + 1} + \frac{3}{x - 5} - \frac{8}{x - 3}\right]$ 29.  $y' = x^x(\ln x + 1)$  31.  $y' = x^{\sin x}[\cos x \ln x + (\sin x)/x]$ 33.  $y' = (\ln x)^x(\ln \ln x + 1/\ln x)$ 35.  $y' = 2x/(x^2 + y^2 - 2y)$ 37.  $f^{(m)}(x) = (-1)^{n-1}(n - 1)!/(x - 1)^n$ 

# Exercises 3.8 ■ page 258



**7.** -0.69 < x < 1.09 **9.** -0.045 < x < 0.055 **11.** (b) -0.344 < x < 0.344 **13.** (a)  $dy = -\sin x \, dx$ (b) dy = -0.025,  $\Delta y \approx -0.02607$ **15.** (a) 270 cm<sup>3</sup> (b) 36 cm<sup>2</sup> **17.**  $\frac{5}{3}\pi \approx 2$  m<sup>3</sup>

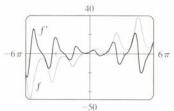
#### Chapter 3 Review = page 260

#### True-False Quiz

**1.** True **3.** True **5.** False **7.** False **9.** True **11.** True

#### Exercises

1.  $y' = 2(7x + 18)(x + 2)^7(x + 3)^5$ 3.  $y' = (9 - 2x)/(9 - 4x)^{3/2}$  5.  $y' = -\sin x \cos(\cos x)$ 7.  $y' = e^{-1/x}(1 + 1/x)$ 9.  $y' = -(\sec^2\sqrt{1-x})/(2\sqrt{1-x})$ 9.  $y' = -(300 \text{ y}^2)^2$   $(310 \text{ y}^2)^2$   $(110 \text{ y}^2)^2$ **15.**  $y' = e^{x+e^3}$  **17.**  $y' = (1 - 2xy^3)/(3x^2y^2 + 6y + 4)$ **19.**  $y' = (2x - 1)/[(x^2 - x) \ln 10]$ **21.**  $y' = \cot x - \sin x \cos x$ 23.  $y' = \cos(\tan\sqrt{1+x^3})(\sec^2\sqrt{1+x^3})\left(\frac{3x^2}{2\sqrt{1+x^3}}\right)$ 25.  $y' = \frac{(x-2)^4(3x^2-55x-52)}{2\sqrt{x+1}(x+3)^8}$ 27. -120 **29.**  $2^{x}(\ln 2)^{n}$ **31.** (a)  $(10 - 3x)/(2\sqrt{5 - x})$ (b)  $y = \frac{7}{4}x + \frac{1}{4}, y = -x + 8$ 10 (c) (4, 4)(1,2) -1010 -10**33.**  $e^{\sin x}(x \cos x + 1)$ 



The sizes of the oscillations of f and f' are linked.

**35.** (a) 2 (b) 44 **37.**  $f'(x) = 2xg(x) + x^2g'(x)$ **39.** f'(x) = 2g(x)g'(x)**41.**  $f'(x) = q'(e^x)e^x$ **43.** f'(x) = g'(x)/g(x)**45.**  $h'(x) = \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[g(x)]^2 + g'(x)[f(x)]^2}$ 47. (-3, 0) $[f(x) + q(x)]^2$ **49.**  $(\pm 2/\sqrt{6}, \pm 1/\sqrt{6})$ 51.  $v(t) = -Ae^{-ct}[c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)],$  $a(t) = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)]$ 53. 4 kg/m 55. (a)  $C'(x) = 2 - 0.04x + 0.00021x^2$ (b) 0.1; the approximate cost of producing the 101st unit (c) C(101) - C(100) = 0.10107(d) About 95.24; at this value of x the marginal cost is minimized. 57. (a) L(x) = 1 + x;  $\sqrt[3]{1 + 3x} \approx 1 + x$ ;  $\sqrt[3]{1.03} \approx 1.01$ (b) -0.23 < x < 0.40**59.**  $(\cos \theta)'|_{\theta=\pi/3} = -\sqrt{3}/2$  **61.**  $\frac{1}{4}$ 

## Focus on Problem Solving = page 264

1.  $(0, \frac{5}{4})$ 3. (a) [-1, 2](b)  $-1/(8\sqrt{3} - x)\sqrt{2 - \sqrt{3} - x}\sqrt{1 - \sqrt{2 - \sqrt{3} - x}})$ 5. (a)  $4\pi\sqrt{3}/\sqrt{11}$  rad/s (b)  $40(\cos\theta + \sqrt{8 + \cos^2\theta})$  cm (c)  $-480\pi \sin\theta(1 + \sin\theta\cos\theta/\sqrt{8 + \cos^2\theta})$  cm/s 9.  $x_T \in (3, \infty), y_T \in (2, \infty), x_N \in (0, \frac{5}{3}), y_N \in (-\frac{5}{2}, 0)$ 11.  $f^{(n)}(x) = n!/(1 - x)^{n+1}$ 13. (b) (i) 53° (ii) 63° (or 117°) 15. *R* approaches the midpoint of the radius *AO*. 17. (1, -2), (-1, 0) 19.  $\sqrt{29}/58$ 

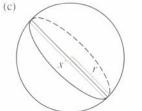
#### **CHAPTER 4**

#### Exercises 4.1 page 272

**1.**  $dV/dt = 3x^2 dx/dt$ 

3. (a) The surface area decreases at  $1 \text{ cm}^2/\text{min}$ .

(b) The rate of decrease of the diameter when the diameter is 10  $\mbox{cm}$ 

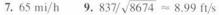


(d)  $S = \pi x^2$ (e)  $1/(20\pi)$  cm/min

5. (a) The plane's altitude is 1 mi and its velocity is 500 mi/h.

(b) The rate at which the distance from the plane to the station is increasing when the plane is 2 mi from the station





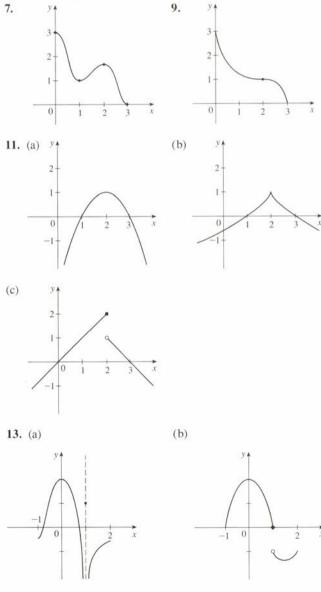
111.6 cm/mir	n 13. 10/√133 ≈	0.87 ft/s
<b>15.</b> $\frac{10}{3}$ cm/min	17. $6/(5\pi)$ ft/min	<b>19.</b> 0.3 m <sup>2</sup> /s
<b>21.</b> 80 cm <sup>3</sup> /min	23. (a) 360 ft/s	(b) 0.096 rad/s
<b>25.</b> 1650/√31 ≈	296 km/h 27. 7	$\sqrt{15}/4 \approx 6.78$ m/s

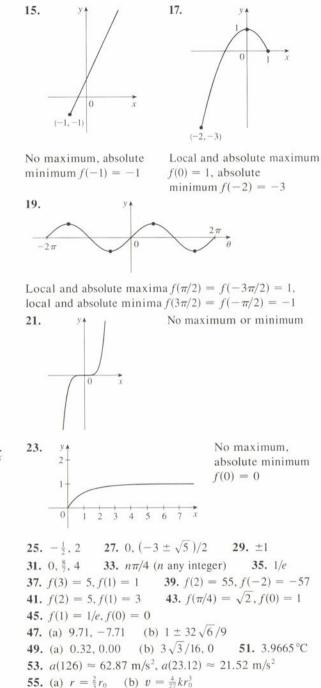
# Exercises 4.2 page 279

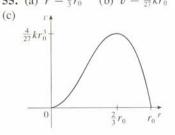
1. Absolute minimum: smallest function value on the entire domain of the function; local minimum at c: smallest function value when x is near c

3. Absolute maximum at b, local maxima at b and e, absolute minimum at d, local minima at d and s

5. Absolute maximum f(4) = 4; absolute minimum f(7) = 0; local maxima f(4) = 4 and f(6) = 3; local minima f(2) = 1 and f(5) = 2







# Exercises 4.3 = page 292

*Abbreviations*: CD, concave downward; CU, concave upward; HA, horizontal asymptote; IP, inflection point; VA, vertical asymptote

1. 0.8, 3.2, 4.4, 6.1

3. (a) I/D Test (b) Concavity Test

(c) Find points at which the concavity changes. 5. x = 1, 7

7. (a) Increasing on  $(-2, \infty)$ , decreasing on  $(-\infty, -2)$ 

(b) No local maximum, local minimum f(-2) = -303
(c) CU on (-∞,∞), no IP

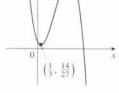
(a) Increasing on (-1,∞), decreasing on (-∞, -1)
(b) No local maximum, local minimum f(-1) = -1/e
(c) CD on (-∞, -2), CU on (-2,∞); IP (-2, -2e<sup>-2</sup>)
11. (a) Increasing on (0, e<sup>2</sup>), decreasing on (e<sup>2</sup>,∞)

(b) Local maximum  $f(e^2) = 2/e$ , no local minimum

(c) CD on  $(0, e^{8/3})$ , CU on  $(e^{8/3}, \infty)$ ; IP  $(e^{8/3}, \frac{8}{3}e^{-4/3})$ 

13. (a) Increasing on (<sup>1</sup>/<sub>3</sub>, 3); decreasing on (-∞, <sup>1</sup>/<sub>3</sub>), (3, ∞)
(b) Local maximum f(3) = 10, local minimum f(<sup>1</sup>/<sub>3</sub>) = <sup>14</sup>/<sub>27</sub>
(c) CD on (<sup>5</sup>/<sub>3</sub>, ∞), CU on (-∞, <sup>5</sup>/<sub>3</sub>); IP (<sup>5</sup>/<sub>3</sub>, <sup>142</sup>/<sub>27</sub>)

(d) See graph at right.



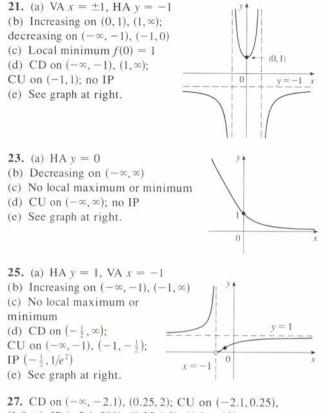
0

(3, 10)

**15.** (a) Increasing on  $(0, \infty)$ , decreasing on  $(-\infty, 0)$ (b) Local minimum f(0) = -1(c) CD on  $(-1, -1/\sqrt{5})$ ,  $(1/\sqrt{5}, 1)$ ; CU on  $(-\infty, -1)$ ,  $(-1/\sqrt{5}, 1/\sqrt{5})$ ,  $(1, \infty)$ ; IP  $(\pm 1, 0)$ ,  $(\pm 1/\sqrt{5}, -\frac{64}{125})$ (d) See graph at right.

**17.** (a) Increasing on  $(-\infty, -3)$ ,  $(-1, \infty)$ , decreasing on (-3, -1)(b) Local maximum f(-3) = 0, local minimum  $f(-1) = -\sqrt[3]{4}$ (c) CD on  $(0, \infty)$ , CU on  $(-\infty, -3)$ , (-3, 0); IP (0, 0)(d) See graph at right. (-1,  $-\sqrt[3]{4}$ )

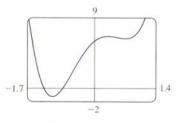
19. (a) Increasing on  $((2n - 1)\pi, 2n\pi)$ , decreasing on  $(2n\pi, (2n + 1)\pi)$ (b) Local maxima  $f(2n\pi) = 2$ , local minima  $f((2n + 1)\pi) = -2$ (c) CD on  $(2n\pi - \frac{2}{3}\pi, 2n\pi + \frac{2}{3}\pi)$ , CU on remaining intervals; IP  $(2n\pi \pm \frac{2}{3}\pi, -\frac{1}{4})$ (d) See graph at right.



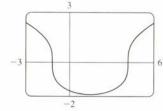
**27.** CD on  $(-\infty, -2.1)$ , (0.25, 2); CU on (-2.1, 0.25),  $(1.9, \infty)$ ; IP (-2.1, 380), (0.25, 1.3), (1.9, -92) **29.** (a) Local and absolute maximum  $f(1) = \sqrt{2}$ , no minimum (b)  $(3 - \sqrt{17})/4$  **31.** When  $t \approx 7.17$  **33.** CD on  $(-\infty, 0.1)$ , CU on  $(0.1, \infty)$ **35.**  $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$  **43.** 17

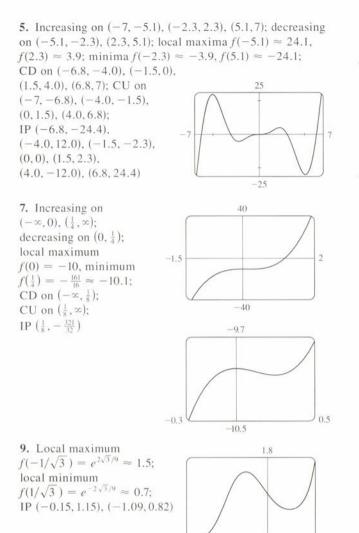
# Exercises 4.4 ■ page 300

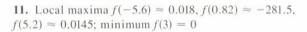
**1.** Increasing on (-1.1, 0.3),  $(0.7, \infty)$ ; decreasing on  $(-\infty, -1.1)$ , (0.3, 0.7); local maximum  $f(0.3) \approx 6.6$ , local minima  $f(-1.1) \approx -1.0$ ,  $f(0.7) \approx 6.3$ ; CD on (-0.5, 0.5), CU on  $(-\infty, -0.5)$ ,  $(0.5, \infty)$ ; IP (-0.5, 2.5), (0.5, 6.5)

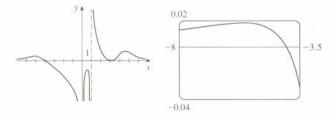


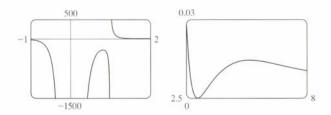
3. Increasing on  $(1.5, \infty)$ , decreasing on  $(-\infty, 1.5)$ ; no local maximum, minimum  $f(1.5) \approx -1.9$ ; CD on  $(-\infty, -1.2)$ ,  $(4.2, \infty)$ ; CU on (-1.2, 4.2); IP (-1.2, 0), (4.2, 0)







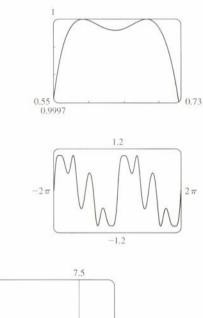




**13.** 
$$f'(x) = -\frac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5},$$
  

$$f''(x) = 2\frac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$$
CU on  $(-\infty, -5.0), (-1, -0.5), (-0.1, 2), (2, 4), (4, \infty);$   
CD on  $(-5, -1), (-0.5, -0.1);$  IP  $(-5.0, -0.005), (-1, 0), (-0.5, 0.00001), (-0.1, 0.0000066)$ 

**15.** Maxima  $f(0.59) \approx 1$ ,  $f(0.68) \approx 1$ ,  $f(1.96) \approx 1$ ; minima  $f(0.64) \approx 0.99996$ ,  $f(1.46) \approx 0.49$ ,  $f(2.73) \approx -0.51$ ; IP (0.61, 0.99998), (0.66, 0.99998), (1.17, 0.72), (1.75, 0.77), (2.28, 0.34)

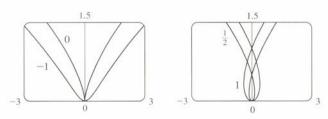




17.

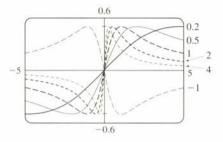
Vertical tangents at (0, 0),  $\left(-\frac{3}{16}, \frac{3}{8}\right)$ , (-8, 6); horizontal tangents at  $\left(-(2\sqrt{3} + 5)/9, -2\sqrt{3}/9\right)$ ,  $\left((2\sqrt{3} - 5)/9, 2\sqrt{3}/9\right)$ 

**19.** For c = 0, there is a cusp; for c > 0, there is a loop whose size increases as *c* increases and the curve intersects itself at (0, c); leftmost point  $(2c\sqrt{3c}/9, c/3)$ , rightmost point  $(-2c\sqrt{3c}/9, c/3)$ 



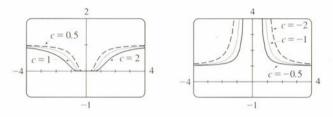
**21.** For c > 0, the maximum and minimum values are always  $\pm \frac{1}{2}$ , but the extreme points and IP move closer to the *y*-axis as *c* increases.

c = 0 is a transitional value: when c is replaced by -c, the curve is reflected in the x-axis.

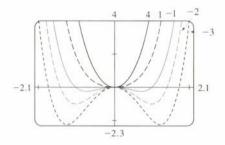


**23.** There is no maximum or minimum, regardless of the value of c. For c < 0, there is a vertical asymptote at x = 0,  $\lim_{x\to 0} f(x) = \infty$ , and  $\lim_{x\to \pm\infty} f(x) = 1$ .

c = 0 is a transitional value at which f(x) = 1 for  $x \neq 0$ . For c > 0,  $\lim_{x\to 0} f(x) = 0$ ,  $\lim_{x\to \pm x} f(x) = 1$ , and there are two IPs, which move away from the *y*-axis as  $c \to \infty$ .



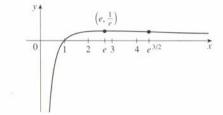
**25.** For  $c \ge 0$ , there is no IP and only one extreme point, the origin. For c < 0, there is a maximum point at the origin, two minimum points, and two IPs, which move downward and away from the origin as  $c \to -\infty$ .



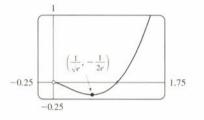
Exercises 4.5 = page 308

5. 1 7. ∞ 1. 1 3. 1 9.  $\frac{1}{2}$ 11. 0 13.  $\frac{2}{3}$ 15. 0 17. 0 19. 0 **21.** 0 23. 1 25.1 **27.**  $e^{-2}$ 29.1 31. 5 33. 1 35. HA y = 0 $(1, \frac{1}{2})$ 

**37.** HA 
$$y = 0$$
, VA  $x = 0$ 

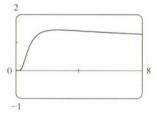






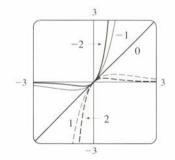
(b)  $\lim_{x\to 0^+} f(x) = 0$ (c) Local minimum  $f(1/\sqrt{e}) = -1/(2e)$ ; CD on  $(0, e^{-3/2})$ ; CU on  $(e^{-3/2}, \infty)$ 

**41.** (a)



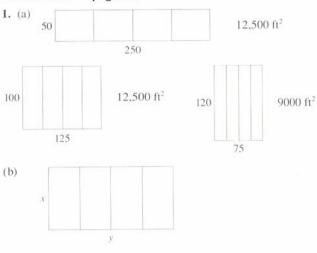
(b)  $\lim_{x\to 0^+} f(x) = 0$  and  $\lim_{x\to \infty} f(x) = 1$ (c) Local maximum  $f(e) = e^{1/e}$ ; IP at  $x \approx 0.58, 4.37$ 

**43.** For c > 0,  $\lim_{x \to \infty} f(x) = 0$  and  $\lim_{x \to -\infty} f(x) = -\infty$ . For c < 0,  $\lim_{x \to \infty} f(x) = \infty$  and  $\lim_{x \to -\infty} f(x) = 0$ . As |c| increases, the maximum and minimum points and the IPs get closer to the origin.



**49.**  $\frac{16}{9}a$ 





(c) A = xy (d) 5x + 2y = 750 (e)  $A = 375x - \frac{5}{2}x^2$ (f) 14,062.5 ft<sup>2</sup>

**3.** 4000 cm<sup>3</sup> **7.** (1.2, -0.6) **9.** Square, side  $\sqrt{2} r$ **11.** 3 cm<sup>2</sup>

13. Width  $60/(4 + \pi)$  ft; rectangle height  $30/(4 + \pi)$  ft

15. (a) All of the wire for the square

(b)  $40\sqrt{3}/(9 + 4\sqrt{3})$  m for the square

17.  $V = 2\pi R^3/(9\sqrt{3})$ 

19. (a)  $\frac{3}{5}s^2 \csc \theta (\csc \theta - \sqrt{3} \cot \theta)$ 

(b)  $\cos^{-1}(1/\sqrt{3}) \approx 55^{\circ}$  (c)  $6s[h + s/(2\sqrt{2})]$ 

21.  $10\sqrt[3]{3}/(1+\sqrt[3]{3})$  ft from the stronger source

**23.**  $y = -\frac{5}{3}x + 10$  **27.** x = 6 in. **29.** 9.35 m

**31.** At a distance  $5 - 2\sqrt{5}$  from A

**33.** (a) About 5.1 km from B

(b) C is close to B; C is close to D;

 $W/L = \sqrt{25 + x^2/x}$ , where x = |BC|

(c)  $\approx 1.07$ ; no such value (d)  $\sqrt{41}/4 \approx 1.60$ 

**35.** (a)  $T_1 = D/c_1, T_2 = (2h \sec \theta)/c_1 + (D - 2h \tan \theta)/c_2, T_3 = \sqrt{4h^2 + D^2}/c_1$ 

(c)  $c_1 \approx 3.85, c_2 \approx 7.66, h \approx 0.42$ 

#### Exercises 4.7 page 325

1. (a) C(0) represents fixed costs, which are incurred even when nothing is produced.

(b) The marginal cost is a minimum there.

(c) C'A



**3.** \$17.40/unit; the cost of producing the 1001st unit is about \$17.40

**5.** (a) \$19,600; \$19.60; \$28/unit (b) 400 units (c) \$16/unit

7. (a)  $c(x) = 3700/x + 5 - 0.04x + 0.0003x^2$ ,  $C'(x) = 5 - 0.08x + 0.0009x^2$ (b) Between 208 and 209 units (c)  $c(209) \approx $27.45/unit$ (d) \$3.22/unit9. 333 units 11. (a) About 200 yards (b) 192 yards 13. (a) p(x) = 19 - (x/3000) (b) \$9.50 15. (a) p(x) = 550 - (x/10) (b) \$175 (c) \$100

#### Exercises 4.8 = page 331

 1.  $x_2 \approx 2.3, x_3 \approx 3$  3.  $\frac{4}{5}$  5. -0.6860

 7. 2.16573677
 9. 1.895494

 11. -2.11490754, 0.25410169, 1.86080585

 13. 0, 1.10914418, 3.69815367

 15. -3.20614267, 1.37506470

 17. 0.15438500, 0.84561500
 19. (b) 31.622777

 25. (0.904557, 1.855277)
 27. 11.28 ft
 29. 0.76286%

#### Exercises 4.9 = page 338

1.  $4x^3 + 3x^2 - 5x + C$ 3.  $-3/(2x^4) + C_1$  if x > 0,  $-3/(2x^4) + C_2$  if x < 05.  $(2t^{7/2}/7) + (4t^{5/2}/5) + C$ 7.  $\tan t + (t^3/3) + C_n, (2n-1)\pi/2 < t < (2n+1)\pi/2$ 11.  $x^5 - (x^6/3) + 4$ 9.  $x^2 + 5\sin^{-1}x + C$ **13.**  $f(x) = (x^4/12) + (x^5/20) + Cx + D$ 17.  $f(x) = x^3 + 3x^2 - 5x + 4$ 15.  $f(x) = 2\ln(-x) + 7$ **19.**  $f(x) = -\ln x + (\ln 2)x - \ln 2$ 21. 10 25. 23. 14 (2, 2)2-E (1, 1)(3, 1) 0 2 -i 27. 1-29. V.A 0.6 0.40.2 0 0.8 0.4 1.2 1.6

**31.**  $s(t) = 2t^{3/2} + 3$ 

**33.** (a)  $s(t) = 450 - 4.9t^2$  (b)  $\sqrt{450/4.9} \approx 9.58$  s (c)  $-9.8\sqrt{450/4.9} \approx -93.9$  m/s (d) About 9.09 s

APPENDIX I ANSWERS TO ODD-NUMBERED EXERCISES A95

**37.** 225 ft **39.**  $\frac{88}{15}$  ft/s<sup>2</sup> **41.** \$742.08 **45.** (a) 22.9125 mi (b) 21.675 mi (c) 30 min 33 s (d) 55.425 mi

#### Chapter 4 Review page 340

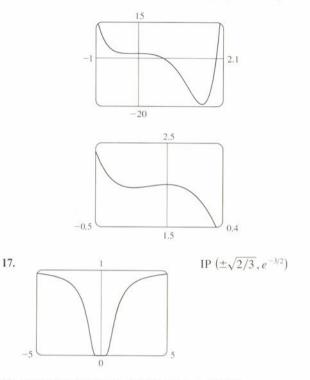
#### **True-False Quiz**

1. False 3. False 5. True 7. False 9. True

#### Exercises

1. Absolute and local maximum f(-2) = 21, local minimum f(2) = -11, absolute minimum f(-5) = -603. Local and absolute minimum  $f(\pi/4) = (\pi/4) - 1$ , absolute maximum  $f(\pi) = \pi$ 5. (a) None (b) Increasing on  $(\frac{1}{4}, \infty)$ , 2 decreasing on  $\left(-\infty, \frac{1}{4}\right)$ (c) Minimum  $f(\frac{1}{4}) = -\frac{27}{256}$ (d) CD on  $(\frac{1}{2}, 1)$ ; CU on  $(-\infty, \frac{1}{2})$ ,  $(1, \infty)$ ; IP  $\left(\frac{1}{2}, -\frac{1}{16}\right)$ , (1, 0) (e) See graph at right. 7. (a) None (b) Increasing on  $\left(-\infty, \frac{3}{4}\right)$ , decreasing on  $\left(\frac{3}{4}, 1\right)$ (c) Maximum  $f(\frac{3}{4}) = \frac{5}{4}$ 0 (d) CD on  $(-\infty, 1)$ (e) See graph at right. **9.** (a) HA y = 0(b) Decreasing on  $(-\infty, -1)$ ,  $(1, \infty)$ π 7 (c) None (d) CD on  $(-\infty, -1)$ , -1 0 CU on  $(1, \infty)$ (e) See graph at right. 11. (a) None (b) Increasing on  $(\frac{1}{4} \ln 3, \infty)$ , decreasing on  $(-\infty, \frac{1}{4} \ln 3)$ (c) Minimum  $f(\frac{1}{4}\ln 3) = 3^{1/4} + 3^{-3/4}$ (d) CU on  $(-\infty,\infty)$ (e) See graph at right. 0 13. Increasing on  $(-\sqrt{3}, 0)$ ,  $(0, \sqrt{3})$ ; decreasing on  $(-\infty, -\sqrt{3}), (\sqrt{3}, \infty);$  local maximum  $f(\sqrt{3}) = 2\sqrt{3}/9,$ minimum  $f(-\sqrt{3}) = -2\sqrt{3}/9$ ; 1.5 CD on  $(-\infty, -\sqrt{6})$ ,  $(0, \sqrt{6})$ ; CU on  $(-\sqrt{6}, 0)$ ,  $(\sqrt{6}, \infty)$ ; IP  $(\sqrt{6}, 5\sqrt{6}/36)$ , 5  $(-\sqrt{6}, -5\sqrt{6}/36)$ 

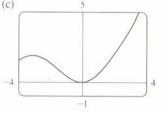
**15.** Increasing on (-0.2, 0),  $(1.6, \infty)$ ; decreasing on  $(-\infty, -0.2)$ , (0, 1.6); local maximum f(0) = 2; minima  $f(-0.2) \approx 1.96$ ,  $f(1.6) \approx -19.2$ ; CD on (-0.1, 1.2); CU on  $(-\infty, -0.1)$ ,  $(1.2, \infty)$ ; IP (-0.1, 2.0), (1.2, -12.1)



**19.** Maximum at x = 0, minima at  $x \approx \pm 0.87$ , IP at  $x \approx \pm 0.52$ 

**21.** For C > -1, *f* is periodic with period  $2\pi$  and has local maxima at  $2n\pi + \pi/2$ , *n* an integer. For  $C \le -1$ , *f* has no graph. For  $-1 < C \le 1$ , *f* has vertical asymptotes. For C > 1, *f* is continuous on  $\mathbb{R}$ . As *C* increases, *f* moves upward and its oscillations become less pronounced. **23.** a = -3, b = 7 **25.**  $-1/(2\pi)$  **27.** 0 **29.**  $-\frac{1}{2}$ 

**31.**  $\frac{1}{3}$  **33.** 400 ft/h **35.** 13 ft/s **39.**  $3\sqrt{3}r^2$  **41.**  $4/\sqrt{3}$  cm from *D*; at *C* **43.** L = C **45.** \$11.50 **47.** -2.063421 **49.**  $e^x - \ln|x| + C_1$  if x < 0,  $e^x - \ln|x| + C_2$  if x > 0 **51.**  $f(x) = 2\tan^{-1}x - 1$  **53.**  $f(x) = (x^5/20) + (x^3/6) + x - 1$  **55.** (b)  $0.1e^x - \cos x + 0.9$ (c) <u>5</u>





-1.5

**61.** (b) About 8.5 in. by 2 in. (c)  $20/\sqrt{3}$  in. by  $20\sqrt{2/3}$  in. **63.** (a)  $\sqrt{800} \approx 28$  ft (b)  $dI/dt = -480k(h-4)/[(h-4)^2 + 1600]^{5/2}$ , where k is the constant of proportionality

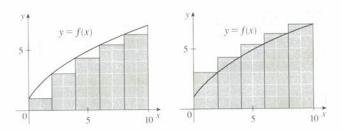
#### Focus on Problem Solving page 345

5. 
$$(-2, 4), (2, -4)$$
  
7.  $\frac{4}{3}$   
11.  $(m/2, m^2/4)$   
13. (a)  $-\tan \theta \left[ \frac{1}{c} \frac{dc}{dt} + \frac{1}{b} \frac{db}{dt} \right]$   
(b)  $\frac{b \frac{db}{dt} + c \frac{dc}{dt} - \left( b \frac{dc}{dt} + c \frac{db}{dt} \right) \sec \theta}{\sqrt{b^2 + c^2 - 2bc \cos \theta}}$   
15. (a)  $x/(x^2 + 1)$  (b)  $\frac{1}{2}$   
19. 11.204 cm<sup>3</sup>/min

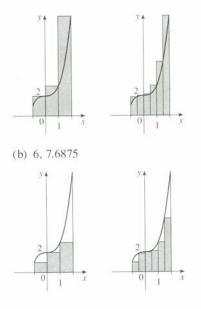
#### CHAPTER 5

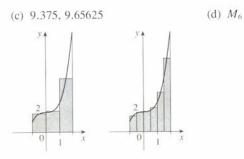
#### Exercises 5.1 = page 359

1. (a) 40, 52



(b) 43.2, 49.23. (a) 15, 12.1875





5. 1.9835, 1.9982, 1.9993; 2 7. (a) Left: 4.5148, 4.6165, 4.6366; right: 4.8148, 4.7165, 4.6966 9. 34.7 ft, 44.8 ft **11.** 155 ft **15.** (a)  $\lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5$ (b)  $n^2(n+1)^2(2n^2+2n-1)/12$  (c)  $\frac{32}{3}$ **17.** sin *b*, 1

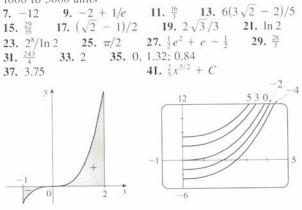
#### Exercises 5.2 ■ page 370

**1.** 0.25 **3.** (a) 4 (b) 6 (c) 10 **5.** 153.1250 **7.** 1.8100 **9.** 1.81001414, 1.81007263, 1.81008347 **11.**  $\int_0^{\pi} \cos x \, dx$  **13.**  $\int_0^1 (2x^2 - 5x) \, dx$  **15.**  $\frac{4}{3}$  **17.** 3.75 **19.**  $\lim_{n \to \infty} \sum_{i=1}^n \left( \sin \frac{5\pi i}{n} \right) \frac{\pi}{n} = \frac{2}{5}$  **21.** (a) 4 (b) 10 (c) -3 (d) 2 **23.** 10 **25.**  $3 + 9\pi/4$  **27.** 0 **29.**  $\int_1^{12} f(x) \, dx$  **31.** -0.8 **33.** 3 **35.**  $e^5 - e^3$ 

#### Exercises 5.3 = page 380

1. The increase in the child's weight (in pounds) between the ages of 5 and 10

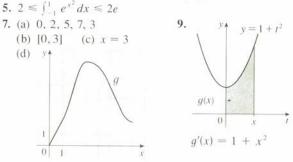
Number of gallons of oil leaked in the first 2 hours
 Increase in revenue when production is increased from 1000 to 5000 units



**43.**  $4x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 + C$  **45.**  $x^2 + \sec x + C$  **47.**  $\frac{4}{3}$  **49.** (a)  $-\frac{3}{2}$  m (b)  $\frac{41}{6}$  m **51.** (a)  $v(t) = \frac{1}{2}t^2 + 4t + 5$  m/s (b)  $416\frac{2}{3}$  m **53.**  $46\frac{2}{3}$  kg **55.** 1.4 mi **57.** \$58,000 **59.** (b) At most 40%;  $\frac{5}{36}$  **61.** 3

#### Exercises 5.4 ■ page 390

1. One process undoes what the other one does. See the Fundamental Theorem of Calculus, page 388.



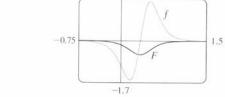
11.  $g'(x) = (x^2 - 1)^{20}$ 13.  $g'(u) = 1/(1 + u^4)$ 15.  $h'(x) = -\sin^4(1/x)/x^2$ 17.  $y' = -\sin(\tan^4 x) \sec^2 x$ 19.  $g'(x) = \frac{-2(2x - 1)}{2x + 1} + \frac{3(3x - 1)}{3x + 1}$ 21.  $\sqrt{257}$ 23. (a) Local maxima at 1 and 5; minima at 3 and 7 (b) 9 (c)  $(\frac{1}{2}, 2), (4, 6), (8, 9)$ (d) See graph at right.

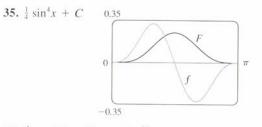
**25.** (a) 
$$-2\sqrt{n}$$
,  $\sqrt{4n-2}$ , *n* any integer > 0  
(b) (0,1),  $(-\sqrt{4n-1}, -\sqrt{4n-3})$ , and  $(\sqrt{4n-1}, \sqrt{4n+1})$ , *n* any integer > 0  
(c) 0.7  
**27.**  $f(x) = \int_{1}^{x} (2^{t}/t) dt$   
**29.**  $f(x) = x^{3/2}$ ,  $a = 9$ 

**31.** (b) Average expenditure over [0, t]; minimize average expenditure.

#### Exercises 5.5 = page 400

1.  $(x^2 - 1)^{100}/200 + C$ 3.  $\frac{1}{4}e^{4x} + C$ 5.  $-1/[2(x^2 + 6x)] + C$ 7.  $(\ln x)^3/3 + C$ 9.  $\frac{2}{3}(x - 1)^{3/2} + C$ 11.  $(2 + x^4)^{3/2}/6 + C$ 13.  $-2/[5(t + 1)^5] + C$ 15.  $(1 + e^x)^{11}/11 + C$ 17.  $\frac{1}{3}\tan 3\theta + C$ 19.  $-\frac{1}{5}\cos^5 x + C$ 21.  $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$ 23.  $\frac{1}{2}\ln|2x - 1| + C$ 25.  $\ln|\ln x| + C$ 27.  $x - e^{-x} + C$ 29.  $\frac{1}{2}\ln|x^2 + 2x| + C$ 31.  $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$ 33.  $\frac{-1}{6(3x^2 - 2x + 1)^3} + C$ 1.7

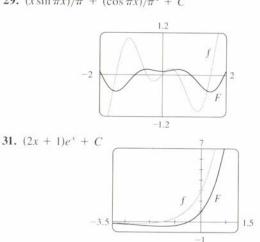




**37.**  $\frac{1}{101}$  **39.**  $\pi/8$  **41.**  $\frac{16}{15}$  **43.** 0 **45.**  $(4\sqrt{2}/3) - (5\sqrt{5}/12)$  **47.** 0 **49.**  $\frac{1}{2}\ln 3$  **51.** 2 **53.**  $\sqrt{3} - \frac{1}{3}$  **55.**  $6\pi$  **57.**  $\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$  **59.**  $\frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + \frac{1}{2}\ln|x| + C$  **63.**  $-\sqrt{x^2 + 4}/(4x) + C$ **65.**  $[5/(4\pi)][1 - \cos(2\pi t/5)]$  L **67.** 5

#### Exercises 5.6 ■ page 407

1.  $(xe^{2x}/2) - (e^{2x}/4) + C$ 3.  $-\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$ 5.  $\frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$ 7.  $x(\ln x)^2 - 2x\ln x + 2x + C$ 9.  $\frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$ 11.  $t^3(3\ln t - 1)/9 + C$ 13.  $e^{2\theta}(2\sin 3\theta - 3\cos 3\theta)/13 + C$ 15. 1 - 2/e17.  $-\frac{1}{2}$ 19.  $(\pi - 12 + 6\sqrt{3})/12$ 21.  $2\ln 4 - \frac{3}{2}$ 23. -125.  $2(\sin \sqrt{x} - \sqrt{x}\cos \sqrt{x}) + C$ 27.  $e^{x^2}[(x^4/2) - x^2 + 1] + C$ 29.  $(x\sin \pi x)/\pi + (\cos \pi x)/\pi^2 + C$ 



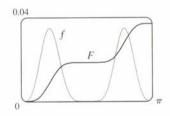
**35.** (b)  $\frac{2}{3}, \frac{8}{15}$  **39.**  $x[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6] + C$ **41.**  $2 - e^{-t}(t^2 + 2t + 2)$  m

#### Exercises 5.7 ■ page 413

1.  $\frac{1}{2}x^2 - x - 4\ln(x^2 + 9) + \frac{8}{3}\tan^{-1}(x/3) + C$ 3.  $\frac{1}{25}e^{-3x}(-3\cos 4x + 4\sin 4x) + C$ 5.  $(-\sqrt{9x^2 - 1}/x) + 3\ln|3x + \sqrt{9x^2 - 1}| + C$ 7.  $\frac{1}{2}[x^2\sin^{-1}(x^2) + \sqrt{1 - x^4}] + C$ 9.  $\frac{x + 2}{2}\sqrt{5 - 4x - x^2} + \frac{9}{2}\sin^{-1}\frac{x + 2}{3} + C$ 

- 11.  $\frac{1}{4} \tan x \sec^3 x + \frac{3}{8} \tan x \sec x + \frac{3}{8} \ln |\sec x + \tan x| + C$ 13.  $\frac{1}{9}\sin^3 x [3\ln(\sin x) - 1] + C$ 15.  $\frac{8}{15}$  17.  $\frac{1}{5} \ln \left| x^5 + \sqrt{x^{10} - 2} \right| + C$ **19.**  $(1 + e^x) \ln(1 + e^x) - e^x + C$ **21.**  $\sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C$ **25.**  $-\frac{1}{4}x(5-x^2)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}+\frac{25}{8}\sin^{-1}(x/\sqrt{5})+C$ 27.  $-\frac{1}{5}\sin^2 x \cos^3 x - \frac{2}{15}\cos^3 x + C$ **29.**  $\frac{1}{10}(1+2x)^{5/2}-\frac{1}{6}(1+2x)^{3/2}+C$ **31.**  $\frac{1}{2} \tan^2 x - \frac{1}{2} \ln(1 + \tan^2 x) + C$  $=\frac{1}{2}\tan^{2}x - \ln|\sec x| + C$  $\frac{2^{x-1}\sqrt{2^{2x}-1}}{\ln 2} - \frac{\ln(\sqrt{2^{2x}-1}+2^x)}{2\ln 2} + C$ 33. **35.**  $F(x) = \frac{1}{2} \ln(x^2 - x + 1) - \frac{1}{2} \ln(x^2 + x + 1);$ maximum at -1, 0.6 minimum at 1; IP at -1.7, 0, and 1.7 4
- **37.**  $F(x) = -\frac{1}{10} \sin^3 x \cos^7 x \frac{3}{80} \sin x \cos^7 x + \frac{1}{160} \cos^5 x \sin x + \frac{1}{128} \cos^3 x \sin x + \frac{3}{256} \cos x \sin x + \frac{3}{256} x$

-1.1



#### Exercises 5.8 ■ page 425

**1.** (a)  $L_2 = 6, R_2 = 12, M_2 \approx 9.8$ (b)  $L_2$  is an underestimate,  $R_2$  and  $M_2$  are overestimates. (c)  $T_2 = 9 < I$  (d)  $L_n < T_n < I < M_n < R_n$ 3. (a)  $T_4 \approx 0.895759$  (underestimate) (b)  $M_4 \approx 0.908907$  (overestimate)  $T_4 < I < M_4$ **5.** (a) 0.746211 (b) 0.747131 (c) 0.746825 7. (a) 0.132465 (b) 0.132857 (c) 0.132727 (b) 0.388849 (c) 0.395802 9. (a) 0.409140 **11.** (a) 1.098004 (b) 1.098709 (c) 1.109031 **13.** (a)  $T_{10} \approx 0.881839, M_{10} \approx 0.882202$ (b)  $|E_T| \le 0.01\overline{3}, |E_M| \le 0.00\overline{6}$ (c) n = 366 for  $T_n$ , n = 259 for  $M_n$ **15.** (a)  $T_{10} \approx 1.719713$ ,  $E_T = -0.001432$ ;  $S_{10} \approx 1.718283, E_s = -0.000001$ (b)  $|E_{\tau}| \le 0.002266$ ,  $|E_{s}| \le 0.0000016$ (c) n = 151 for  $T_n$ , n = 107 for  $M_n$ , n = 8 for  $S_n$ 17. (a) 2.8 (b) 7.954926518 (c) 0.287 (d) 7.954926521 (e) The actual error is much smaller (f) 10.9 (g) 7.953789422 (h) 0.0592 (i) The actual error is smaller (j)  $n \ge 50$ 

n	$L_n$	$R_n$	$T_n$	$M_n$
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

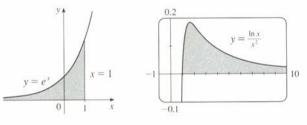
n	$E_L$	$E_R$	$E_T$	$E_M$
4	0.109375	-0.140625	-0.015625	0.007813
8	0.058594	-0.066406	-0.003906	0.001953
16	0.030273	-0.032227	-0.000977	0.000488

Observations are the same as after Example 1.

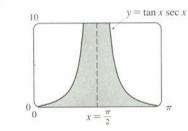
**21.** (a) 11.5 (b) 12 (c) 11.6 **23.** 8.57 mi **25.** 37.73 ft/s **27.** 29.3%

#### Exercises 5.9 ■ page 435

**1.** (a) Infinite interval (b) Infinite discontinuity (c) Infinite discontinuity (d) Infinite interval **3.**  $\frac{1}{2} - 1/(2t^2)$ ; 0.495, 0.49995, 0.4999995; 0.5 5. 1 7.  $\frac{1}{2}$  9. Divergent 11. 0 13. Divergent 17.  $e^2/4$ 19. Divergent 15. Divergent 21. Divergent **23.**  $2\sqrt{3}$ 25. Divergent 29.  $\frac{5}{3}$  31.  $-\frac{1}{4}$ 27. Divergent 35. 1 33. e

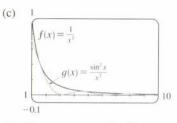


37. Divergent



<b>39.</b> (a)	t	$\int_1^t \left[ (\sin^2 x) / x^2 \right] dx$
Ī	2	0.447453
	5	0.577101
	10	0.621306
	100	0.688479
	1,000	0.672957
	10,000	0.673390

It appears that the integral is convergent.



**41.** Convergent **43.** Convergent **45.** Divergent **47.**  $\pi$  **49.** 1/(1 - p), p < 1**53.** (a) y + 1 + y = F(t) + y = F

(b) The rate at which the fraction F(t) increases as t increases

(in hours)

- (c) 1; all bulbs burn out eventually
- **55.** 8264.5 years **59.** 1000

#### Chapter 5 Review = page 437

#### True-False Quiz

**1.** True **3.** False **5.** True **7.** True **9.** True **11.** False

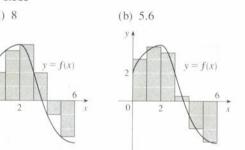
#### Exercises

1. (a) 8

VA

2

0



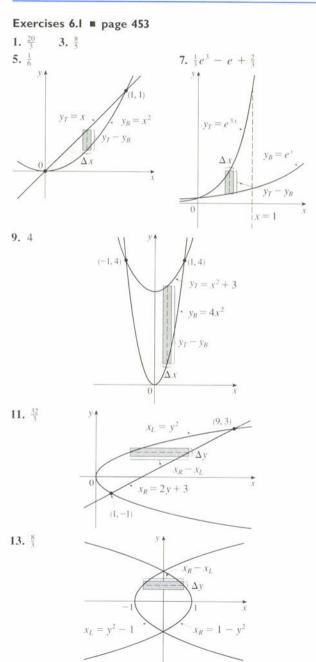
**3.**  $\frac{1}{2} + \frac{\pi}{4}$  **5.** 3 **7.**  $f = c, f' = b, \int_0^x f(t) dt = a$ 9.  $\frac{9}{10}$  11.  $\frac{1}{2} \ln 2$  13.  $\frac{1209}{28}$  15. 2 **17.**  $(1/\pi)(e^{\pi}-1)$  **19.**  $x \sec x - \ln|\sec x + \tan x| + C$ **21.**  $\ln(e^x + 1) + C$  **23.**  $\frac{1}{2}e^x(\cos x + \sin x) + C$ **25.**  $2e^{\sqrt{x}} + C$  **27.**  $2\sqrt{1 + \sin x} + C$ 29.  $\frac{64}{5}$ **31.**  $F'(x) = \sqrt{1 + x^4}$  **33.**  $g'(x) = 3x^5/\sqrt{1 + x^9}$ **35.**  $\frac{1}{2} \left[ e^x \sqrt{1 - e^{2x}} + \sin^{-1}(e^x) \right] + C$ **37.**  $\frac{1}{4}(2x+1)\sqrt{x^2+x+1}$  +  $\frac{3}{8} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + C$ **39.** (a) 1.090608 (overestimate) (b) 1.088840 (underestimate) (c) 1.089429 (unknown) **41.** (a) 0.0067 (b) 0.003 **43.** (a) 3.8 (b) 1.7867, 0.000646 (c)  $n \ge 30$ **45.**  $\frac{1}{24}$  **47.** Divergent **49.** 2 **51.** Convergent **53.** (a) 29.16 (b) 29.5 **55.** 44.4%

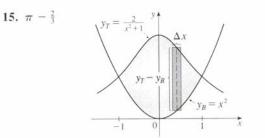
**57.**  $Ce^{-x^2/(4kt)}/\sqrt{4\pi kt}$ **59.**  $f(x) = e^{2x}(2x+1)/(1-e^{-x})$ 

#### Focus on Problem Solving page 444

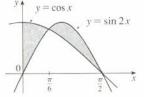
**1.** About 1.85 in. from the center **3.**  $\pi/2$  **5.** 1 **7.**  $e^{-2}$  **9.** Does not exist **11.** [-1, 2] **13.**  $\sqrt{1 + \sin^4 x} \cos x$  **15.** 0

#### CHAPTER 6



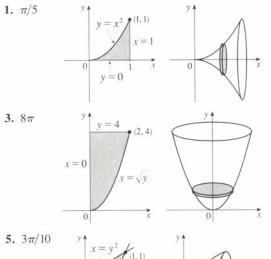


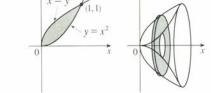
**17.** -1.02, 1.02; 2.70 **19.** 0, 0.70; 0.08 **21.** 118 ft **23.** 84 m<sup>2</sup> **25.**  $\frac{1}{2}$  **9** ft  $y = \cos x$ 



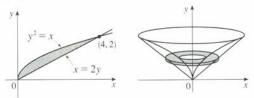
**27.**  $\pi ab$  **29.**  $(e^{\pi/2} - 1)/2$  **31.**  $24\sqrt{3}/5$  **33.**  $\pm 6$ **35.**  $4^{2/3}$  **37.**  $f(t) = 3t^2$ **41.**  $0 < m < 1; m - \ln m - 1$ 

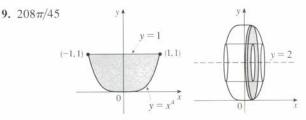
#### Exercises 6.2 page 462





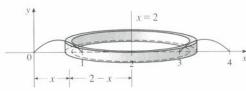
#### **7.** 64π/15



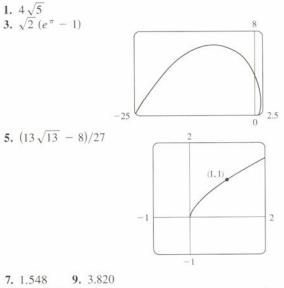


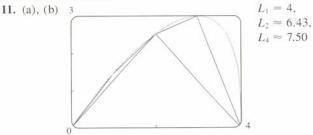
**11.**  $832\pi/21$  **13.** 0, 0.747; 0.132 **15.** 1072 cm<sup>3</sup> **17.**  $\pi r^2 h/3$  **19.**  $\pi h^2 [r - (h/3)]$  **21.**  $2b^2 h/3$  **23.** 10 cm<sup>3</sup> **25.** 24 **27.** 2 **29.**  $\frac{1}{3}$  **31.** (a)  $8\pi R \int_0^r \sqrt{r^2 - y^2} \, dy$  (b)  $2\pi^2 r^2 R$  **33.** (b)  $\pi r^2 h$  **35.**  $\frac{5}{12}\pi r^3$  **37.**  $8 \int_0^r \sqrt{R^2 - y^2} \sqrt{r^2 - y^2} \, dy$  **39.**  $\pi/15$ **41.**  $\pi/2$   $y^*$ 





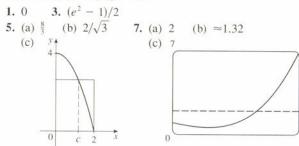
#### Exercises 6.3 ■ page 468





(c)  $\int_{0}^{4} \sqrt{1 + [4(3 - x)/(3(4 - x)^{2/3})]^2} dx$  (d) 7.7988 **13.** (81 ln 3)/512 **15.**  $\ln(\sqrt{2} + 1)$  **17.** 29.36 **21.** (a) 15  $0 \le t \le 4\pi$  (b)  $\approx 294$ -15

#### Exercises 6.4 page 472



**11.**  $(50 + 28/\pi)$  °F  $\approx 59$  °F **13.** 6 kg/m **15.**  $5/(4\pi) \approx 0.4$  L

#### Exercises 6.5 ■ page 482

**1.**  $\frac{5030}{24}$  ft-lb **3.**  $\frac{15}{4}$  ft-lb **5.** (a)  $\frac{25}{24} \approx 1.04$  J (b) 10.8 cm **7.** 625 ft-lb **9.** 650,000 ft-lb **11.**  $\approx 2.45 \times 10^3$  J **13.** (a)  $\approx 1.06 \times 10^6$  J (b) 2.0 m **17.** (a)  $Gm_1m_2[(1/a) - (1/b)]$  (b)  $\approx 8.50 \times 10^9$  J **19.**  $6.5 \times 10^6$  N **21.**  $1.56 \times 10^3$  lb **23.** (a)  $5.63 \times 10^3$  lb (b)  $5.06 \times 10^4$  lb (c)  $4.88 \times 10^4$  lb (d)  $3.03 \times 10^5$  lb **25.** 40, 12,  $(1, \frac{10}{3})$  **27.** (1/(e-1), (e+1)/4)**29.**  $(0, \pi/8)$  **31.**  $\frac{4}{3}, 0, (0, \frac{2}{3})$  **33.** (b)  $(\frac{1}{2}, \frac{2}{5})$ 

#### Exercises 6.6 ■ page 488

**1.** \$14,516,000 **3.** \$388,280,000 **5.** \$316.29 **7.** \$4166.67 **9.** \$112,500 **11.**  $1.19 \times 10^{-4} \text{ cm}^3/\text{s}$  **13.**  $\frac{1}{9} \text{ L/s}$ 

#### Exercises 6.7 ■ page 494

(a) ∫<sup>200</sup><sub>100</sub> f(t) dt is the probability that a randomly chosen battery will have a lifetime between 100 and 200 hours.
 (b) ∫<sup>∞</sup><sub>200</sub> f(t) dt is the probability that a randomly chosen battery will have a lifetime of at least 200 hours.

**3.** (a)  $f(x) \ge 0$  for all x and  $\int_{-\infty}^{\infty} f(x) dx = 1$  (b) 5 **7.** (a)  $e^{-4/2.5} \approx 0.20$  (b)  $1 - e^{-2/2.5} \approx 0.55$  (c) If you area's correct within 10 minutes of f(x) = 0

(c) If you aren't served within 10 minutes, you get a free hamburger.

**9.**  $\approx$ 44.3% **11.**  $\approx$ 0.9545 **13.**  $2/\pi$ ,  $1/\pi$ 

#### Chapter 6 Review ■ page 496

#### Exercises

**1.** 108 **3.** (a) 0.38 (b) 0.87 **5.** (a)  $2\pi/15$  (b)  $\pi/6$  (c)  $8\pi/15$  **7.** (-2, 3);  $\frac{81}{20}$  **9.** 36 **11.**  $125\sqrt{3}/3$  m<sup>3</sup> **13.**  $2(5\sqrt{5}-1)$  **15.** 3.2 J **17.** (a)  $8000\pi/3$  ft-lb (b) 2.1 ft **19.**  $\approx 458$  lb **21.** \$7166.67 **23.** f(x) **25.** (a)  $1 - e^{-3/8} \approx 0.31$  (b)  $e^{-5/4} \approx 0.29$ (c)  $8 \ln 2 \approx 5.55$  min

#### Focus on Problem Solving a page 499

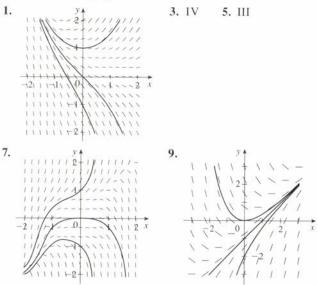
1.  $f(x) = \sqrt{2x/\pi}$ 3. (b) 0.2261 (c) 0.6736 m (d) (i)  $1/(105\pi) \approx 0.003$  in/s (ii)  $370\pi/3 \ s \approx 6.5$  min 7. (a)  $P(z) = P_0 + g \int_0^z \rho(x) dx$ (b)  $(P_0 - \rho_0 gH) (\pi r^2) + \rho_0 gH e^{L/H} \int_{-r}^r e^{x/H} \cdot 2\sqrt{r^2 - x^2} dx$ 11.  $\ln(\pi/2)$ 

#### CHAPTER 7

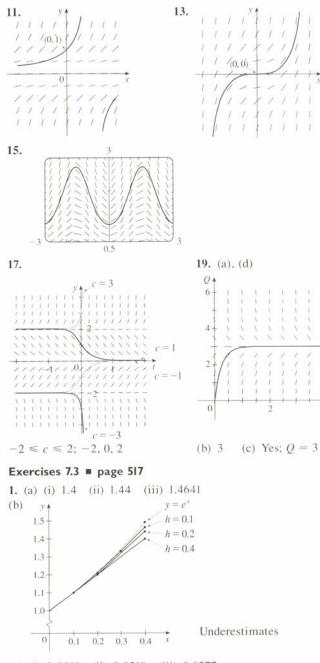
#### Exercises 7.1 = page 508

3. (a)  $\pm 3$  5. (b) and (c) 7. (a) It must be either 0 or decreasing (c) y = 0 (d) y = 1/(x + 2)9. (a) 0 < P < 4200 (b) P > 4200 (c) P = 0, P = 420011. (a) At the beginning; stays positive, but decreases (c) P(t)M P(0) 0t

#### Exercises 7.2 ■ page 513



5  $\mu = -\ln(C - \frac{1}{2}e^{2t})$  7  $\nu = \tan(x - 1)$ 



(c) (i) 0.0918 (ii) 0.0518 (iii) 0.0277
It appears that the error is also halved (approximately).
3. 2, 2.75, 3.5, 4.25 5. 1.8371
7. (a) (i) 3 (ii) 2.3928 (iii) 2.3701 (iv) 2.3681
(c) (i) -0.6321 (ii) -0.0249 (iii) -0.0022
(iv) -0.0002
It appears that the error is also divided by 10
(approximately).
9. 2.77 C

#### Exercises 7.4 page 524

**1.** y = -1/(x + C) or y = 0 **3.**  $x^2 - y^2 = C$ 

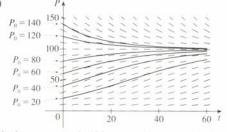
**35.** (a)  $dA/dt = k\sqrt{A} (M - A)$ (b)  $A = M[(Ce^{\sqrt{M}kt} - 1)/(Ce^{\sqrt{M}kt} + 1)]^2$ , where  $C = (\sqrt{M} + \sqrt{A_0})/(\sqrt{M} - \sqrt{A_0})$  and  $A_0 = A(0)$ [If  $A_0 = 0$ , then C = 1.] **37.** (b)  $y(t) = (\sqrt{6} - \frac{1}{144}t)^2$  (c)  $144\sqrt{6}$  s  $\approx 5$  min 53 s

#### Exercises 7.5 = page 534

1. About 235 3. (a)  $500 \times 16^{t/3}$  (b)  $\approx 20,159$  (c)  $(3 \ln 60)/\ln 16 \approx 4.4$  h 5. (a) 1403 million, 1746 million (b) 2208 million (c) 3667 million; wars in first half of century, increased life expectancy in second half 7. (a)  $Ce^{-0.0005t}$  (b)  $-2000 \ln 0.9 \approx 211$  s **9.** (a)  $50 \times 2^{-t/0.00014}$  (b)  $\approx 1.57 \times 10^{-20}$  mg (c)  $\approx 4.5 \times 10^{-5}$  s 11. ≈2500 yr **13.** (a)  $dy/dt = ky, y(0) = 110; y(t) = 110e^{kt}$ (b)  $\approx 137 \,^{\circ}F$  (c)  $\approx 116 \, \text{min}$ **15.** (a)  $\approx 64.5 \text{ kPa}$  (b)  $\approx 39.9 \text{ kPa}$ 17. (a) (i) \$3828.84 (ii) \$3840.25 (iii) \$3850.08 (iv) \$3851.61 (v) \$3852.01 (vi) \$3852.08 (b) dA/dt = 0.05A, A(0) = 3000**19.** (a)  $P(t) = (m/k) + (P_0 - m/k)e^{kt}$  (b)  $m < kP_0$ (c)  $m = kP_0, m > kP_0$  (d) Declining

#### Exercises 7.6 ■ page 544

**1.** (a) 100; 0.05 (b) Where *P* is close to 0 or 100; on the line P = 50;  $0 < P_0 < 100$ ;  $P_0 > 100$  (c)  $P_{\bullet}$ 



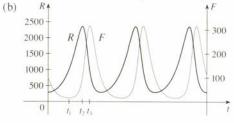
Solutions approach 100; some increase and some decrease, some have an inflection point but others don't; solutions with  $P_0 = 20$  and  $P_0 = 40$  have inflection points at P = 50(d) P = 0, P = 100; other solutions move away from P = 0and toward P = 100**3.** (a)  $3.23 \times 10^7$  kg (b) ≈1.55 yr 5. (a) dP/dt = 0.00377P(1 - P/100)(b) In billions: 5.49, 7.81, 27.72 (c) In billions: 5.48, 7.61, 22.41 7. (a) dy/dt = ky(1 - y) (b)  $y = y_0/[y_0 + (1 - y_0)e^{-kt}]$ (c) 3:36 P.M 11. (a) Fish are caught at a rate of 15 per week. (c) P = 250, P = 750(b) See part (d) (d) P 1200 800 400 80 120 /  $0 < P_0 < 250; P \rightarrow 0; P_0 = 250; P \rightarrow 250;$  $P_0 > 250: P \to 750$ (e)  $P(t) = \frac{250 - 750ce^{t/25}}{1 - ce^{t/25}},$ 1200 where  $c = \frac{1}{11}, -\frac{1}{9}$ 120 0 13. (b) P 1400 1200 1000 800 600 400 200 0 20 40 60 80 100 1  $0 < P_0 < 200; P \rightarrow 0; P_0 = 200; P \rightarrow 200;$  $P_0 > 200: P \to 1000$ 

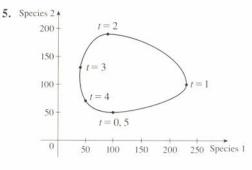
(c)  $P(t) = \frac{m(K - P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}$  **15.** (a)  $P(t) = P_0 e^{(k/r)[\sin(rt - \phi) + \sin \phi]}$ (b) Does not exist

#### Exercises 7.7 = page 551

(a) x = predators, y = prey; growth is restricted only by predators, which feed only on prey.
 (b) x = prey, y = predators; growth is restricted by carrying capacity and by predators, which feed only on prey.
 (a) The rabbit population starts at about 300, increases

to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.



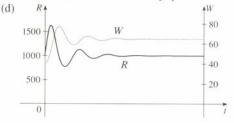


9. (a) Population stabilizes at 5000

(b) (i) W = 0, R = 0: Zero populations

(ii) W = 0, R = 5000: In the absence of wolves, the rabbit population is always 5000.

(iii) W = 64, R = 1000: Both populations are stable.

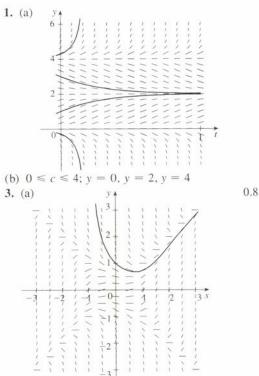


Chapter 7 Review = page 553

#### True-False Quiz

1. False 3. False 5. True

#### Exercises



(b) 0.7568

(c) y = x and y = -x; there is a local maximum or minimum 5.  $y = ke^{x+x^3/3} - 2$ 7.  $y = \sqrt{(\ln x)^2 + 4}$ 9.  $y^2 - 2\ln|y| + x^2 = K$ 11. (a)  $1000e^{(\ln 9)t/2} = 1000 \times 3'$  (b) 27,000 (c)  $(\ln 2)/\ln 3 \approx 0.63 \text{ h}$ 13. (a)  $C_0e^{-kt}$  (b)  $\approx 100 \text{ h}$ 

**15.** (a)  $L(t) = L_{\infty} - [L_{\infty} - L(0)]e^{-kt}$ 

(b)  $L(t) = 53 - 43e^{-0.2t}$  17. 15 days

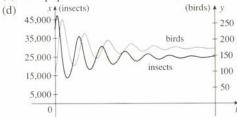
19. (a) Stabilizes at 200,000

(b) (i) x = 0, y = 0: Zero populations

(ii) x = 200,000, y = 0: In the absence of birds, the insect population is always 200,000.

(iii) x = 25,000, y = 175: Both populations are stable.

(c) The populations stabilize at 25,000 insects and 175 birds.



#### Focus on Problem Solving page 556

**1.**  $f(x) = \pm 10e^x$ **5.** (a) 9.8 h (b) 31,900 $\pi \approx 100,000$  ft<sup>2</sup>; 6283 ft<sup>2</sup>/h (c) 5.1 h

7. (b) 
$$f(x) = (x^2 - L^2)/(4L) - (L/2)\ln(x/L)$$
 (c) No

#### **CHAPTER 8**

#### Exercises 8.1 = page 567

**1.** (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms  $a_n$  approach 8 as n becomes large.

(c) The terms  $a_n$  become large as n becomes large.

**3.**  $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$ ; yes;  $\frac{1}{2}$ 5.  $a_n = 1/2^n$ 9.0 7.  $a_n = (n + 2)/(n + 3)^2$ 11. 1 17. Diverges (to ∞) **13.** Diverges (to  $\infty$ ) 15. Diverges 23. 0 25. 0 27. Diverges 19. 0 21. 0 33. (a) Divergent (b) Convergent 29.  $\pi/4$ 31. 0 **35.** (b)  $(1 + \sqrt{5})/2$ 39. Increasing

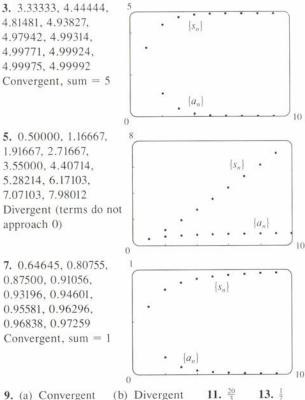
**35.** (b)  $(1 + \sqrt{5})/2$  **37.** Decreasing **39.** Increasin **41.** Convergent by the Monotonic Sequence Theorem;  $5 \le L < 8$ 

**43.**  $(3 + \sqrt{5})/2$  **45.** 62

#### Exercises 8.2 m page 577

**1.** (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

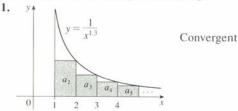


9. (a) Convergent (b) Divergent 11.  $\frac{1}{3}$  13. 15. Divergent 17.  $\frac{17}{36}$  19. Divergent 21.  $\frac{3}{4}$ 

**23.**  $\frac{3}{2}$  **25.**  $\sin 1$  **27.** Divergent **29.**  $\frac{5}{9}$  **31.**  $\frac{307}{999}$ **33.** 2 < x < 4, 1/(4 - x) **35.** |x| > 1, x/(x - 1)**37.**  $\frac{1}{4}$  **39.**  $a_1 = 0$ ,  $a_n = 2/[n(n + 1)]$  for n > 1, sum = 1 **41.** (a)  $S_n = D(1 - c^n)/(1 - c)$  (b) 5 **43.**  $(\sqrt{3} - 1)/2$  **45.** 1/[n(n + 1)]**47.** The series is divergent. **51.**  $\{s_n\}$  is bounded and increasing. **53.** (a)  $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$ **55.** (a)  $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}; [(n + 1)! - 1]/(n + 1)!$  (c) 1

#### Exercises 8.3 page 588

Abbreviations: C, convergent, D, divergent



**3.** (a) Nothing (b) Convergent **5.** *p*-series; geometric series; b < -1; -1 < b < 1 **7.** C **9.** D **11.** D **13.** C **15.** D **17.** C **19.** D **21.** D **23.** p > 1 **25.** (a) 1.54977, error  $\leq 0.1$  (b) 1.64522, error  $\leq 0.005$ (c) n > 1000**27.** 2.6124 **29.** 0.567975, error  $\leq 0.0003$  **35.** Yes

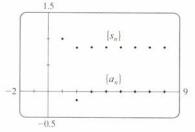
#### Exercises 8.4 page 595

**1.** (a) A series whose terms are alternately positive and negative

(b)  $0 < b_{n+1} \le b_n$  and  $\lim_{n \to \infty} b_n = 0$ , where  $b_n = |a_n|$ (c)  $|R_n| \le b_{n+1}$ 3. C 5. D 7. C 9. Underestimate 11. p > 0

**13.** 7 **14.** p > 0

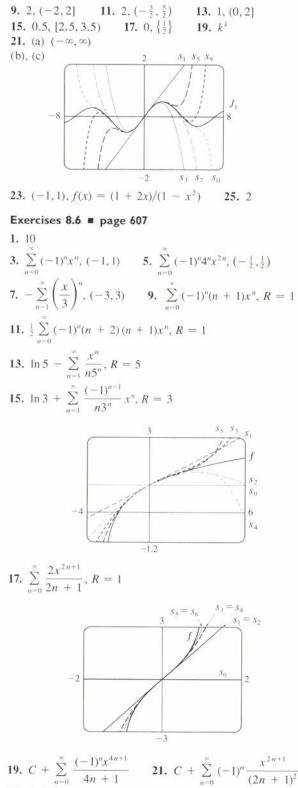
15. 0.8415



**17.** 0.6065 **19.** No **21.** No **23.** Yes **25.** Yes **27.** Yes **29.** Diverges **31.** (a) and (d) **35.** (a)  $\frac{661}{960} \approx 0.68854$ , error < 0.00521 (b)  $n \ge 11, 0.693109$ 

#### Exercises 8.5 ■ page 602

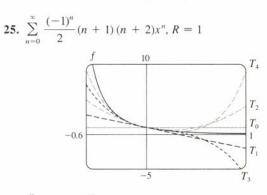
**1.** A series of the form  $\sum_{n=0}^{\infty} c_n (x-a)^n$ , where x is a variable and a and the  $c_n$ 's are constants **3.** (a) Yes (b) No **5.** 1, [-1, 1) **7.**  $\infty$ ,  $(-\infty, \infty)$ 



**19.**  $C + \sum_{n=0}^{2} \frac{4n+1}{4n+1}$  **21.**  $C + \sum_{n=0}^{2} (-1) \frac{(2n+1)}{(2n+1)}$ **23.** 0.199936 **25.** 0.000065 **27.** 0.09531 **29.** (b) 0.920 **33.** [-1, 1], [-1, 1), (-1, 1)

#### Exercises 8.7 = page 618

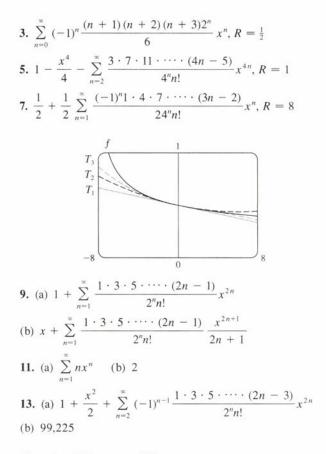
1. 
$$b_8 = f^{(8)}(5)/8!$$
  
3.  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$   
5.  $\sum_{n=0}^{\infty} (-1)^n (n+1)x^n, R = 1$   
7.  $\sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n, R = \infty$   
9.  $\sum_{n=0}^{\infty} (-1)^n (x-1)^n, R = 1$   
11.  $\sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2} (x-\pi/4)^n}{\sqrt{2n!}}, R = \infty$   
15.  $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}, R = \infty$   
17.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n)!}, R = \infty$   
19.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n+1} (2n+1)!}, R = \infty$   
21.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}, R = \infty$   
23.  $1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n!} x^n, R = 1$ 



27.  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, 0.09531$ 29.  $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$ 31.  $C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n n! (3n+1)} x^{3n+1}$ 33. 0.310 35. 0.09998750 37.  $\frac{1}{3}$  39.  $\frac{1}{120}$ 41.  $1 - \frac{3}{2}x^2 + \frac{25}{4}x^4$  43.  $-x + \frac{1}{2}x^2 - \frac{1}{3}x^3$  45.  $e^{-x^4}$ 47.  $1/\sqrt{2}$  49.  $e^x - 1$ 

## Exercises 8.8 ■ page 622

**1.**  $1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{2^n n!} x^n, R = 1$ 



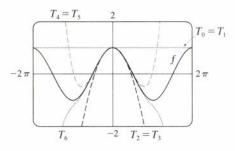
#### Exercises 8.9 = page 630

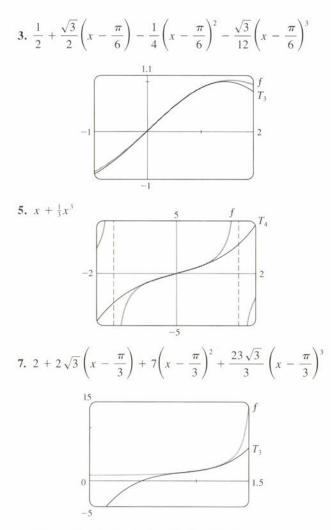
1. (a)  $T_0(x) = 1 = T_1(x), T_2(x) = 1 - \frac{1}{2}x^2 = T_3(x),$   $T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 = T_5(x),$  $T_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$ 

(b)

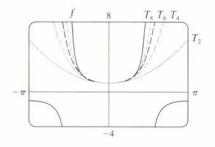
x	f	$T_1$	$T_2 = T_3$	$T_4 = T_5$	$T_6$
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
$\pi^{2}$	-1	1	-3.9348	0.1239	-1.2114

(c) An *n* increases,  $T_n(x)$  is a good approximation to f(x) on a larger and larger interval.





**9.**  $T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8$ 



**11.** (a)  $1 + \frac{1}{2}x$  (b) 0.00125 **13.** (a)  $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{24\sqrt{2}}\left(x - \frac{\pi}{4}\right)^4 + \frac{1}{120\sqrt{2}}\left(x - \frac{\pi}{4}\right)^5$ (b) 0.00033 **15.** (a)  $1 + x^2$  (b) 0.00006 **17.** 0.57358 **19.** 3 **21.** -1.037 < x < 1.037**23.** 21 m, no

#### Exercises 8.10 = page 637

1. 
$$\sum_{n=0}^{\infty} c_0 \frac{6^n}{n!} x^n = c_0 e^{6x}$$
3. 
$$\sum_{n=0}^{\infty} \frac{c_0}{3^n n!} x^{3n} = c_0 e^{x^3/3}$$
5. 
$$c_0 \sum_{n=0}^{\infty} \left( -\frac{3}{2} \right)^n \frac{1}{n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-6)^n n!}{(2n+1)!} x^{2n+1}$$
7. 
$$\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$$
9. 
$$x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2 5^2 \cdots (3n-1)^2}{(3n+1)!} x^{3n+1}$$

#### Chapter 8 Review page 637

#### True-False Quiz

 1. False
 3. False
 5. False
 7. False
 9. True

 11. True
 13. False
 15. True

#### Exercises

**1.** C,  $\frac{1}{2}$  **3.** D **5.** D **7.** C,  $e^{12}$  **9.** D **11.** C **13.** D **15.** C **17.** C **19.** 8 **21.**  $\pi/4$  **23.**  $\frac{4111}{3330}$  **25.** 0.9721 **27.** 0.18976224, error < 6.4 × 10<sup>-7</sup> **31.** 3, [-3,3] **33.** 0.5, [2.5, 3.5)

35. 
$$\frac{1}{2} + \frac{\sqrt{3}}{2} \left( x - \frac{\pi}{6} \right) - \frac{1}{2} \frac{1}{2!} \left( x - \frac{\pi}{6} \right)^2 - \frac{\sqrt{3}}{2} \frac{1}{3!} \left( x - \frac{\pi}{6} \right)^3 + \dots$$
$$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{(2n)!} \left( x - \frac{\pi}{6} \right)^{2n} + \frac{\sqrt{3}}{(2n+1)!} \left( x - \frac{\pi}{6} \right)^{2n+1} \right]$$
37. 
$$\sum_{n=0}^{\infty} (-1)^n x^{n+2}, 1 \qquad 39. - \sum_{n=1}^{\infty} \frac{x^n}{n}, 1$$
41. 
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}, \infty$$
43. 
$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{n! 2^{6n+1}} x^n, 16$$
45. 
$$\ln |x| + C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$
47. (a) 
$$1 + \frac{1}{2} (x - 1) - \frac{1}{8} (x - 1)^2 + \frac{1}{16} (x - 1)^3$$
(b) 1.5 (c) 0.000006

**49.** 1 **51.**  $\sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$ 

#### Focus on Problem Solving ■ page 640

**1.** 15!/5! = 10,897,286,400 **3.** (b) 0 if x = 0,  $(1/x) - \cot x$  if  $x \neq n\pi$ , *n* an integer **5.** (a)  $s_n = 3 \cdot 4^n$ ,  $l_n = 1/3^n$ ,  $p_n = 4^n/3^{n-1}$  (c)  $2\sqrt{3}/5$ **9.** (-1,1),  $(x^3 + 4x^2 + x)/(1 - x)^4$ 

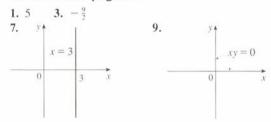
#### APPENDIXES

#### Exercises A page A6

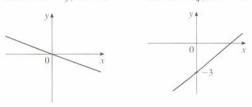
<b>1.</b> 18 <b>3.</b> $5 - \sqrt{5}$	5. $2 - x$
7. $ x+1  = \begin{cases} x+1\\ -x-1 \end{cases}$	for $x \ge -1$ for $x < -1$ <b>9.</b> $x^2 + 1$
11. (−2,∞)	<b>13.</b> [−1,∞)
-2 0	-1 0
<b>15.</b> (0, 1]	<b>17.</b> $(-\infty, 1) \cup (2, \infty)$
 0 1 →	
<b>19.</b> $(-\sqrt{3},\sqrt{3})$	<b>21.</b> (−∞, 1]
$-\sqrt{3}$ 0 $\sqrt{3}$	0 1
<b>23.</b> $(-1,0) \cup (1,\infty)$	<b>25.</b> $(-\infty, 0) \cup (\frac{1}{4}, \infty)$
-1 0 1	$\xrightarrow{0} 1$
	4

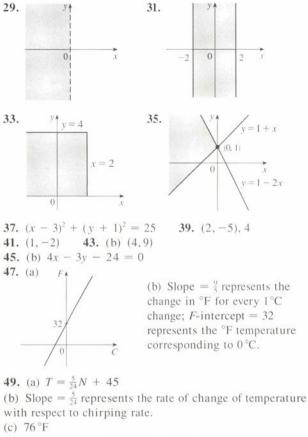
**27.**  $10 \le C \le 35$  **29.** (a)  $T = 20 - 10h, 0 \le h \le 12$ (b)  $-30 \degree C \le T \le 20 \degree C$  **31.**  $2, -\frac{4}{3}$  **33.** (-3,3) **35.** (3,5) **37.**  $(-\infty, -7] \cup [-3,\infty)$ **39.** [1.3, 1.7] **41.**  $x \ge (a + b)c/(ab)$ 

#### Exercises B = page Al7



**11.** y = 6x - 15 **13.** 5x + y = 11 **15.** y = 3x - 2 **17.** y = 3x - 3 **19.** y = 5 **21.** x + 2y + 11 = 0 **23.** 5x - 2y + 1 = 0**25.**  $m = -\frac{1}{3}, b = 0$  **27.**  $m = \frac{3}{4}, b = -3$ 



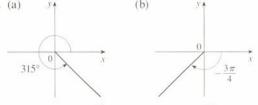


**51.** (a) P = 0.43d + 15 (b) 196 ft **59.**  $y_{\pm}^{*}$ 



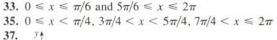
#### Exercises C = page A30

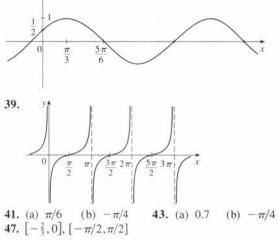
- **1.** (a)  $7\pi/6$  (b)  $\pi/20$  **3.** (a)  $720^{\circ}$  (b)  $-67.5^{\circ}$ **5.**  $3\pi$  cm **7.**  $\frac{2}{3}$  rad =  $(120/\pi)^{\circ}$
- 9. (a)  $y_1$  (b)



**11.**  $\sin(3\pi/4) = 1/\sqrt{2}$ ,  $\cos(3\pi/4) = -1/\sqrt{2}$ ,  $\tan(3\pi/4) = -1$ ,  $\csc(3\pi/4) = \sqrt{2}$ ,  $\sec(3\pi/4) = -\sqrt{2}$ ,  $\cot(3\pi/4) = -1$ **13.**  $\cos\theta = \frac{4}{5}$ ,  $\tan\theta = \frac{3}{4}$ ,  $\csc\theta = \frac{5}{3}$ ,  $\sec\theta = \frac{5}{4}$ ,  $\cot\theta = \frac{4}{3}$ 

- **15.** 5.73576 cm **17.** 24.62147 cm **27.**  $(4 + 6\sqrt{2})/15$
- **29.**  $\pi/3$ ,  $5\pi/3$  **31.**  $\pi/6$ ,  $\pi/2$ ,  $5\pi/6$ ,  $3\pi/2$





#### Exercises D = page A38

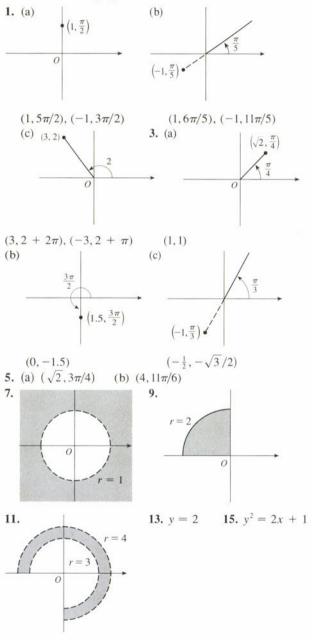
4/7 (or any smaller positive number)
 0.6875 (or any smaller positive number)
 0.11, 0.012 (or smaller positive numbers)
 (a) √1000/π cm (b) Within approximately 0.0445 cm
 (c) Radius; area; √1000/π; 1000; 5; ≈0.0445
 N ≥ 13 15. (a) x > 100 17. (a) 0 (b) 9, 11

Exercises F = page A50 1.  $\frac{A}{2x+1} + \frac{B}{x-2}$ 3.  $\frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \frac{D}{r-1}$  5.  $1 + \frac{A}{r-1} + \frac{B}{r+1}$ 7.  $\frac{A}{x} + \frac{Bx + C}{x^2 + 2}$  9.  $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + x + 2}$ 11.  $\frac{x^2}{2} - x + \ln|x + 1| + C$ 13.  $\ln 3 + 3 \ln 6 - 3 \ln 4 = \ln \frac{81}{9}$ **15.**  $2 \ln 2 + \frac{1}{2}$  **17.**  $4 \ln 6 - 3 \ln 5$ **19.**  $2\ln|x| + 3\ln|x + 2| + (1/x) + C$ **21.**  $\ln\sqrt{3} - (\sqrt{3}\pi/18)$ 23.  $\ln(x-1)^2 + \ln \sqrt{x^2+1} - 3 \tan^{-1}x + C$ **25.**  $\frac{1}{3} \ln |x - 1| - \frac{1}{6} \ln (x^2 + x + 1) - \frac{1}{6} \ln (x^2 + x + 1)$  $\frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C$ **27.**  $\frac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}(x/\sqrt{2}) + C$ **29.**  $-\frac{1}{2}\ln 3 \approx -0.55$  **31.**  $\ln \frac{3}{2}$ **33.** (a)  $\frac{24,110}{1000} - \frac{1}{1000} - \frac{668}{1000} - \frac{1}{1000}$  $\frac{24,110}{4879} \frac{1}{5x+2} - \frac{668}{323} \frac{1}{2x+1} - \frac{9438}{80,155} \frac{1}{3x-7} +$ 1 22,098x + 48,935260,015  $x^2 + x + 5$ 

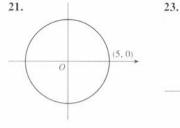
(b) 
$$\frac{4822}{4879} \ln|5x + 2| - \frac{334}{323} \ln|2x + 1| - \frac{3146}{80,155} \ln|3x - 7| + \frac{11,049}{260,015} \ln(x^2 + x + 5) + \frac{75,772}{260,015\sqrt{19}} \tan^{-1}\frac{2x + 1}{\sqrt{19}} + C$$

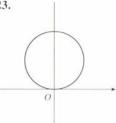
The CAS omits the absolute value signs and the constant of integration.

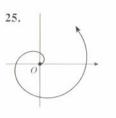
Exercises G.I = page A59

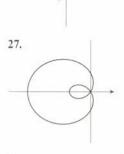


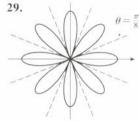
**17.**  $r \sin \theta = 5$  **19.** r = 5

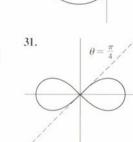


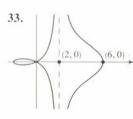










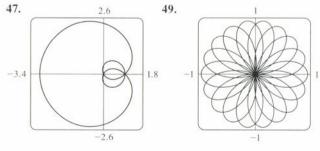


**35.**  $1/\sqrt{3}$  **37.**  $-2/\pi$ **39.** Horizontal at  $(1, 3\pi/2), (1, \pi/2), (\frac{2}{3}, \alpha), (\frac{2}{3}, \pi - \alpha), (\frac{2}{3}, \pi + \alpha), (\frac{2}{3}, 2\pi - \alpha),$  where  $\alpha = \sin^{-1}(1/\sqrt{6});$  vertical at  $(1, 0), (1, \pi), (\frac{2}{3}, 3\pi/2 - \alpha), (\frac{2}{3}, 3\pi/2 + \alpha), (\frac{2}{3}, \pi/2 - \alpha), (\frac{2}$ 

**41.** Horizontal at  $(\frac{3}{2}, \pi/3)$ ,  $(\frac{3}{2}, 5\pi/3)$ , and the pole; vertical at (2, 0),  $(\frac{1}{2}, 2\pi/3)$ ,  $(\frac{1}{2}, 4\pi/3)$ 

**43.** Center (b/2, a/2), radius  $\sqrt{a^2 + b^2}/2$ 

**45.** (a) For c < -1, the loop begins at  $\theta = \sin^{-1}(-1/c)$  and ends at  $\theta = \pi - \sin^{-1}(-1/c)$ ; for c > 1, it begins at  $\theta = \pi + \sin^{-1}(1/c)$  and ends at  $\theta = 2\pi - \sin^{-1}(1/c)$ .

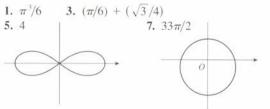


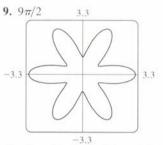
**51.** By counterclockwise rotation through angle  $\pi/6$ ,  $\pi/3$ , or  $\alpha$  about the origin

53. (a) A rose with n loops if n is odd and 2n loops if n is even(b) Number of loops is always 2n

**55.** For 0 < a < 1, the curve is an oval, which develops a dimple as  $a \rightarrow 1^-$ . When a > 1, the curve splits into two parts, one of which has a loop.

#### Exercises G.2 = page A65





11.  $\pi/20$ 13.  $\pi - (3\sqrt{3}/2)$ 15.  $(9\sqrt{3}/8) - (\pi/4)$ 17.  $(4\pi/3) + 2\sqrt{3}$ 19.  $(\pi - 2)/8$ 21.  $(\pi/2) - 1$ 23.  $(\pi + 3\sqrt{3})/4$ 

- **25.**  $(\frac{1}{2}, \pi/3)$ ,  $(\frac{1}{2}, 5\pi/3)$ , and the pole
- **27.**  $(\sqrt{3}/2, \pi/3), (\sqrt{3}/2, 2\pi/3)$ , and the pole
- **29.** Intersection at  $\theta \approx 0.89$ , 2.25; area  $\approx 3.46$
- **31.**  $\sqrt{1 + (\ln 2)^2} (4^{\pi} 1)/\ln 2$
- **33.**  $\frac{8}{3}[(\pi^2 + 1)^{3/2} 1]$  **35.** 2.422

#### Exercises H = page A75

1. 10 - i**3.** 13 - i **5.** 12 - 7i **7.**  $-\frac{1}{2} + \frac{1}{2}i$ **11.** -i **13.** 5i **15.** 3 - 4i, 5 **17.** 4i, 49.  $\frac{1}{2} - \frac{1}{2}i$ 19.  $\pm \frac{3}{2}i$ **21.**  $4 \pm i$  **23.**  $-\frac{1}{2} \pm (\sqrt{7}/2)i$ **25.**  $3\sqrt{2} \left[ \cos(3\pi/4) + i \sin(3\pi/4) \right]$ **27.**  $5\left\{\cos\left[\tan^{-1}\left(\frac{4}{3}\right)\right] + i\sin\left[\tan^{-1}\left(\frac{4}{3}\right)\right]\right\}$ **29.**  $4[\cos(\pi/2) + i\sin(\pi/2)], \cos(-\pi/6) + i\sin(-\pi/6),$  $\frac{1}{2} [\cos(-\pi/6) + i \sin(-\pi/6)]$ **31.**  $4\sqrt{2} \left[\cos(7\pi/12) + i\sin(7\pi/12)\right]$ ,  $(2\sqrt{2}) [\cos(13\pi/12) + i\sin(13\pi/12)],$  $\frac{1}{4} [\cos(\pi/6) + i \sin(\pi/6)]$ 33. -1024**35.**  $-512\sqrt{3} + 512i$ **37.**  $\pm 1$ ,  $\pm i$ ,  $(1/\sqrt{2})(\pm 1 \pm i)$ **39.**  $\pm(\sqrt{3}/2) + \frac{1}{2}i, -i$ Im A Im A 0 Re Re 0 **43.**  $(-1/\sqrt{2}) + (1/\sqrt{2})i$ 45.  $-e^2$ 41. i 47.  $\cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta$ ,

 $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ 



. . .

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## TABLE OF INTEGRALS

#### BASIC FORMS

$$\begin{aligned} 1. \int u \, dv &= uv - \int v \, du &= 6. \int \sin u \, du &= -\cos u + C &= 11. \int \csc u \cot u \, du &= -\csc u + C &= 16. \int \frac{du}{\sqrt{a^2 - u^2}} &= \sin^{-1} \frac{u}{a} + C \\ 2. \int u^n \, du &= \frac{1}{n+1} u^{n+1} &= 7. \int \cos u \, du &= \sin u + C &= 12. \int \tan u \, du &= \ln|\sec u| + C &= 17. \int \frac{du}{a^2 + u^2} &= \frac{1}{a} \tan^{-1} \frac{u}{a} + C \\ + C, &n \neq -1 &= 1 \\ 3. \int \frac{du}{u} &= \ln|u| + C &= 8. \int \sec^2 u \, du &= \tan u + C &= 13. \int \cot u \, du &= \ln|\sin u| + C &= 18. \int \frac{du}{u\sqrt{u^2 - a^2}} &= \frac{1}{a} \sec^{-1} \frac{u}{a} + C \\ 4. \int e^u \, du &= e^u + C &= 9. \int \csc^2 u \, du &= -\cot u + C &= 14. \int \sec u \, du &= \ln|\sec u + \tan u| + C &= 19. \int \frac{du}{a^2 - u^2} &= \frac{1}{2a} \ln\left|\frac{u + a}{u - a}\right| + C \\ 5. \int a^u \, du &= \frac{1}{\ln a} a^u + C &= 10. \int \sec u \tan u \, du &= \sec u + C &= 15. \int \csc u \, du &= \ln|\csc u - \cot u| + C &= 20. \int \frac{du}{u^2 - a^2} &= \frac{1}{2a} \ln\left|\frac{u - a}{u + a}\right| + C \end{aligned}$$

### FORMS INVOLVING $\sqrt{a^2 + u^2}, a > 0$

21. 
$$\int \sqrt{a^2 + u^2} \, du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$$
  
22. 
$$\int u^2 \sqrt{a^2 + u^2} \, du = \frac{u}{8} (a^2 + 2u^2) \sqrt{a^2 + u^2} - \frac{a^4}{8} \ln(u + \sqrt{a^2 + u^2}) + C$$
  
23. 
$$\int \frac{\sqrt{a^2 + u^2}}{u} \, du = \sqrt{a^2 + u^2} - a \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C$$
  
24. 
$$\int \frac{\sqrt{a^2 + u^2}}{u^2} \, du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln(u + \sqrt{a^2 + u^2}) + C$$
  
25. 
$$\int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{a^2 + u^2}) + C$$

## FORMS INVOLVING $\sqrt{a^2-u^2}, a>0$

$$30. \int \sqrt{a^2 - u^2} \, du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$31. \int u^2 \sqrt{a^2 - u^2} \, du = \frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$32. \int \frac{\sqrt{a^2 - u^2}}{u} \, du = \sqrt{a^2 - u^2} - a \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$

$$33. \int \frac{\sqrt{a^2 - u^2}}{u^2} \, du = -\frac{1}{u} \sqrt{a^2 - u^2} - \sin^{-1} \frac{u}{a} + C$$

$$34. \int \frac{u^2 \, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

## FORMS INVOLVING $\sqrt{u^2-a^2}, a > 0$

$$39. \int \sqrt{u^2 - a^2} \, du = \frac{u}{2} \sqrt{u^2 - a^2} - \frac{a^2}{2} \ln|u + \sqrt{u^2 - a^2}| + C$$

$$40. \int u^2 \sqrt{u^2 - a^2} \, du = \frac{u}{8} (2u^2 - a^2) \sqrt{u^2 - a^2} - \frac{a^4}{8} \ln|u + \sqrt{u^2 - a^2}| + C$$

$$41. \int \frac{\sqrt{u^2 - a^2}}{u} \, du = \sqrt{u^2 - a^2} - a \cos^{-1} \frac{a}{u} + C$$

$$42. \int \frac{\sqrt{u^2 - a^2}}{u^2} \, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln|u + \sqrt{u^2 - a^2}| + C$$

26. 
$$\int \frac{u^2 du}{\sqrt{a^2 + u^2}} = \frac{u}{2} \sqrt{a^2 + u^2} - \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$$
  
27. 
$$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln \left| \frac{\sqrt{a^2 + u^2} + a}{u} \right| + C$$
  
28. 
$$\int \frac{du}{u^2 \sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$$
  
29. 
$$\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 + u^2}} + C$$

35. 
$$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$
  
36. 
$$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2u}\sqrt{a^2 - u^2} + C$$
  
37. 
$$\int (a^2 - u^2)^{3/2} du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$
  
38. 
$$\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$$

43. 
$$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln |u + \sqrt{u^2 - a^2}| + C$$
  
44. 
$$\int \frac{u^2 du}{\sqrt{u^2 - a^2}} = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 - a^2}| + C$$
  
45. 
$$\int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$$
  
46. 
$$\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2 \sqrt{u^2 - a^2}} + C$$

## TABLE OF INTEGRALS

#### FORMS INVOLVING a + bu

$$41. \int \frac{u \, du}{a + bu} = \frac{1}{b^2} (a + bu - a \ln |a + bu|) + C$$

$$48. \int \frac{u^2 \, du}{a + bu} = \frac{1}{2b^3} [(a + bu)^2 - 4a(a + bu) + 2a^2 \ln |a + bu|] + C$$

$$49. \int \frac{du}{u(a + bu)} = \frac{1}{a} \ln \left| \frac{u}{a + bu} \right| + C$$

$$50. \int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left| \frac{a + bu}{u} \right| + C$$

$$51. \int \frac{u \, du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2} \ln |a + bu| + C$$

$$52. \int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2} \ln \left| \frac{a + bu}{u} \right| + C$$

$$53. \int \frac{u^2 \, du}{u(a + bu)^2} = \frac{1}{b^3} \left( a + bu - \frac{a^2}{a + bu} - 2a \ln |a + bu| \right) + C$$

$$54. \int u \sqrt{a + bu} \, du = \frac{2}{15b^2} (3bu - 2a) (a + bu)^{3/2} + C$$

$$55. \int \frac{u \, du}{\sqrt{a + bu}} = \frac{2}{3b^2} (bu - 2a) \sqrt{a + bu} + C$$

#### TRIGONOMETRIC FORMS

63.  $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$ 64.  $\int \cos^2 u \, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$ 65.  $\int \tan^2 u \, du = \tan u - u + C$ 66.  $\int \cot^2 u \, du = -\cot u - u + C$ 67.  $\int \sin^3 u \, du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$ 68.  $\int \cos^3 u \, du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$ 69.  $\int \tan^3 u \, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$ 70.  $\int \cot^3 u \, du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$ 71.  $\int \sec^3 u \, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$ 72.  $\int \csc^3 u \, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$ 73.  $\int \sin^n u \, du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u \, du$ 74.  $\int \cos^n u \, du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u \, du$ 75.  $\int \tan^n u \, du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u \, du$ 

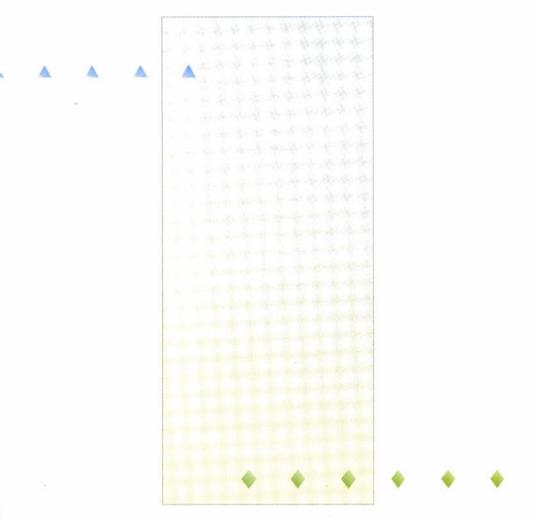
51. 
$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}} \right| + C, \text{ if } a > 0$$
$$= \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bu}{-a}} + C, \text{ if } a < 0$$
58. 
$$\int \frac{\sqrt{a+bu}}{u} du = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}}$$
59. 
$$\int \frac{\sqrt{a+bu}}{u^2} du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2} \int \frac{du}{u\sqrt{a+bu}}$$
60. 
$$\int u^n \sqrt{a+bu} du$$
$$= \frac{2}{b(2n+3)} \left[ u^n (a+bu)^{3/2} - na \int u^{n-1} \sqrt{a+bu} du \right]$$
61. 
$$\int \frac{u^n du}{\sqrt{a+bu}} = \frac{2u^n \sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)} \int \frac{u^{n-1} du}{\sqrt{a+bu}}$$
62. 
$$\int \frac{du}{u^n \sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{du}{u^{n-1} \sqrt{a+bu}}$$

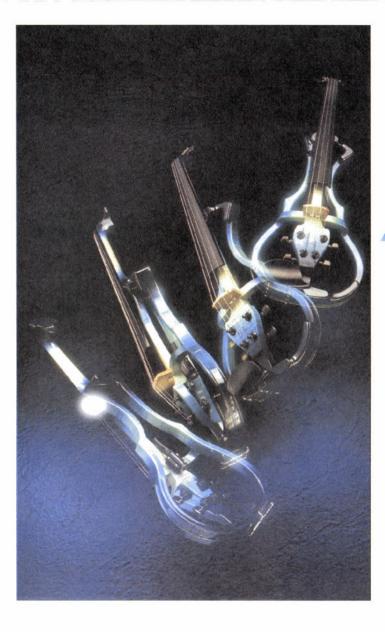
76. 
$$\int \cot^{n} u \, du = \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u \, du$$
  
77. 
$$\int \sec^{n} u \, du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$$
  
78. 
$$\int \csc^{n} u \, du = \frac{-1}{n-1} \cot u \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} u \, du$$
  
79. 
$$\int \sin au \sin bu \, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$$
  
80. 
$$\int \cos au \cos bu \, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$$
  
81. 
$$\int \sin au \cos bu \, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$$
  
82. 
$$\int u \sin u \, du = \sin u - u \cos u + C$$
  
83. 
$$\int u \cos u \, du = \cos u + u \sin u + C$$
  
84. 
$$\int u^{n} \sin u \, du = -u^{n} \cos u + n \int u^{n-1} \cos u \, du$$
  
85. 
$$\int u^{n} \cos u \, du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} u \cos^{m} u \, du$$
  
86. 
$$\int \sin^{n} u \cos^{m} u \, du = -\frac{\sin^{n-1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m} \int \sin^{n} u \cos^{m-2} u \, du$$

# CALCULUS

## **Concepts AND Contexts**

## SINGLE · VARIABLE · · · · · · ·





#### About the Cover

The photograph shows an electric violin made by David Bruce Johnson.

An acoustic violin, with its sound hole in the shape of an integral sign, became a symbol of James Stewart's previous calculus textbooks. Stewart plays both an 18th-century French violin and the blue electric violin that appears on the cover of this book.

The electric violin reflects the increased use of technology in calculus instruction, as well as a more informal approach to the subject. The quadruple image symbolizes the use of the Rule of Four throughout the book—four ways of looking at the same object.

## ALGEBRA

#### **ARITHMETIC OPERATIONS**

$a(b \pm a) = ab \pm aa$	a c ad + bc
a(b+c) = ab + ac	$\frac{d}{b} + \frac{d}{d} = \frac{d}{bd}$
	<u>a</u>
a + c = a + c	$b \_ a \lor d \_ ad$
$b = b + \overline{b}$	$\frac{1}{c} = \frac{1}{b} \times \frac{1}{c} = \frac{1}{bc}$
	$\overline{d}$

#### **EXPONENTS AND RADICALS**

$x^m x^n = x^{m+n}$	$\frac{x^m}{x^n} = x^{m-n}$
$(x^m)^n = x^{mn}$	$x^{-n} = \frac{1}{x^n}$
$(xy)^n = x^n y^n$	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
$x^{1/n} = \sqrt[n]{x}$	
$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$	$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^n$
$\sqrt[m]{\sqrt{x}} = \sqrt[n]{\sqrt{x}} = \sqrt[mn]{x}$	$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

#### FACTORING SPECIAL POLYNOMIALS

 $x^{2} - y^{2} = (x + y)(x - y)$  $x^{3} + y^{3} = (x + y)(x^{2} - xy + y^{2})$  $x^{3} - y^{3} = (x - y)(x^{2} + xy + y^{2})$ 

#### **BINOMIAL THEOREM**

$$(x + y)^{2} = x^{2} + 2xy + y^{2} \qquad (x - y)^{2} = x^{2} - 2xy + y^{2}$$
$$(x + y)^{3} = x^{3} + 3x^{2}y + 3xy^{2} + y^{3}$$
$$(x - y)^{3} = x^{3} - 3x^{2}y + 3xy^{2} - y^{3}$$
$$(x + y)^{n} = x^{n} + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^{2}$$
$$+ \dots + \binom{n}{k}x^{n-k}y^{k} + \dots + nxy^{n-1} + y^{n}$$
where  $\binom{n}{k} = \frac{k(k-1)\cdots(k-n+1)}{1\cdot 2\cdot 3\cdots n}$ 

#### QUADRATIC FORMULA

If  $ax^2 + bx + c = 0$ , then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

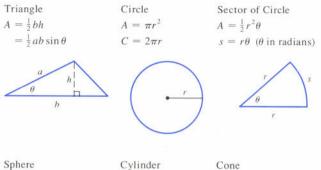
#### **INEQUALITIES AND ABSOLUTE VALUE**

If a < b and b < c, then a < c. If a < b, then a + c < b + c. If a < b and c > 0, then ca < cb. If a < b and c < 0, then ca > cb. If a > 0, then |x| = a means x = a or x = -a|x| < a means -a < x < a|x| > a means x > a or x < -a

## GEOMETRY

#### **GEOMETRIC FORMULAS**

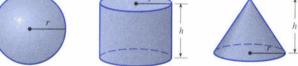
Formulas for area A, circumference C, and volume V:



 $V = \frac{4}{3}\pi r^3$  $A = 4\pi r^2$ 







#### DISTANCE AND MIDPOINT FORMULAS

Distance between  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of 
$$\overline{P_1P_2}$$
:  $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ 

#### LINES

Slope of line through  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ :

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through  $P_1(x_1, y_1)$  with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope *m* and *y*-intercept *b*:

y = mx + b

#### CIRCLES

Equation of the circle with center (h, k) and radius r:

$$(x - h)^{2} + (y - k)^{2} = r^{2}$$

#### TRIGONOMETRY

#### ANGLE MEASUREMENT

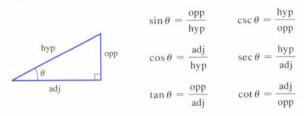
 $\pi$  radians = 180°

 $1^\circ = \frac{\pi}{180} \text{ rad} \qquad 1 \text{ rad} = \frac{180^\circ}{\pi}$ 

 $s = r\theta$ 

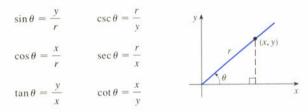
 $(\theta \text{ in radians})$ 

#### RIGHT ANGLE TRIGONOMETRY

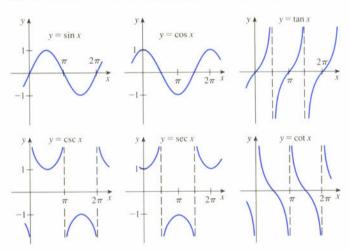


θ

#### TRIGONOMETRIC FUNCTIONS



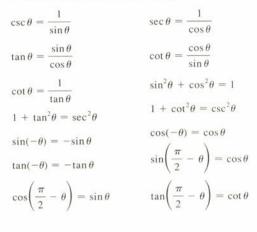
#### GRAPHS OF THE TRIGONOMETRIC FUNCTIONS



#### TRIGONOMETRIC FUNCTIONS OF IMPORTANT ANGLES

θ	radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	0	1	0
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90°	$\pi/2$	1	0	

#### FUNDAMENTAL IDENTITIES

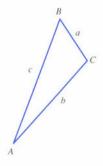


#### THE LAW OF SINES

 $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ 

#### THE LAW OF COSINES

 $a<sup>2</sup> = b<sup>2</sup> + c<sup>2</sup> - 2bc \cos A$   $b<sup>2</sup> = a<sup>2</sup> + c<sup>2</sup> - 2ac \cos B$  $c<sup>2</sup> = a<sup>2</sup> + b<sup>2</sup> - 2ab \cos C$ 



#### ADDITION AND SUBTRACTION FORMULAS

 $\sin(x + y) = \sin x \cos y + \cos x \sin y$  $\sin(x - y) = \sin x \cos y - \cos x \sin y$  $\cos(x + y) = \cos x \cos y - \sin x \sin y$  $\cos(x - y) = \cos x \cos y + \sin x \sin y$  $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$  $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$ 

#### DOUBLE-ANGLE FORMULAS

 $\sin 2x = 2\sin x \cos x$   $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$  $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$ 

#### HALF-ANGLE FORMULAS

$\sin^2 x =$	$1 - \cos 2x$	$\cos^2 x =$	$1 + \cos 2x$
	2		2